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Plane Geometry

WELCHONS
KRICKENBERGER
PEARSON

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Plane Geometry

By A. M. WELCHONS
W. R. KRICKENBERGER
and HELEN R. PEARSON
The Arsenal Technical High School
INDIANAPOLIS, INDIANA

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FOREWORD

This *Plane Geometry* is based upon the earlier editions of the Welchons-Krickenberger plane geometries. This new text retains those features of the older ones which have proved themselves most desired by teachers and best adapted to the needs of pupils. We have added new features to keep the book in line with the latest accepted thinking in the field of geometry teaching and learning.

In this text considerable attention is given to helping pupils understand the nature of deductive proof—the role of definitions and the undefined terms, the meaning of and use of assumptions, the meaning of logical sequence, and the requirements of necessary and sufficient conditions. Attention is given to helping pupils see how experimentation, induction, and deduction must supplement each other in scientific reasoning. Attention is also given to helping pupils discover proofs for themselves. Opportunities are provided whereby pupils may practice deductive reasoning in problem situations similar to those they meet in everyday living. And throughout the book pupils are encouraged to think of geometry as a logical system of thought which relates each theorem to other theorems.

The new text helps pupils see the interrelationships which exist between geometry and other fields of mathematics—arithmetic, algebra, trigonometry, analytic geometry, and solid geometry. A section on approximate numbers helps to make clear the role of these numbers in the field of geometry.

The skills acquired in earlier mathematics courses in arithmetic and algebra are maintained and extended. This work is an integral part of the course and is introduced where it is most useful in geometry.

Illustrations and problem materials showing the uses of geometry include the newer discoveries and engineering feats.

Special Features of the Text

1. Creation of Interest. Since interest is one of the greatest motivating forces in any study, we have attempted throughout the book to interest pupils in the study of geometry. The illustrations show applications of geometry to art, nature, industry, and engineering. In the problem material there is an application of geometry to various trades. At many

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points in the book the usefulness of geometry as an aid to everyday reasoning is explained, and the pupil is encouraged to strive for clear thinking in his daily life. Brief references to the history of geometry also add interest.

Color, used as an effective teaching aid, also adds to the appeal of the book.

2. Gradual Approach to Formal Geometry. In the early pages of the book pupils are led to see the need for basing a conclusion on proof rather than on observation and measurement. Proof is introduced gradually throughout the first three chapters. The role of the undefined terms, definitions, and assumptions is discussed. First proofs consist of simple one-step if-then-because relationships, many of them equations.

3. Standard Requirements Met. The most important propositions and corollaries are marked with two stars; the subsidiary propositions and corollaries are marked with one star; and the less important propositions and corollaries, including the theorems in optional subject matter, are unmarked. This system of marking is the result of a study of the relative importance of the propositions, in which the authors took into consideration the reports of leading organizations.

4. Unusual Flexibility. By organizing the text into eighteen chapters we have provided for a high degree of flexibility and adaptability to various classes and curricula. In some places footnotes have been provided to show possible variations in sequence of topics. For example, see page 111.

Any or all of the last four chapters may be omitted without destroying any sequence. Footnotes in the preceding chapters designate appropriate times of presenting Chapters 16 and 17.

The abundance of exercises in the text also increases its adaptability.

5. Provision for Individual Differences. To provide for individual differences, three levels of work are provided. Exercises marked *A* are for all pupils, those marked *B* offer additional work for pupils having more ambition and interest, and those marked *C* are a challenge to the exceptional pupils.

A minimum course consists of the proofs of the propositions and corollaries marked with two stars, an informal treatment of the propositions and corollaries marked with one star, and some of the *A* exercises.

A medium course consists of the proofs of all starred propositions and corollaries, exercises in *A* groups, and some exercises in *B* groups.

A maximum course includes the proofs of additional propositions and

corollaries and some exercises in the *A*, *B*, and *C* groups. Some of the optional work can be assigned. No class can be expected to do all the exercises in this text during one school year, but some exceptional pupils may be able to do so.

6. Planning the Proof. The logical procedure in discovering a proof of a theorem or original exercise involves a combination of analytic and synthetic methods. A plan of procedure is described on page 104. Most of the theorems proved in the book also contain a section "Planning the Proof." Both the procedure outlined on page 104 and these "Planning the Proof" sections of the various proofs are recommended as aids to the pupil in attacking a proof.

7. Summary of Bases for Proof. At the close of each chapter (except Chapters 1, 2, 15–18) there is a summary of the statements of that chapter which may be used as bases for other proofs. A complete summary is given on pages 563–573. The complete summary may be used in place of the chapter summaries if desired. For example, in § 287, instead of writing, "We shall use § 97 a," we might have written, "We shall use § 567 b."

8. Proper Stress on Theorems, Problems, and Corollaries. A proper balance of theorems, corollaries, construction problems, and numerical applications is maintained throughout the text. An undue emphasis on any particular type of work is thus prevented.

9. Preparation for Theorems. Where desirable, the groups of *A* exercises include developments of theorems which appear later. An example is Ex. 3, p. 111. By this procedure the pupils are encouraged to discover proofs and are not thrust into new situations without proper preparation.

10. Treatment of Corollaries. The corollaries in this text are corollaries in the strictest sense and are not theorems listed as corollaries to reduce the number of theorems. They are, in most cases, based upon theorems and corollaries which immediately precede.

11. Reasons Required for Constructions. Reasons are required for all constructions, thus preventing errors that might otherwise result. For example, such statements as "Construct $CD \perp AB$ and bisecting $\angle ACB$" are usually prevented when pupils give reasons for their constructions.

12. Training in Solution of Original Exercises. Various means are used to teach pupils how to solve original exercises successfully. These are as follows:

a. By the algebraic proofs in Chapter 3.
b. By training in planning the proof.
c. By the gradation of exercises. For example, see one-step exercises on pages 62 and 63, and two-step exercises on page 64.
d. By the use of examples. See pages 65, 66, and 156, 157.
e. By the use of exercises after theorems.
f. By explicit directions for (1) proving corollaries (§ 70); (2) proving theorems (§§ 81, 201, 408); (3) indirect proofs (§ 102); (4) constructions (§§ 175, 374, 375); (5) proving locus theorems (§ 342).

13. Applications. The text is well supplied with interesting applications which can be comprehended by most pupils without any extensive study of other subjects. Many pupils will find Chapter 18, on aeronautics, of special interest.

14. Reasoning Applied to Life Situations. The sections on the application of the reasoning of geometry to life situations are optional. Many teachers will wish to use at least some of this material.

15. Word Lists. To improve spelling and give reviews of word meaning, many word lists are included.

16. Tests. The text includes 42 tests at the ends of chapters. These tests may be used to test the pupil's mastery of subject matter but they serve better as reviews and self-measuring tests.

The Authors

CONTENTS

CHAPTER		PAGE
1	Lines and Angles	4
2	Inductive Reasoning	30
3	Deductive Reasoning	44
4	Triangles	76
5	Parallel and Perpendicular Lines	128
6	Constructions	184
7	Polygons	200
8	Areas of Polygons	236
9	Circles, Angles, and Arcs	284
10	Measurement of Angles and Arcs	320
11	Loci	346
12	Proportion and Proportional Line Segments	380
13	Similar Polygons	402
14	Regular Polygons and the Circle	450
15	Elements of Trigonometry	492
16	Inequalities	504
17	Coordinate Geometry	518
18	Additional Topics	540

1. Aeronautics, 541

2. Geometric Relations; Orthographic Projection, 557

Summary of Principal Bases for Proof 563

Tables 574

Index 577

Symbols and Abbreviations

The following symbols and abbreviations are used in this text. Each should be learned when it first occurs in the subject matter.

SYMBOLS

$\angle$ angle
$\longrightarrow$ approaches as a limit; tends to
$\frown$ arc
$\odot$ circle
$=$ (is) equal to; equals; equal
$\neq$ is not equal to
$>$ is greater than
$<$ is less than
$\cong$ (is) congruent to; congruent
$\parallel$ (is) parallel to; parallel
$\perp$ (is) perpendicular to; perpendicular
$\doteq$ is equal in degrees to
$\sim$ (is) similar to; similar
$\square$ parallelogram
§ section
$\therefore$ therefore
$\triangle$ triangle
x', x'', . . . x-prime, x-second, . . .
x_1, x_2, . . . x-sub-one, x-sub-two, . . .

The plural of any symbol representing a noun is formed by adding the letter *s*. Thus Ⓢ means *circles*.

ABBREVIATIONS

adj., adjacent
alt., alternate
Asmt., assumption
comp., complementary
Const., construction
Cor., corollary
corr., corresponding
cos, cosine
Def., definition
Ex., exercise
ext., exterior
Fig., figure
Hyp., hypothesis
Iden., identity
int., interior
isos., isosceles
opp., opposite
Prob., problem
quad., quadrilateral
rect., rectangle
rt., right
sin, sine
st., straight
supp., supplementary
tan, tangent
Th., theorem

S.A.S., If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

A.S.A., If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

S.S.S., If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.

Plane Geometry

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A few years from now you will find opportunities such as those suggested on these pages awaiting you. Will you be prepared?

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This young designer says, "I studied all the mathematics offered in high school, including algebra, geometry, and trigonometry."

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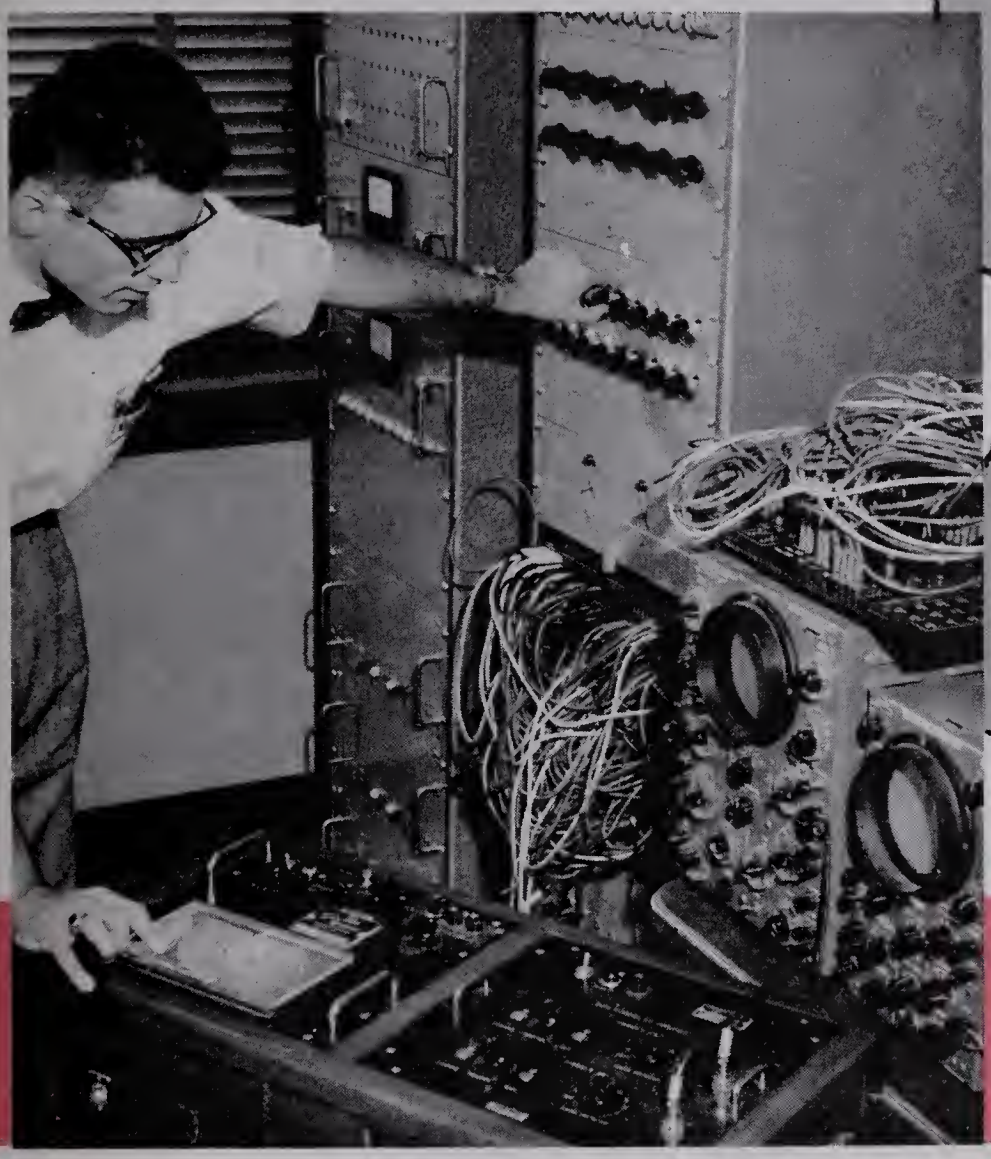

An engineer working on a nuclear submarine boiler level control system says, "My college courses included advanced mathematics."

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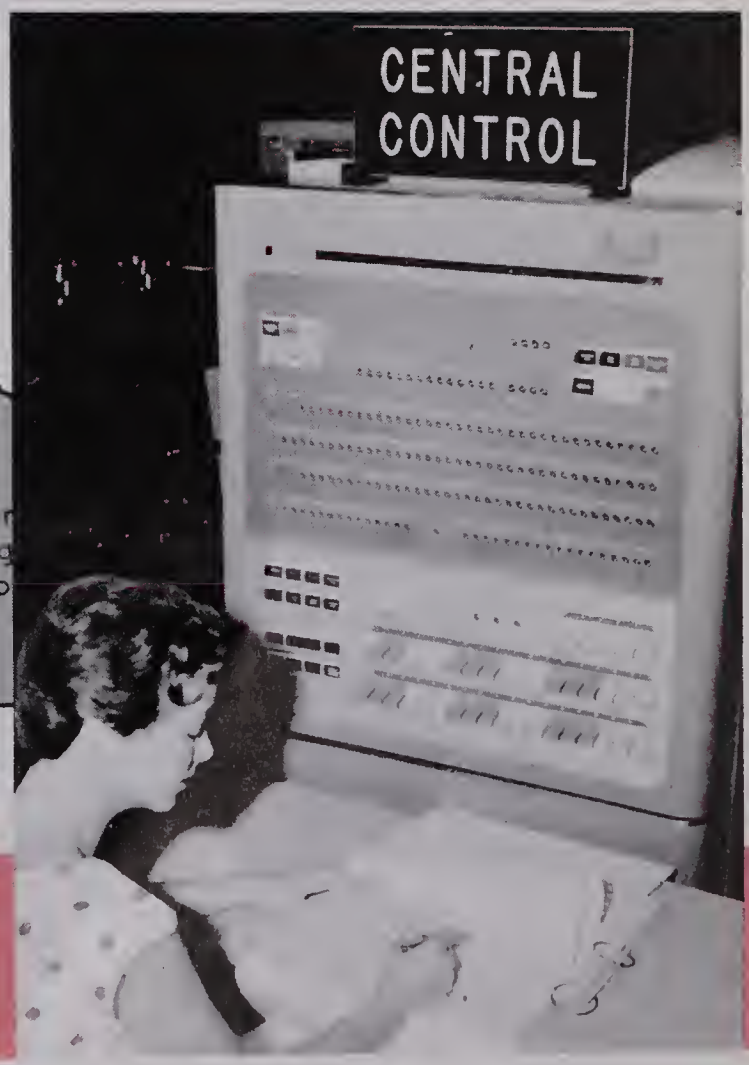

The numerical analyst at the IBM 704 Computer says, "I suggest four years of mathematics in high school."

Photos courtesy of General Electric Company

CHAPTER 1

Lines and Angles

ALL YOUR LIFE
you have been studying geometry informally,
for all your life you have been studying
the sizes and shapes of things.
That is what geometry is—
the accurate study of size and shape.
Now you are ready
to organize your ideas of geometry
into a logical system of thought.

Logical thinking is correct and productive thinking. Building a logical system of thought means correctly combining ideas in such a way that they produce other ideas. In a sense it is the process that we sometimes refer to as "putting two and two together." If one of your friends says, "By putting two and two together, I have decided that my parents are giving me a watch for my birthday," you know that through combining various bits of evidence your friend has been able to draw his conclusion. No one piece of evidence would have been sufficient to give him the idea, but the combination of several pieces did produce it.

Of course, if we combine wrong ideas, or if we combine correct ideas in a wrong way, our conclusions are worthless. In this course in geometry you will learn to weigh ideas for correctness, and you will learn to combine them correctly. By correctly combining the ideas of size and shape which you have already accumulated, you will be able to discover new ideas. These ideas, in turn, can be used to produce still other ideas until you have built a whole system of logical thought about geometry.

The geometric ideas you discover will be of great value to you if you are planning a career in engineering, drafting, architecture, astronomy, or any one of a large number of other occupations. But the process of building a logical system of thought is important, also—to all of you regardless of what field of work you enter. Studying how to make your thinking about geometry logical can help to make all of your thinking logical. This is important, for much of your success and happiness will depend upon your ability to think logically.

1. Importance of Definitions

If you would learn to think accurately, first give attention to understanding the exact meanings of the words you use. Muddled ideas of meanings can only lead to muddled thinking. Imagine, for example, the confusion that would result if the referee in a basketball game had no clear idea of what constitutes a foul.

In geometry, such key words as *triangle*, *angle*, *perpendicular*, and *circle* should be defined accurately. Definitions give us a solid foundation upon which to base further thinking.

2. Undefined Terms

In defining a term, we express its meaning by use of simpler (better understood) terms which have already been defined. These latter terms, in turn, can be defined in still other simple terms, and so on, but obviously

the process cannot go on forever. Eventually we must come to a stopping place consisting of a few terms so well understood that there are no simpler terms to use in explaining them. For example, if we look in the dictionary for the meaning of the word "straight," we find words expressing the idea "not curved," and if we look for "curved," we find words expressing the idea "not straight." Unless we know the meaning of one or the other of these words, the dictionary is of little help. We allow such terms to remain *undefined terms*, for any attempt to explain them results in either the kind of circular reasoning we have just illustrated, or the use of terms more complicated than the term being explained.

In geometry, such terms as *point*, *line*, *plane*, and *solid* are generally considered undefined terms. All of us know what a point is and what a line is.

We represent a *point* on paper by a dot, though the dot is not a real geometric point since it has some size. A point, as we mean it in geometry, has no length, breadth, or thickness, that is, no size. A point cannot be seen. To name a point, we place a capital letter near the dot which represents it; thus, $\cdot A$ represents a point which we have named A.

We represent a geometric *line* by drawing a line on paper. Unless otherwise stated, the word "line" means "straight line." A straight line is represented by a straight mark and is named by any two points on it, or by a small letter near it. The line shown here is read "line AB" or "line l." A *straight line* extends infinitely far in both directions; that is, it has no ends.

Although we represent a line by a mark on paper or the blackboard, we must remember that a geometric line cannot be seen because it has no width or thickness. A geometric line has only one dimension—length.

To draw a straight line, we use an instrument known as a *straightedge*. An ordinary ruler is a combined straightedge and device for measuring lengths.

We represent a *curved line*, or *curve*, by a curved mark, such as CD.

In this course in geometry you will have little occasion to represent a *plane* because the surface of the paper or blackboard upon which you write and draw is a portion of a plane. A plane has length and width, but no thickness. Both the length and width of a plane are infinite. "Infinite" means "becoming large beyond any fixed bound." Can you

The engineers who plan and build today's great hydroelectric systems found geometry a basic part of their training. This powerhouse is situated on the Saguenay River and has a turbine capacity of 1,200,000 horsepower.

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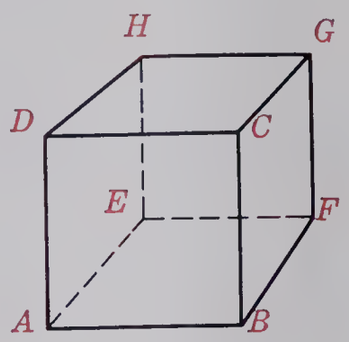
imagine the length and width of the front cover of your geometry book extended without end?

A *solid* has three dimensions—length, width, and thickness. Such solids as blocks, bricks, and crystals are *physical solids* because they have color, weight, and other physical properties. The picture at the left below



represents a physical solid. If we disregard the physical properties of a solid and think only of its size and shape, we are considering a *geometric solid*. A geometric solid cannot be seen, but we sometimes represent such a solid by drawing its outline as shown at the right above.

EXERCISES

A

1. Draw two lines, AB and CD, which *intersect* (cross each other) in E. How large is point E?

2. Place a point P on paper. Draw a line RS through P. How many other lines can you draw through P? How many lines can be drawn through any point?

3. Place two points, A and B, on paper. With a straightedge draw a line through them. Try to draw another line through them. What is your conclusion about the number of lines that can be drawn through two points?

4. Draw lines AB and CD which intersect (cut each other) in point E. Do you think AB and CD can intersect in another point if they are extended?

5. When men set fence posts, why do they first set the end posts?

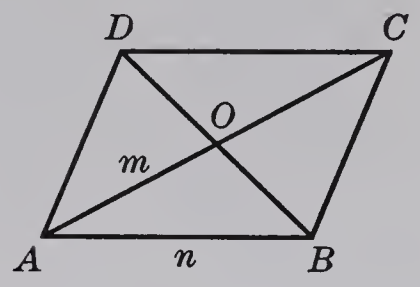
6. Draw a curve through two points, A and B. Can you draw another curved line through these points? How many curved lines can be drawn through two points?

7. What kind of line is formed when a piece of paper is folded?

8. Name line m in the diagram in two other ways.

9. Name line AB using one letter.

10. How many lines are represented in the diagram?



Ex. 8–10

3. Definitions

By building upon the undefined terms we can obtain definitions. As we have already indicated, a *definition* of a term is a statement of the meaning of the term. A good definition

a. Names the term to be defined;

b. Places it in the smallest class to which it belongs;

c. Gives the definite characteristics which distinguish it from other members of that class;

d. Is reversible.

We may define *line segment* by saying: A line segment is a definite part of a line. Since "definite" means "with distinct limits," our definition is a short way of saying that a line segment is a part of a line which begins at one point on a line and ends at another.

Thus, A and B are the end points of the segment AB which is a part of line l.

Notice that in our definition we mention the term "line segment" which we are defining; we place "line segment" in the class, "part of a line"; and we distinguish a line segment from any other part of a line by saying that it is a "definite" part.

When we say that a definition is reversible, we mean that its subject and predicate are interchangeable. Thus, the definition for line segment can be expressed in either of the two forms:

1. A line segment is a definite part of a line.
2. A definite part of a line is a line segment.

We may define *ray* by saying: A ray is a part of a line extending infinitely far in one direction from a point. Thus, ray m extends infinitely far in one direction from A.

Do you see that while "line segment" and "ray" both fall into the class "part of a line," the definitions distinguish "line segment" from "ray" by saying that a line segment is a "definite" part of a line while a ray is an "infinite" part?

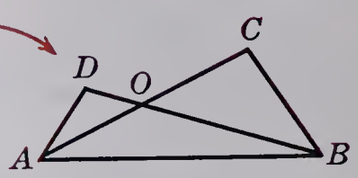
A *broken line* is a line composed of line segments. $ABCDE$ is a broken line.

EXERCISES

A

1. How many end points has a line segment? a ray? a line?

2. Name the nine line segments in the diagram.



3. For each of the following terms (1) state the smallest class into which the term falls; (2) state the definite characteristics which distinguish the term from other members of that class; (3) write a definition for the term.

TERM	SMALLEST CLASS	DISTINGUISHING CHARACTERISTICS
plane geometry		
study hall		
lead pencil		
equation		
dime		

4. State each of the following definitions reversed.

a. A polygon is a closed broken line.

b. A triangle is a polygon having three sides.

c. An isosceles triangle is a triangle having two equal sides.

4. Using Definitions

When John Burgess left Longmont High School he was listed as a sophomore because he had earned eight high-school credits in Longmont High School, where a sophomore was defined as a pupil with at least eight and not more than seventeen credits. When John entered Morton High School the following week, the principal said, "John, we shall have to list you as a freshman until you have earned an extra credit, for in this school

Bates College, Lewiston, Maine

In this college debating workshop key words are carefully defined.

we define a sophomore as a pupil who has earned at least nine and not more than seventeen credits."

Even though Longmont and Morton High Schools differed in their definition for "sophomore," we should not say that one was right and the other wrong. The important thing is not that the definitions were different, but that each definition was consistent with the curricular program within that particular school, and that having defined "sophomore" each school was consistently basing its further acting and thinking upon that definition.

Had Bill Morris transferred his ten Longmont credits to Morton, how would he have been classified in Morton? Had Sue Adams transferred her eight Morton credits to Longmont, how would she have been classified in Longmont? Had Ted Shore entered Morton with twenty high-school credits, would he have been classified as a sophomore? Do you see that the answers to these questions are found within the definition for "sophomore" used by the school under consideration? Do you see that a definition makes definite the characteristics of a term?

Throughout this book you will find many definitions. Read each carefully for each definition makes definite a particular term that you will use. Remember that the system of geometric thought that we are build-

ing in this book is based upon these definitions as they are given. Changing a definition might cause changes in our system of thought just as changing Morton's definition of "sophomore" might cause changes in the classification of many Morton pupils. Notice that we are not saying that the definitions in the book are the only definitions possible; we are saying that for the system of thought we are building, we are using the definitions stated.

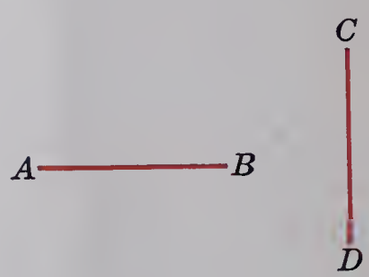
5. Distance between Two Points

In geometry the distance between two points is the length of the line segment joining them.

6. Equal Line Segments

Two line segments are *equal* if they can be made to coincide (fit together exactly). Since all definitions are reversible, we know that two equal line segments can be made to *coincide*.

If we wish to test the approximate equality of two line segments such as AB and CD, we can cut the paper, separating the two segments, and try to make one segment fit exactly on the other.



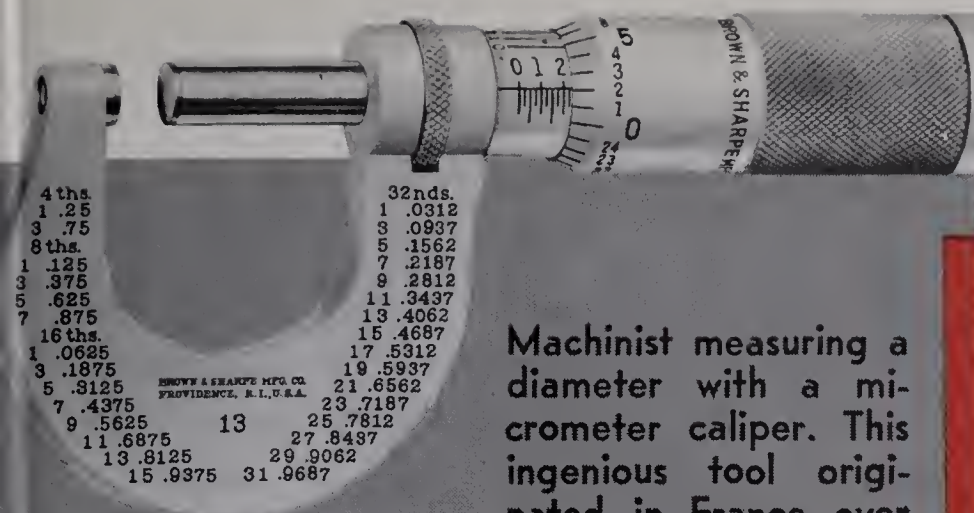
A better method of testing the approximate equality of the two segments is to use compasses.

Open the compasses so that one of the points can be placed on A and the other on B. Without changing the distance between the points, place the points of the compasses on the segment CD. If one of the points of the compasses falls on C and the other appears to fall on D, the two line segments are approximately equal. We say "approximately equal" because our eyes cannot detect very small differences. Do you see that if the two segments differed by $\frac{1}{1000}$ of an inch, the difference would not be detected by either of the procedures outlined?

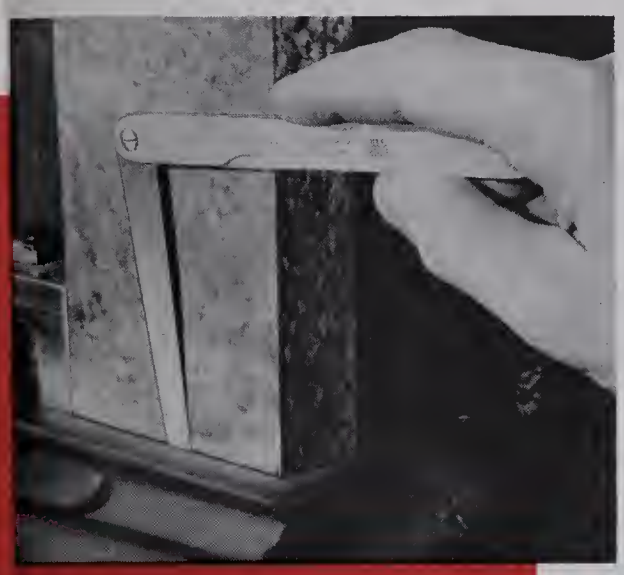
7. Measurement Is Approximate

No one can measure the length of an object and be sure that the measurement is correct. Yet the object can be measured to any degree of precision desired.

The degree of precision in any measurement depends upon the person making the measurement and upon the delicacy of the measuring in-

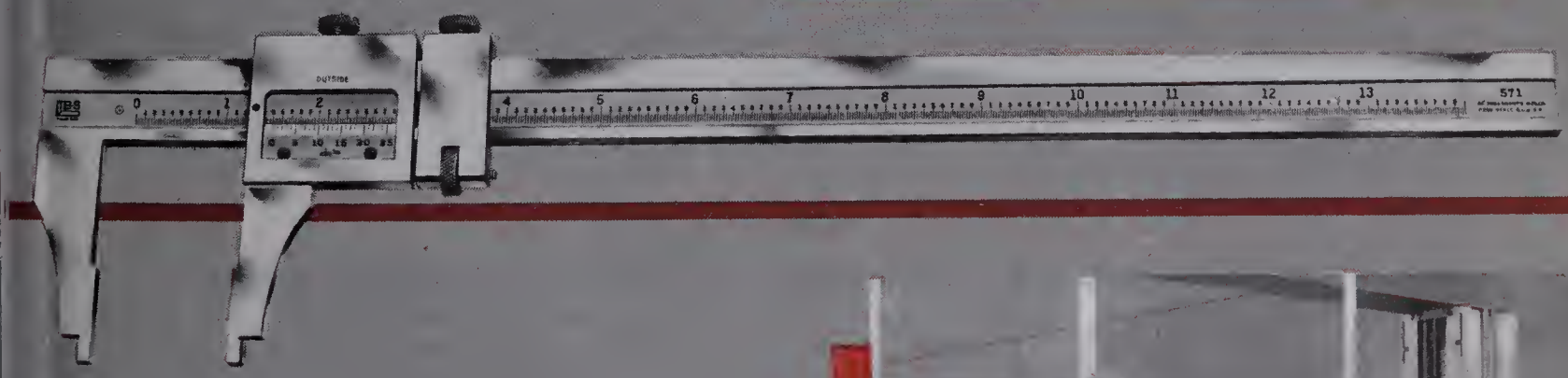


Machinist measuring a diameter with a micrometer caliper. This ingenious tool originated in France over 100 years ago.



This thickness gage has 21 blades, varying in thickness from .004" to .025" by .001" steps. It is being used to measure the space between the block and its enclosure.

The vernier caliper is still another tool used for measurements to the nearest thousandth of an inch. It uses the principle of the vernier scale, invented in 1631 by Pierre Vernier.



The picture shows a king-size caliper, accurate to within one ten-thousandth of an inch. It is used in the automobile industry to measure full-size clay models of tomorrow's cars.

courtesy of Brown & Sharpe Mfg. Co. and Ford Motor Company

strument. With an ordinary yardstick a measurement can be made to the nearest eighth of an inch. A machinist, using a micrometer caliper (commonly referred to as a "mike"), can measure objects to the nearest thousandth of an inch. The auto mechanic uses thickness gauges to measure the gaps in spark plugs. Measurements made to the thousandth of an inch are precise but they are not exact. Since any measurement is approximate it is not correct to say that two line segments are equal because they have the same measurement.

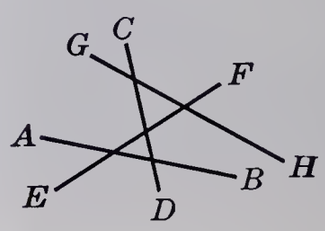
EXERCISES

A

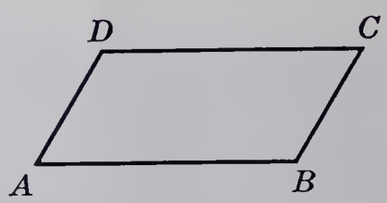
1. Using a ruler, find the length of this page to the nearest inch; to the nearest eighth of an inch. Can you find the length to the nearest sixteenth of an inch?

2. Find the length of the second line of print in Ex. 1. Express the length to the nearest eighth of an inch.

3. Using compasses, determine which line segments in the figure at the right are approximately equal. Which is the shortest segment? the longest?

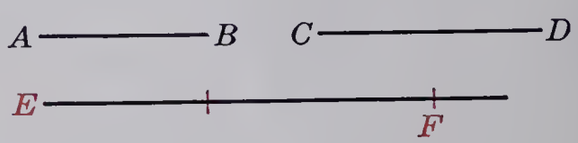


4. Using a ruler, measure each of the four sides of the polygon $ABCD$. Which sides appear to be equal?

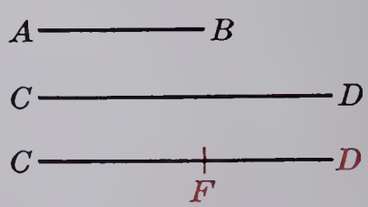


8. How to Add and Subtract Line Segments

Two line segments can be added by placing them end to end on the same line. In the figure the segment EF is the sum of the segments AB and CD.



One line segment can be subtracted from a longer line segment by placing the smaller segment upon the larger with one end of the smaller falling upon one end of the larger. In the figure, FD is the remainder when AB is subtracted from CD.

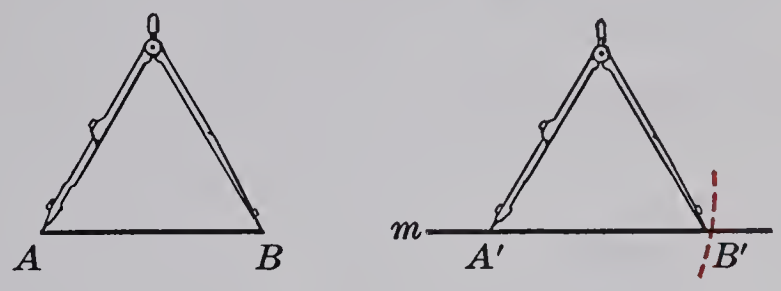


9. Constructions in Geometry

In this book we shall use the word "construct" to mean the making of an accurate drawing which meets certain requirements. Thus, the segments in the construction described below meet the requirements for being called equal line segments. Traditionally, the only instruments allowed in making constructions are a straightedge and compasses.

To construct a line segment equal to a given line segment we proceed as follows:

Let AB be the given line segment. Suppose you are required to construct another line segment equal to AB. Draw a line m and choose any point A' on it. Adjust the compasses so that the end points coincide with the points A and B of the given line. Without changing the opening between the legs of the compasses, set one of the points on A' and draw an arc which intersects m at point B'. Then $A'B' = AB$.



Of course, our construction is subject to many of the weaknesses of measurement. It cannot be made perfectly. When we say, "Then $A'B' = AB$," we only mean that if we could carry out the steps of the construction perfectly, then $A'B'$ would equal AB because they would meet the requirements of our definition for equal line segments.

EXERCISES

A

1. On your paper draw a line segment RS. Construct a line segment $R'S'$ equal to RS.

2. Construct a line segment equal in length to $2\,a$, $a + b$, $a - b$, $2\,a + 3\,b$.

3. Name the five line segments in AD.

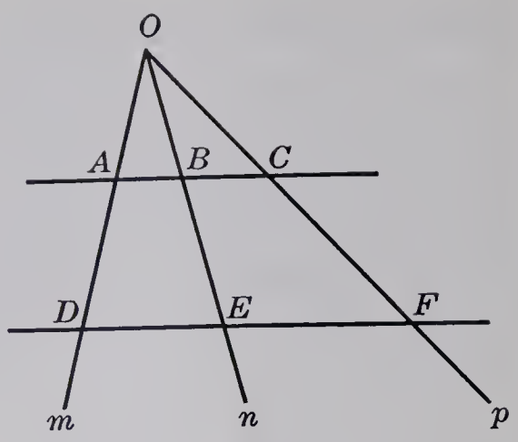
Using the drawing for Ex. 3, complete the following:

4. $AC = AB +$ __?__.

5. $AD = AB +$ __?__ $+$ __?__.

6. $AC = AD -$ __?__.

7. Using two letters, name line m in the diagram in three ways; name line n in three ways.



8. In this figure, $DF = DE +$ __?__.

9. $OD = OA +$ __?__.

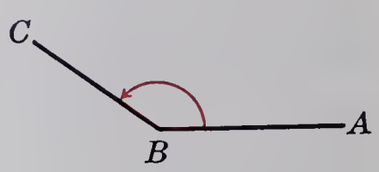
10. $OE =$ __?__ $+ BE$.

11. $DE = DF -$ __?__.

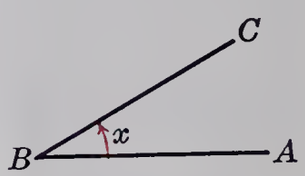
12. $OF =$ __?__ $+ CF$.

10. Angles

An *angle* ($\angle$) is a figure formed by two rays drawn from the same point. The rays are called the *sides* of the angle and their point of meeting is called the *vertex* of the angle. Thus AB and BC are the sides of $\angle ABC$ and B is the vertex.

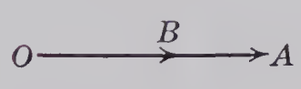


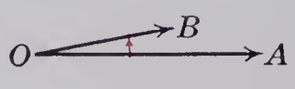
There are three common ways of naming an angle: (1) by a capital letter at its vertex, as $\angle B$; (2) by a small letter within the angle, as $\angle x$; and (3) by three capital letters, the middle letter being the vertex letter and the other two being the names of points on the sides of the angle, as $\angle ABC$. In naming an angle by three letters, you should always remember that the vertex letter is the middle letter. When confusion might arise, always name an angle by three letters.

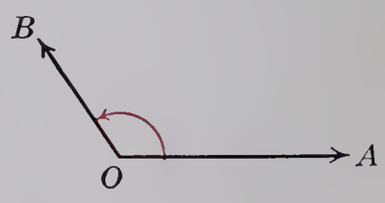


11. Size of Angles

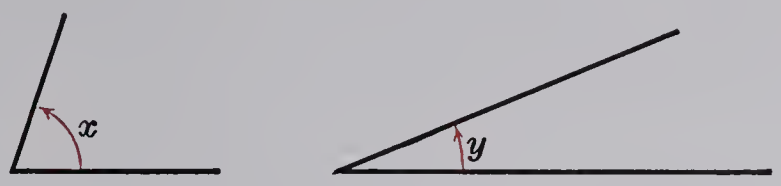
Suppose that the ray OB lies upon the ray OA. Now let the ray OB rotate counterclockwise about O. As it rotates, it generates $\angle AOB$.





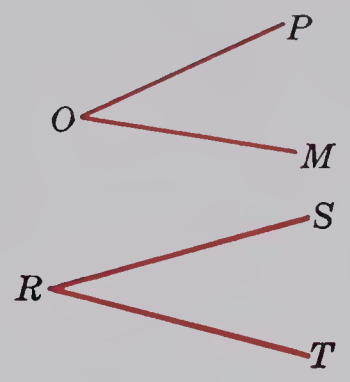


Can you see that the size of an angle depends upon the amount of rotation of one of its sides? We say that the *size* of an angle depends only upon the



amount of opening between its sides. Thus $\angle x$ is greater than $\angle y$, even though the sides of $\angle y$ are longer than those of $\angle x$.

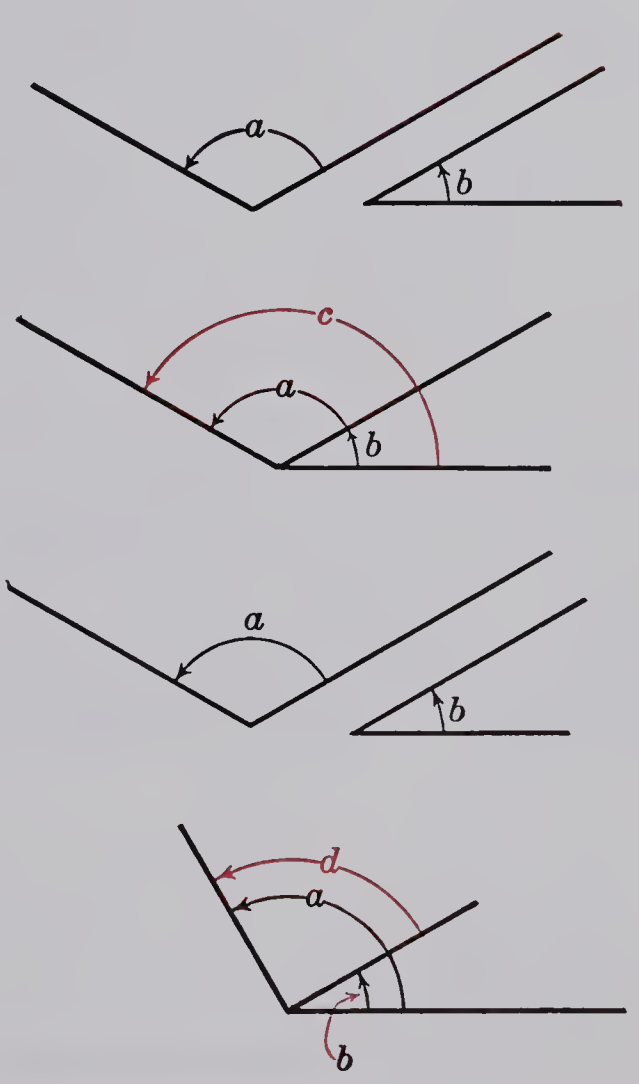
Two angles are equal if they can be made to coincide. Since a definition is reversible, two angles which are equal can be made to coincide. Thus if the sides of $\angle MOP$ can be made to fall on the sides of $\angle TRS$, the angles are equal; and if $\angle MOP$ equals $\angle TRS$, the sides of $\angle MOP$ can be made to fall on the sides of $\angle TRS$.



12. Adding and Subtracting Angles

Two angles may be added geometrically by placing them side by side so that the vertex and a side of one coincide with the vertex and a side of the other and the common side is between the angles. Thus, $\angle a$ and $\angle b$ may be added by placing them as shown in the second drawing at the right. In the drawing, $\angle c = \angle a + \angle b$.

Two angles may be subtracted geometrically by placing one angle on the other so that the vertex and a side of one coincide with the vertex and a side of the other and the common side is not between the angles. Thus, $\angle b$ may be subtracted from $\angle a$ by placing the angles as shown in the lower drawing at the right. In the drawing, $\angle d = \angle a - \angle b$.

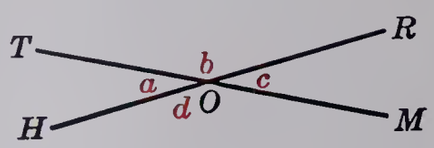


EXERCISES

A

1. Does the size of an angle depend upon the length of its sides? upon the amount of opening between its sides? Can a 30° angle have sides equal to the sides of a 40° angle?

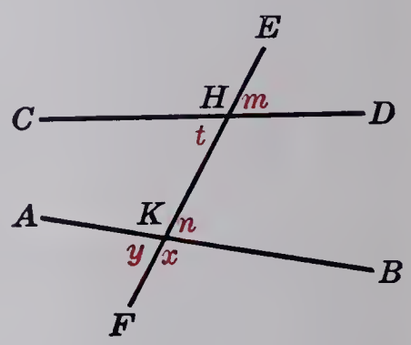
2. Read each of these angles by naming one letter; by naming three letters.



3. Make an angle equal to $\angle TOH$ by folding a piece of paper. Compare the angle you made with $\angle MOR$.

4. Draw an angle ABC. Draw an angle approximately twice as large as $\angle ABC$ and label it $\angle DEF$.

5. In the figure read by naming three letters: $\angle m$; $\angle n$; $\angle x$; $\angle y$; $\angle t$.



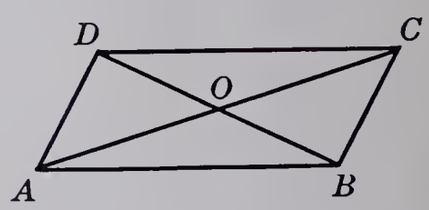
6. Draw a square. Draw a line from each vertex to the opposite vertex. How many such lines can you draw? Why? How many angles are in the completed figure?

7. Draw a five-pointed star (pentagram). How many angles are formed inside the figure? Do any of these angles seem to be equal?

8. Draw a line segment AB. Then draw another line segment CD meeting AB at C so that $\angle ACD$ appears to equal $\angle BCD$.

Complete:

9. $\angle ADC = \angle ADO +$ __?__.
10. $\angle DCB = \angle DCO +$ __?__.
11. $\angle ABO +$ __?__ $= \angle ABC$.
12. $\angle BAO = \angle BAD -$ __?__.
13. $\angle AOC = \angle DOC +$ __?__.
14. $BD =$ __?__ $+$ __?__.



Ex. 9-16

15. Using the compasses, determine which line segment in the figure above is approximately equal to segment DO; to AD; to AO; to AB.

16. Which angle in the same figure do you think is equal to $\angle AOD$? to $\angle AOB$?

17. Draw two angles which have a common vertex but no common side.

Aero Service Corporation, Philadelphia

If you were to fly over Washington, D.C., you might be able to see the patterns formed by its streets, as in the photograph above. Can you see how the streets radiate from the Capitol? Notice the angles formed by the streets in different sections of the city. A rectangular street plan provides convenient building sites, but a radial plan provides better communication.

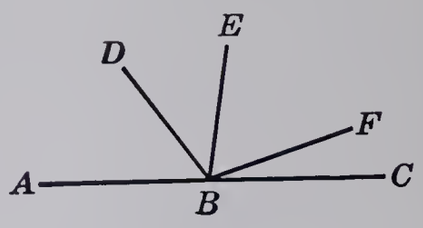
18. Draw two angles which have a common vertex and a common side between them.

19. Which of the two angles, ABF and EBC, is the larger?

20. If $\angle DBE$ is added to $\angle EBF$, what angle is the result?

21. If $\angle CBF$ is subtracted from $\angle CBD$, what angle is the remainder?

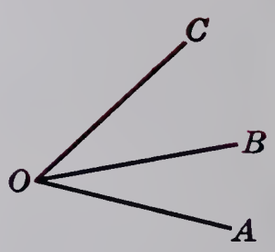
22. $\angle ABD + \angle DBE = \angle$ __?___.



Ex. 19-22

13. Adjacent Angles

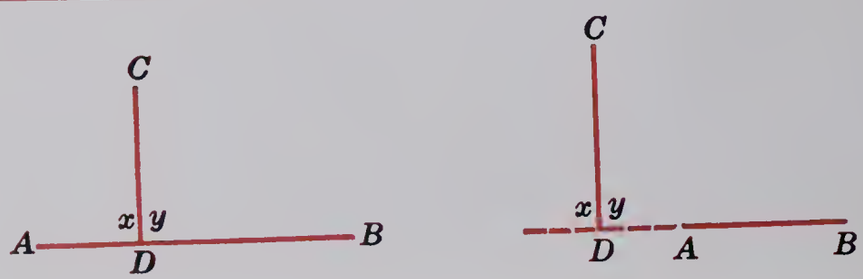
Two angles which have a common vertex and a common side between them are *adjacent angles* (adj. ∡s). In the figure, ∡ AOB and BOC are adjacent angles. Their common vertex is O and their common side OB. Notice that OB is between the angles. ∡ AOB and AOC have what common side? Is it between the angles? Are ∡ AOB and AOC adjacent?



14. Perpendiculars and Right Angles

Two lines are *perpendicular* (⊥) to each other if one of them forms two equal adjacent angles with the other. Thus CD in the figure below is perpendicular to AB ($CD \perp AB$) and $AB \perp CD$.

The equal adjacent angles which are formed by a line and its perpendicular are called *right angles* (rt. ∡s). Thus $\angle x$ and $\angle y$ are right angles.

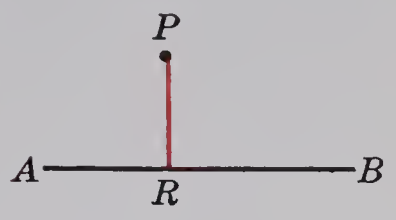


The *foot* of a perpendicular to a line is the point where the perpendicular meets the line. D is the foot of CD, a perpendicular to the line AB. It is also the foot of the perpendicular AB to the line CD.

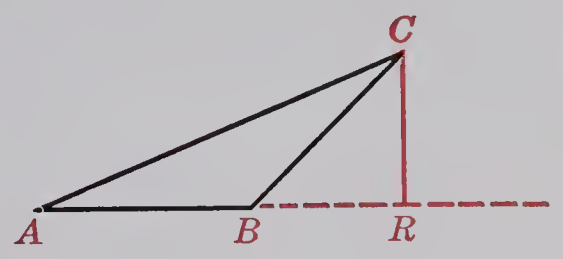
The angles at the corners of this page are right angles, and the side edge is perpendicular to the edges adjacent to it.

15. Distance from a Point to a Line

The distance from a point to a line is the length of the perpendicular *from* the point *to* the line. Thus the distance from P to AB is the length of PR.

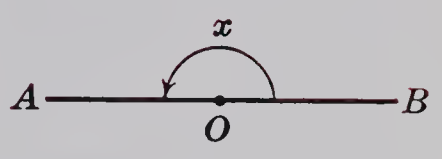


To find the distance from the vertex of a triangle to the base means to find the distance from the vertex to the line of which the base is a segment. Do you see that in some cases the base must be extended as in the triangle at the right? In the drawing, CR is the distance from the vertex, C, to the base, AB.

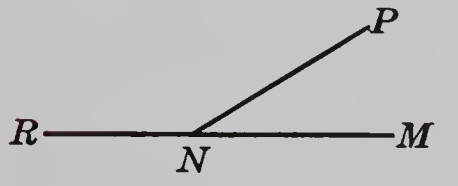


16. Angles Classified as to Size

A *straight angle* (st. $\angle$) is an angle whose sides extend in opposite directions from the vertex, forming a straight line. In the figure, $\angle x$, or $\angle BOA$, is a straight angle. Since two adjacent right angles (§ 14) form a straight angle, a *right angle* is half of a straight angle.



An *acute angle* is an angle less than a right angle, and an *obtuse angle* is an angle greater than a right angle but less than a straight angle. In the figure, $\angle MNP$ is an acute angle and $\angle PNR$ is an obtuse angle.



An *oblique angle* is an angle that is either acute or obtuse. The sides of an oblique angle are *oblique* to each other. Two lines that meet are either oblique or perpendicular to each other.

EXERCISES

A

1. Sketch and label an acute $\angle ABC$; a rt. $\angle EFH$; an obtuse $\angle PQR$.

2. A right angle is what part of a straight angle?

3. Name four articles in your home which contain right angles.

4. In this figure, $CD \perp AB$.

a. What kind of angle is $\angle ADC$? $\angle DCB$?

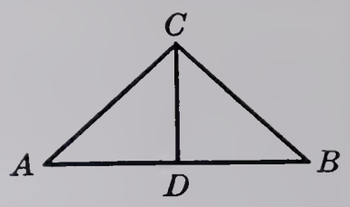
b. Name two pairs of adjacent angles.

c. What is the foot of the perpendicular CD?

d. To what line is AB perpendicular?

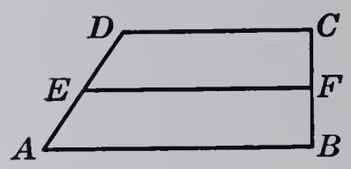
e. What angle has the same size as $\angle ADC$?

f. Which segment represents the distance from C to AB?



5. What kind of angle is formed by the hands of a clock at 2 o'clock? at 3 o'clock? at 6 o'clock? at 7 o'clock?

6. Name four angles of this figure that seem to be right angles. Name two obtuse angles; two acute angles.

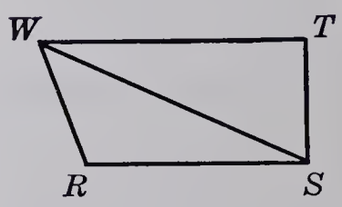


Ex. 6

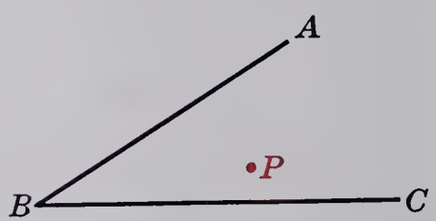
7. In this figure which angle is adjacent to $\angle WSR$? to $\angle SWT$?

8. Is $\angle RWT$ adjacent to $\angle SWT$? Why?

9. With a ruler draw two right angles. Do you think they are equal?



10. Draw an angle ABC. Locate a point P within the angle; then draw line segments which represent the distance from P to the sides of the angle. What is the relationship of the line segments to the sides AB and BC?



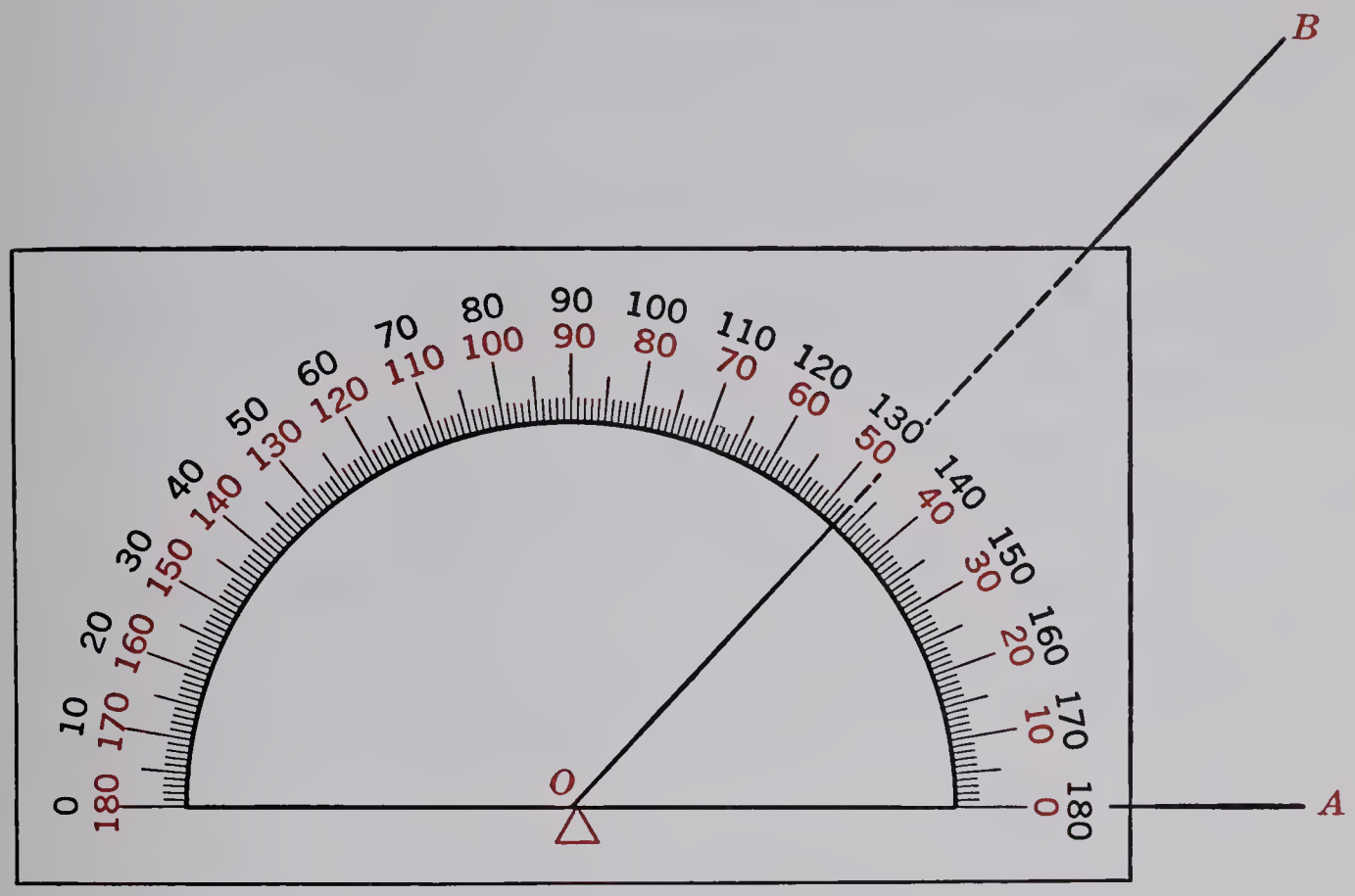
17. Measurement of Angles

A common unit for measuring angles is the *degree* (°), which is one ninetieth of a right angle. A right angle contains 90° and a straight angle contains 180°. For greater accuracy in measurement the degree is divided into 60 equal parts, each of which is called a *minute*. Mariners and engineers divide the minute into 60 equal parts called *seconds*. Thus 60 seconds (60″) = 1 minute (1′), and 60′ = 1°.

18. How to Use the Protractor

The *protractor* is an instrument used in measuring angles. A simple protractor which is graduated in degrees is shown on page 23. To measure an angle AOB with the protractor, place the protractor upon the angle so that the wedge point is on O, the vertex of the angle, and the diameter

of the semicircle lies on OA, one side of the angle. Next find the scale reading of the point where the other side of the angle intersects the semicircle. In the diagram, this reading is 47°. This reading gives the number



of degrees in the angle. If the sides of the angle are too short to cut the circle, they should be extended.

To draw an angle of any given size, say 37°, draw a line segment to be used as a side of the angle. Then select a point on this segment for the vertex of the angle. Place the protractor so that the wedge point of the protractor falls on this point and the diameter of the protractor semicircle falls on the line segment. Make a dot on the paper (or blackboard) opposite the 37° division mark on the inner set of degree readings. Then draw a line from this point to the point chosen for the vertex. To draw an angle formed in the opposite direction (∠), use the outer set of degree readings.

EXERCISES

A

1. Draw an angle of 36° and label it $\angle x$.

2. From any point P draw two rays forming an angle of 47°. On each ray locate a point 2 inches from P. Letter these points M and N. Draw MN. Measure $\measuredangle PMN$ and PNM.

3. Draw two nonadjacent angles of 21° and 59° respectively.

4. Draw any triangle ABC ($\triangle ABC$). Measure each of its angles. What seems to be the sum of its angles?

5. Draw a quadrilateral $ABCD$ and measure each of its four angles. What seems to be the sum of its angles?

6. Draw a line segment, AB, 3 inches long. Place a point C on the line halfway between A and B. Draw a perpendicular to AB at C. Locate a point D on this perpendicular so that CD is 3 inches in length. Draw AD and BD. Measure $\angle CAD$ and $\angle CBD$. Which angle appears to be the larger?

Add:

7. 18° 45′ and 32° 30′.

8. 7° 18′ 32″ and 15° 16′ 43″.

9. 19° 27′ 42″ and 42° 46′ 15″.

Subtract:

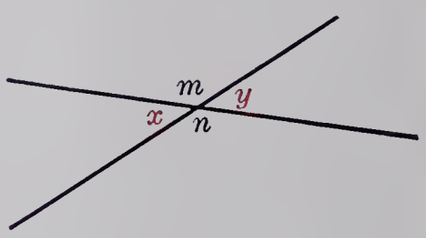
10. 7° 23′ from 25° 42′.

11. 31° 16′ 40″ from 45° 20′ 15″.

12. 18° 32′ 25″ from 90°.

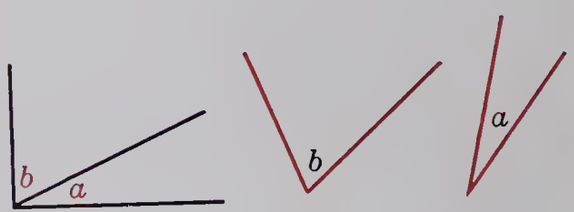
19. Vertical Angles

Two nonadjacent angles which are formed by two intersecting straight lines are called *vertical angles*. In the figure, $\angle x$ and $\angle y$ form one pair of vertical angles and $\angle m$ and $\angle n$ form another pair of vertical angles.



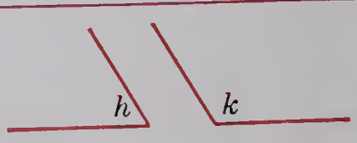
20. Complementary Angles

Two angles are *complementary* (comp.) when their sum is a right angle (90°). In the figure, $\angle a$ and $\angle b$ are complementary angles, $\angle a$ being the complement of $\angle b$, and $\angle b$ being the complement of $\angle a$.



21. Supplementary Angles

Two angles are *supplementary* (supp.) when their sum is a straight angle (180°). In the figure, $\angle h$ and $\angle k$ are supplementary, $\angle h$ being the supplement of $\angle k$, and $\angle k$ being the supplement of $\angle h$.



EXERCISES

A

In the following exercises use arithmetic or algebra:

1. Find the complement of an angle of 30°; of 40°; of 19.2°; of 32° 18′; of $x°$; of $(2x-18)°$; of $(3x+7)°$.

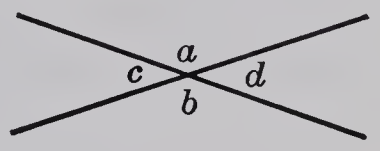
2. Find the supplement of an angle of 60°; of 45°; of 37°; of 118°; of 30° 45′; of $x°$; of $x°+40°$; of $(3a+45)°$.

3. One of two complementary angles is twice the other. Find the angles.

4. Find the angle which is four times as large as its complement.

5. Find the angle which exceeds twice its complement by 8°.

6. In the figure, $\angle a = 140°$. How large is $\angle c$? $\angle d$? $\angle b$? Are the two adjacent angles equal? Are the two vertical angles equal?



7. Find the angle whose supplement is three times as large as its complement.

22. Bisectors

A line segment is *bisected* when it is separated into two equal parts. The *bisector* of a line segment may be a point or a line. Thus the line segment AB (Fig. 1) is bisected by the point C, and the segment DE (Fig. 2) by the line MN. The point at which a line segment is bisected is called its *midpoint*.

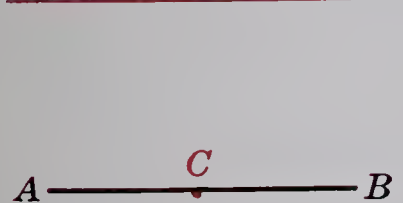


Fig. 1

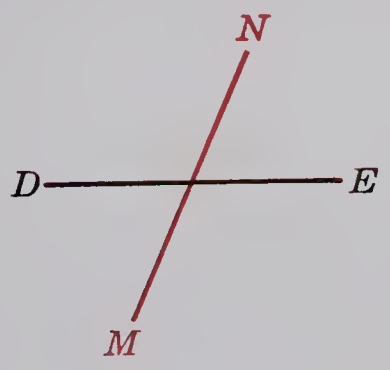


Fig. 2

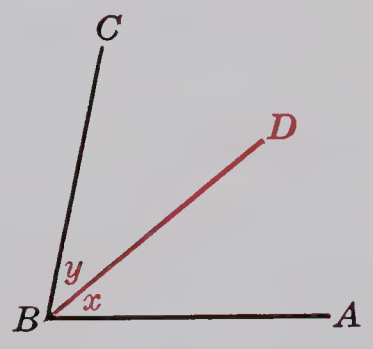


Fig. 3

The *bisector of an angle* is the line which meets the vertex of the angle and separates the angle into two equal angles. BD is the bisector of $\angle ABC$ (Fig. 3). It separates $\angle ABC$ into the two equal angles $\angle x$ and $\angle y$.

EXERCISES

A

1. Do you think every line segment can be bisected? every angle?

2. Draw a line segment AB and bisect it by folding the paper so that the two end points of the segment coincide. At how many points may a line segment be bisected?

3. Are handkerchiefs ever bisected when they are ironed? Explain.

4. How can you bisect a string without using a ruler? How can you divide a string into four equal parts without using a ruler?

5. How many bisectors may an angle have?

6. Cut a piece of paper so that its shape is irregular. Fold the paper to form a straight line. Fold the paper to erect a perpendicular to this straight line.

7. Starting with an irregular piece of paper, fold the paper to form a right angle. Bisect the right angle, forming an angle of 45°.

8. Fold an irregular piece of paper to form a square.

9. Draw a line segment AB. Choose any point C in AB and a point D not in AB. Draw CD. Using the protractor, bisect $\angle ACD$ and $\angle BCD$. Approximately what is the size of the angle formed by the two bisectors?

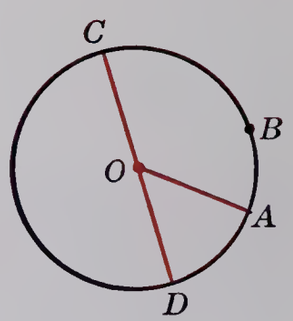
10. Study the result of Ex. 9 and state a relationship which appears to be true concerning the bisectors of two supplementary adjacent angles.

11. One of two complementary angles is seven-eighths as large as the other. How many degrees are in each angle?

12. Find two angles whose sum is 40° and whose difference is 8°.

23. Circle

A *circle* (⊙) is a plane closed curve all points of which are equidistant from a point within called the *center*. A *radius* (plural, *radii*) of a circle is the line segment joining the center to any point on the circle, such as OA. A *diameter* of a circle is a line segment (such as CD) through the center and having both its end points on the circle. An *arc of a circle* is any part of the circle, such as arc AB (written $\overset{\frown}{AB}$). A circle is drawn by using a pair of compasses.



EXERCISES

A

1. Draw a circle whose center is O. Draw a line through O. At how many points does the line intersect the circle? What is the greatest number of points at which a line may intersect a circle? the least number?

2. Draw any circle O. Choose any two points, A and B, on the circle. Draw AB. Does AB pass through the center? If AB should pass through the center, what would be its name?

3. Can you draw a line touching a circle in one point?

4. Draw two circles (a) which do not intersect; (b) which intersect in two points; (c) which touch at one point. Can two circles intersect in three points?

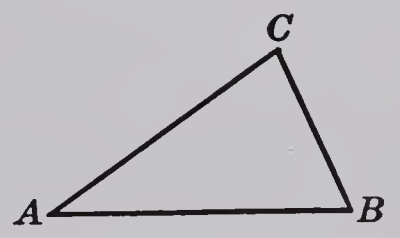
5. Draw a circle whose center is O. Draw a line through O, intersecting the circle in points A and B. What is the name of the line segment AO? of BO? of AB? How many diameters of the circle can be drawn?

6. Draw two unequal circles which intersect in A and B. Label the centers O and O'. Draw AB and OO' intersecting in C. Draw AO, BO, AO', and BO'. Does $OB = OA$? Does $O'B = O'A$? Does $OC = O'C$? Does $AC = CB$?

REVIEW QUESTIONS — Chapter 1

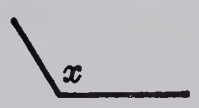
1. How many dimensions has a point? a line? a plane?

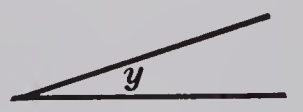
2. Name $\angle A$ with three letters. Name $\angle C$ with three letters. Is side AB a line or a line segment?



3. What word means "to divide into two equal parts"?

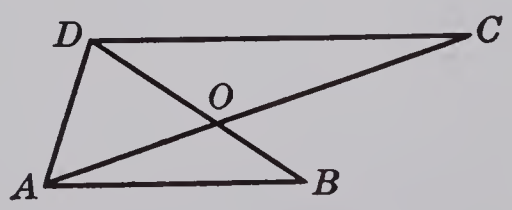
4. Which angle is larger, $\angle x$ or $\angle y$?





5. How many straight lines can be drawn through one point? through two points? through three points?

6. In the drawing



$AO + OC =$ __?__.
$DB - DO =$ __?__
$\angle DAO + \angle OAB = \angle$ __?__
What word describes $\angle DAO$ and $\angle OAB$? $\angle AOB$ and $\angle DOC$?

7. Change the following definition by reversing it: An acute angle is an angle less than a right angle.

8. What are the qualities of a good definition?

9. What are equal line segments? equal angles?

10. What is the difference between a line segment and a ray?

11. How many points can lie on one line? How many points are common to two intersecting lines?

12. What word describes two angles whose sum is a right angle? whose sum is a straight angle?

13. What word describes two angles which have a common vertex and a common side between them?

14. What word describes an angle which is equal to its supplement? which is less than its supplement? which is greater than its supplement?

15. How many end points has a line? a line segment? a circle?

16. What word describes the line which divides an angle into two equal parts?

17. If line a is perpendicular to line b, is line b perpendicular to line a?

18. State the definition of perpendicular lines. State the definition reversed.

19. State the definition of right angles. State the definition reversed.

20. What word describes the point in which a line segment is bisected?

WORD LIST

At this time you should be able to spell and use correctly in sentences each of the following words:

acute
adjacent
angle
arc
bisect
center
circle
compasses
complementary
construction
curve
geometry
intersection
length
midpoint
oblique
obtuse
perpendicular
plane
point
protractor
radius
ray
segment
straight
supplementary
undefined
vertex

TEST 1 — True-False Statements

[Eight Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number.

1. A line has only one dimension.
2. One half of a straight angle is a right angle.
3. One half of an obtuse angle is an acute angle.
4. At 3 o'clock the hands of a clock form an obtuse angle.
5. A degree contains 60 minutes.
6. The longer the sides of an angle, the larger the angle.
7. Two angles which have a common vertex and a common side are adjacent angles.
8. The sides of a right angle are perpendicular to each other.
9. Two adjacent angles are either complementary or supplementary.
10. If $\angle x = 57°$ and $\angle y = 33°$, the angles are supplementary.
11. By carefully measuring with a protractor we can determine whether an angle is a right angle.
12. The distance from a point to a line is the length of the perpendicular from the point to the line.
13. A straightedge is used to measure line segments.
14. Two straight lines can intersect in only one point.
15. The sides of an oblique angle are oblique to each other.
16. A degree is 1/90 of a right angle.
17. To intersect means to divide into two equal parts.
18. A definition is reversible.
19. A diameter of a circle joins any two points of the circle.
20. A straight line has definite length.

CHAPTER 2

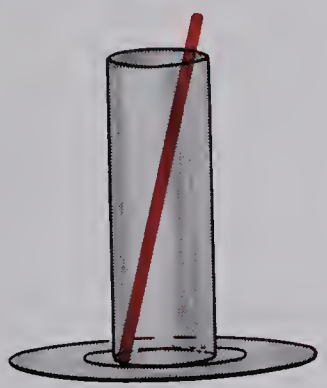
Inductive Reasoning

Before we proceed further
with building a logical system
of geometric thought,
let us pause to consider
the nature of the geometric thinking
you have done thus far in your life.

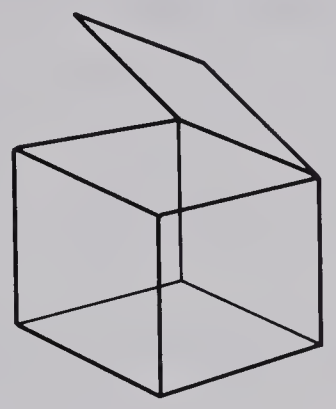
24. Discovering Information by Observation

You have formed many concepts of geometry through observation. Like all of us, you have observed the roundness of balls, wheels, pennies, and thousands of other objects. You have studied from rectangular books, eaten ice cream from cones, and sealed the triangular flaps of envelopes. You have marveled at the beautiful shapes of snowflakes and the delicate polygon-like spirals of a spider's web. Such experiences have given you ideas of size and shape and relationships. We could not get along without observation.

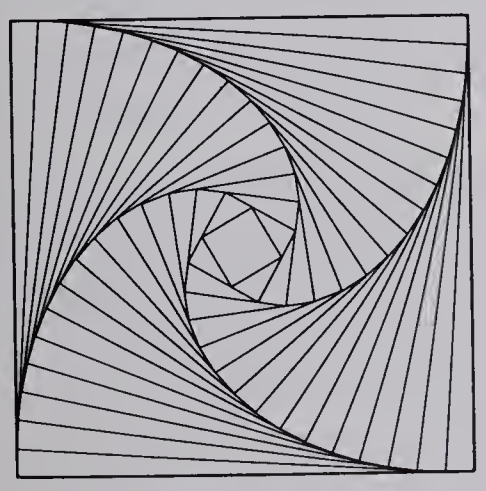
But indispensable as observation is, you know that it can lead us into false conclusions. For example, our eyes can play tricks on us as they do in the following drawings.



Which is greater, the width of the plate or the height of the glass?*



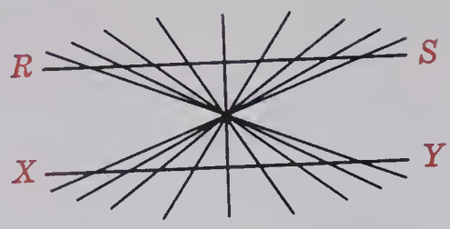
In how many ways can you make the box open?*



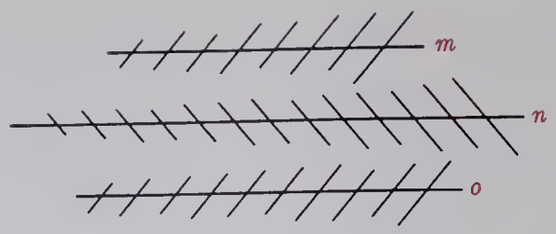
Can you believe that this complicated figure is one continuous broken line coiled up inside the square?*

Are there six blocks or seven in this figure?

*Adapted from illustrations by Fred H. Lyon, in *Experiments in Optical Illusion*, by Nelson F. Beeler and Franklyn M. Branley. Copyright 1951 by Thomas Y. Crowell Company, publishers. Used by permission.



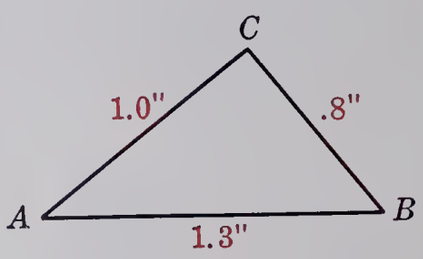
Are RS and XY straight?



Are m, n, and o parallel?

25. Discovering Information through Measurement

You have discovered geometric information through measurement. Perhaps, for example, you have measured the sides of a triangle and have added your measurements to find the perimeter of the triangle as shown at the right.

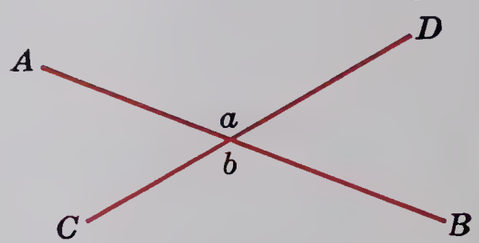


$p = 1.3 + 0.8 + 1.0$
$p = 3.1''$

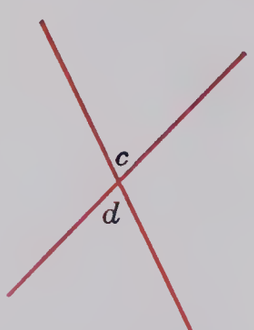
You realize, of course, that any perimeter found by measurement is probably not the exact perimeter because it is impossible to make exact measurements.

26. Discovering Information through Experimentation

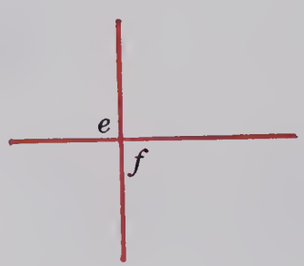
Perhaps you have combined observation and measurement to give yourself other geometric information, as Mike Harris did. In looking at the intersecting lines at the right, Mike thought that $\angle a$ appeared equal to $\angle b$. Using a protractor he tested his theory by measuring both angles. He found that each contained approximately 130°.



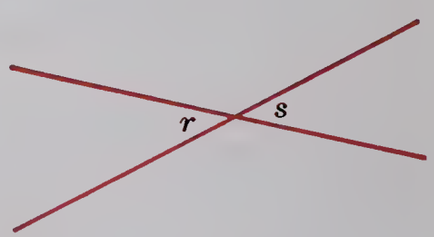
Mike then wondered whether vertical angles are always equal. He drew several pairs of intersecting lines as shown below, then measured a pair of vertical angles in each drawing and entered the results in tables.



$\angle c = 70°$
$\angle d = 70°$



$\angle e = 90°$
$\angle f = 90°$



$\angle r = 40°$
$\angle s = 40°$

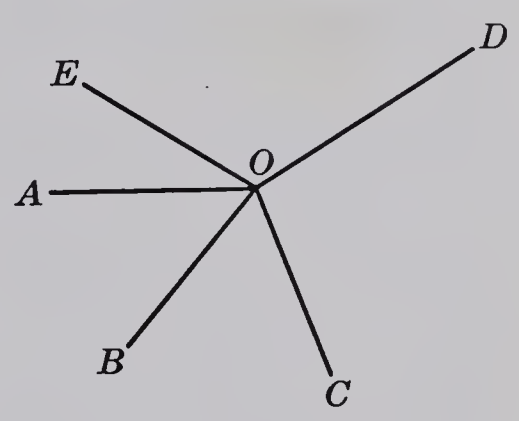
After he had performed the experiment several times, Mike decided that vertical angles are probably always equal when two lines intersect. Being a careful thinker, he inserted the word "probably" because he knew that there was a faint possibility that more precise measurements would show the angles slightly unequal. He realized, too, that he had not measured the vertical angles for all possible intersecting lines. He knew that if sometime even one pair of vertical angles were found to be unequal, his theory would have to be discarded.

Mike felt that his experiment was valuable in that it had helped him to discover what appeared to be an important geometric principle, but he was shrewd enough not to be led into unwarranted conclusions. The following exercises will give you a chance to discover similar geometric principles which appear to be true. Try to be as careful in your thinking as Mike was in his.

EXERCISES

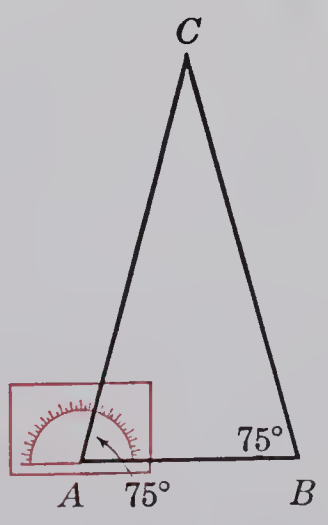
A

1. Find the number of degrees in $\angle AOB$; $\angle BOC$; $\angle COD$; $\angle DOE$; $\angle EOA$. What seems to be the sum of all the angles about O? about any point?



2. Draw several triangles. Find the sum of the angles of each triangle. What appears to be the sum of the angles of any triangle? Can you be absolutely sure that the sum is always what your measurement shows it to be?

3. Using your protractor, draw a triangle having two 75° angles. Measure the sides. Do your measurements show any sides equal? Draw another triangle having two 40° angles and measure its sides. Are any sides equal? Draw several triangles each having two equal angles and measure the sides. What principle seems to be true?



Ex. 3

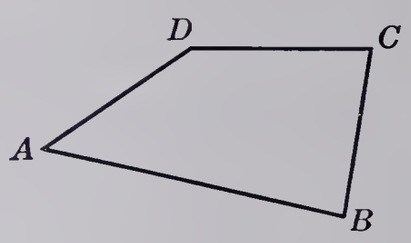
4. On paper place any three points which are not in the same straight line and letter them A, B, and C respectively. Draw AB, BC, and CA. Measure the three line segments and find the sum of the lengths of the two shorter segments. Is this sum greater than the length of the longest segment?

Complete: "The sum of two sides of a triangle is --?-- than the third --?--."

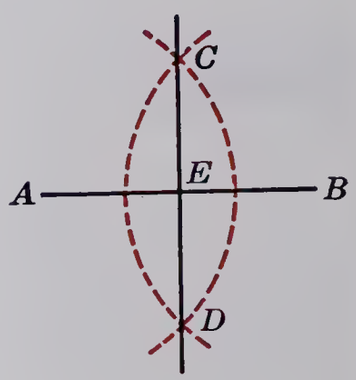
If you should find the above statement true for one thousand triangles, would you know conclusively that it is true for all triangles?

B

5. $ABCD$ is a quadrilateral. Find the sum of its angles. Draw another quadrilateral and find the sum of its angles. What fact have you apparently discovered? Test your conclusion by measuring the angles of a third quadrilateral.



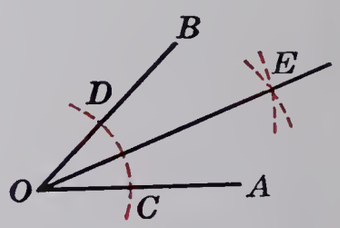
6. Draw a line segment AB as shown at the right. Place the point of your compasses at A and with a radius more than half of AB draw an arc as shown. Keeping the same radius, place a point of the compasses at B and draw an arc cutting the first arc at C and D. Using your straightedge, draw line CD. Let E represent the point where CD intersects AB.



What seems to be true concerning the lengths of AE and BE? Test your conclusion by measuring AE and BE. What appears to be true of the angles in the drawing? Test your conclusion by measuring the angles with a protractor.

Make the same construction several times. Do you reach the same conclusions about the segments and the angles each time you make the construction?

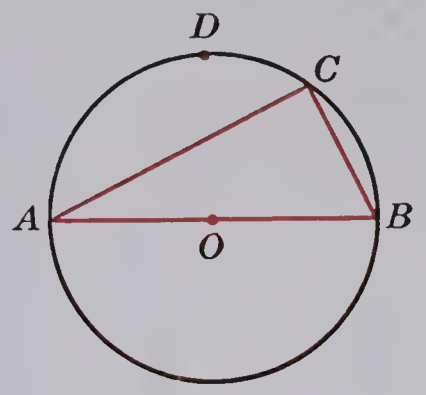
7. Draw an angle AOB as shown at the right. With O as the center, draw an arc intersecting OA in C and OB in D. With C and D as centers and with equal radii, draw arcs intersecting in E. Draw OE.



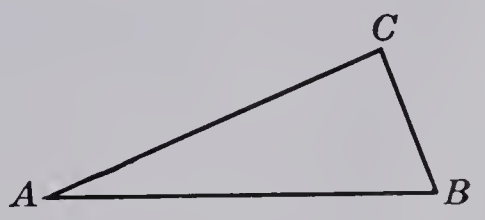
What appears to be true of the two angles AOE and EOB? Test your conclusion by measuring the angles with a protractor. Repeat the construction for angles of various sizes.

Can you state a fact which you infer from the construction you have made? Can you be absolutely sure that your conclusion is correct?

8. With O as center draw a circle. Draw a diameter AB. Choose any point C on the circle and draw AC and BC. What kind of angle does $\angle C$ appear to be? Test your conclusion by measurement. Choose another point D on the same circle. Draw AD and BD. Measure $\angle D$. Choose two other points on the circle and join them to A and B. Measure each angle formed. Can you state a principle which seems to be true?



9. Draw a triangle ABC having $\angle A$ smaller than $\angle B$. Measure sides BC and AC. Which is longer? Draw a triangle with $\angle A$ larger than $\angle B$ and determine whether AC or BC is the longer. Draw several triangles in which either $\angle A$ is larger than $\angle B$ or $\angle B$ is larger than $\angle A$. What relationship seems to exist between the sides and the angles?



27. Inductive Reasoning

Have you noticed that in each of the exercises above you arrived at a general truth by investigating a number of particular cases? Such reasoning is known as *inductive reasoning*. Much of the geometry you have learned to this point in your life has been learned inductively.

Inductive reasoning has made a large contribution to civilization. Geometry was first studied by induction, and each year new facts are discovered inductively. Induction has been used to discover laws and principles in the arts and in sciences such as chemistry, physics, astronomy, and medicine. Through its use we have acquired knowledge of the weather, crops, and the behavior of human beings.

But we must remember that inductive reasoning does not prove the conclusions obtained by it, for it has two sources of error:

First, it may depend upon measurement and observation, neither of which can be said to be absolutely accurate.

Second, inductive reasoning arrives at the generalization before all possible cases have been studied. Any exception to a generalization obtained through inductive reasoning proves that the generalization is false. For example, the pupils in a certain class concluded that all rainbows appear in the east. Later they knew their generalization was false, for one of the pupils saw a rainbow in the west.

United States Steel Corporation

The picture shows a research scientist working with an instrument called a mass spectrometer. This instrument is used to analyze gases and vapors by smashing their molecules into fragments and weighing the resulting atoms and radicals. Would you say that a person who wants to be a research scientist should try to develop a superior reasoning ability?

EXERCISES

A

1. A farmer in Ohio experimented on the use of lime in growing corn. In each hill of one row he placed a small amount of lime. In the next row he used no lime. The row with lime gave the better yield. The next year he repeated the experiment and obtained the same result. Should he conclude that lime benefits his corn crop?

2. In a Western city some people say that if it rains on Easter Sunday, it will rain on each of the next seven Sundays. How do you think this conclusion was made? Do you believe that it is valid?

3. Ten boys went camping and six of them became ill. To determine the cause of their illness, they checked five of the possible causes as follows:

Boys	Ill	Chicken	Oranges	Sand-wiches	Drinking Water	Pool
Frank Wilson	Yes	Yes	Yes	Yes	Yes	Yes
Eugene Webb	No	No	Yes	Yes	Yes	Yes
Bob Jones	Yes	Yes	Yes	No	Yes	Yes
Blair Wilson	Yes	Yes	No	Yes	Yes	Yes
Edson Frank	Yes	Yes	Yes	Yes	Yes	No
Bill Flick	No	No	No	No	Yes	No
John Eiler	Yes	Yes	Yes	Yes	Yes	Yes
Joe Gregg	No	No	Yes	Yes	Yes	Yes
Forrest Black	No	No	No	Yes	Yes	Yes
Glen Morris	Yes	Yes	Yes	No	Yes	Yes

Tell what you think was the cause of the illness and give the reason for your answer.

4. Suppose a scientist, after finding a compound which he believed was a preventative for the common cold, took some of the compound. Take for granted that he had no colds during the time for which the compound was supposedly active. Would he have proved that the compound was a cold preventative?

Suppose he next gave the compound to ten volunteers, and suppose they, too, had no colds for that period of time. Would the compound have been proved a preventative? To how many people do you think

the compound should have been given before it could be declared a reasonably certain preventative for colds, or else could be discarded as ineffective? Could the compound ever be declared one hundred per cent effective as a cold preventative? Why?

28. Need for Proof

Since inductive reasoning does not produce proof, we like to supplement it with another kind of reasoning—deductive reasoning. You will see, when you study the next chapter, that you have been doing deductive reasoning all your life, but you will also see that there is still much to learn about it.

29. The Growth of Geometry

Geometry had its beginning either in Egypt or Babylonia. This early geometry consisted of observation and measurement. People used geometry in finding volumes and areas, in doing simple surveying, in planning buildings, and in decoration.

The earliest documents of Egyptian geometry which we have were written on papyrus paper about 1550 B.C. by Aāhmesu, commonly called Ahmes.* This manuscript, now in the British Museum, was copied from another manuscript, now lost, which was written about 2300 B.C. Ahmes's work consists of arithmetic and geometric problems dealing with the volumes of barns, the areas of rectilinear figures, and the area of the circle. The rule for finding the area of a circle as given by Ahmes is, in substance, given by the formula $A = 3.1605\, r^2$. Herodotus, the Greek historian, tells us that about 1400 B.C. Ramses II divided Egypt into rectangular plots of ground and assigned them to his subjects. This indicates that the early Egyptians knew something of surveying.

From Egypt, and perhaps Babylonia, this geometry of measurement was carried into Greece and Asia Minor. Thales of Miletus (640–546 B.C.) accumulated much wealth as a merchant and spent his later years in travel and study. While studying in Egypt he became interested in geometry. Upon his return to Greece he taught geometry to his friends, who made many contributions to the subject.

*Pronunciation of the proper names in this section will be found in the Index at the end of the book.

Famous mathematicians of ancient times lived and worked in the places shown. The places on the map are numbered to correspond to the key showing the persons associated with each place.

The greatest pupil of Thales was Pythagoras, who was born about 580 B.C., probably on the island of Samos, near Asia Minor. He was one of the most remarkable men of antiquity and gave to us many of the proofs of theorems found in our geometries of today. He is said to have been the founder of the *Pythagoreans*, a society of about 300 members interested in mathematics and philosophy. The members of the society were divided into two groups, the mathematicians and the listeners. Before one could become a member of the mathematicians, he first had to be a good listener. Pythagoras is said to have been the first one to prove that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

During the two hundred years following the death of Pythagoras, the Greeks, who were interested in geometry more for its own sake than for its practical value, discovered and proved a large number of new proposi-

tions. Among these notable scholars who contributed to this new science appear the names of Antiphon, Hippias, Plato (429–348 B.C.), Eudoxus (408–355 B.C.), Aristotle (384–322 B.C.), and Hippocrates. From 600 B.C. to 300 B.C. the study of geometry consisted in research, discovery, and proof. During this time there were few textbooks and the propositions were not arranged in logical order.

Greece reached its pinnacle as a world power during the reign of Alexander the Great (336 B.C.–323 B.C.). After his death the empire was divided and Ptolemy became the ruler of Egypt. Under Ptolemy's rule, Alexandria became the center of learning of the world. One of the teachers of mathematics at the University of Alexandria was Euclid. It was while teaching at Alexandria that Euclid wrote the first great textbook of geometry (about 300 B.C.). This geometry text of Euclid was called "Elements." It contained the first systematic, orderly, and logical arrangement of geometry and was divided into thirteen chapters, called *books*. The "Elements" contained most of our present plane geometry and some solid geometry. The substance of Books I and II is probably the work of Pythagoras; that of Book III, of Hippocrates; and that of Book V, of Eudoxus, who also contributed much to the science of solid geometry.

The greatest contributor to solid geometry was Archimedes (287–212 B.C.), the famous mathematician and physicist of Syracuse. At the time of his death, the plane and solid geometry, as given in our high-school texts, was almost completed.

Since the death of Archimedes, geometry has been enriched by many prominent mathematicians. Among these are Apollonius (260–200 B.C.), Hipparchus (about 140 B.C.), Heron, Descartes (1596–1650 A.D.), Gauss (1777–1855 A.D.), Saccheri (1667–1733 A.D.), Lobachevsky (1793–1856 A.D.), Bolyai (1802–1860 A.D.), and Riemann (1826–1866 A.D.).

REVIEW QUESTIONS — Chapter 2

1. The people of ancient times believed that the earth is flat and that the sun travels around the earth. Do you think that they arrived at these beliefs through observation, through measurement, or through experimentation?

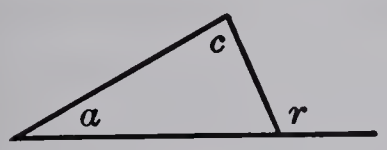
2. Are the following statements true or false? Explain.

a. We can make exact measurements. **b.** Seeing is believing.

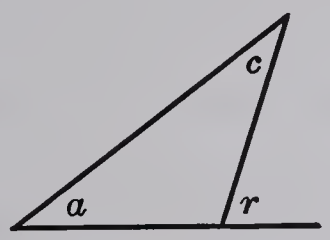
c. If we combine observation and measurement to perform geometric experiments, we can be sure of the truth of the conclusions we reach through the experiments.

3. A foreigner landing in New York found it raining. It rained the next day, and the next, and the next. Had the foreigner a right to conclude that in New York it rains every day? Explain.

4.



I



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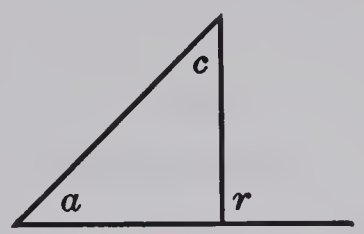


III

In each of the figures above measure $\angle r$, $\angle a$, and $\angle c$; then enter the results in a table like the one below.

	I	II	III
$\angle a$	?	?	?
$\angle c$	?	?	?
$\angle a + \angle c$	?	?	?
$\angle r$	?	?	?

What appears to be true of $\angle r$ as compared with the sum of $\angle a$ and $\angle c$? Draw several other triangles, lettering them as shown above. In these drawings does the same relationship appear to be true of $\angle r$ and the sum of $\angle a$ and $\angle c$? Have you proved that $\angle r$ bears a certain relationship to the sum of $\angle a$ and $\angle c$?

5. Geometry originated in either __?__ or __?__.

6. Geometry was first studied by the method of __?__ and __?__.

7. Who made the first orderly arrangement of geometry?

8. When a scientist observes and measures in order to reach a tentative conclusion he is reasoning __?__.

WORD LIST

Make sure that you can spell and use correctly the following words:

conclusion	experiment	measurement	principle
Euclid	inductive	observation	reasoning

TEST 2

In Exercises 1–3 choose the words (in parentheses) which make the statement true.

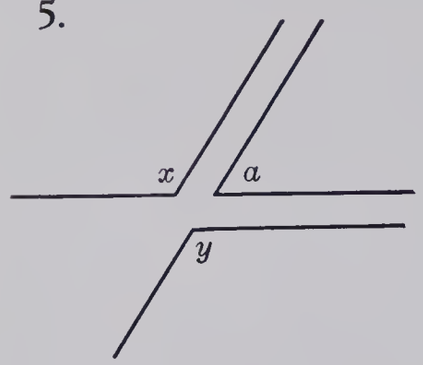
1. We make the most use of inductive reasoning in (proving truths, determining possible new truths).

2. We (can, cannot) make exact measurements.

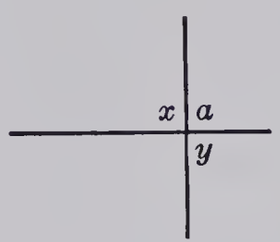
3. We (can, cannot) depend upon our eyes to give us exact impressions.

4. Fred wanted to know the exact length of line segment AB so he took it to a friend who was able to make measurements to the nearest thousandth of an inch. The friend found by measuring that AB was 4.089 inches long. Was that the exact length? Explain.

5.



I



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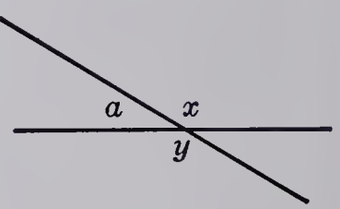


III

In each of the drawings above $\angle x$ is the supplement of $\angle a$ and $\angle y$ is the supplement of $\angle a$. Measure $\angle x$ and $\angle y$ for each drawing and enter the measurements in a table like the one below. What appears to be true about supplements of an angle? Have you proved that this principle is actually true?

	I	II	III
$\angle x$	?	?	?
$\angle y$	?	?	?

6. Using your protractor, draw a triangle having one angle which contains 90°. Measure the other two angles and find their sum. Repeat the experiment with a triangle of different shape but still containing a 90° angle. What appears to be true about the sum of the two angles which are not right angles? Have you proved this to be a fact?

COMPREHENSIVE TEST

[8 Minutes]

In the first column you will find descriptive phrases for the terms listed alphabetically in the second column. Write the numbers 1–15 on your paper and after each number write the term which you think belongs with the description bearing that number. Some terms are not described.

DESCRIPTIVE PHRASE	TERM
1. A definite part of a line	arc
2. Reasoning in which we arrive at generalizations through study of special cases	adjacent angles
3. A line joining two points on a circle and passing through the center	bisect
4. A line joining the center of a circle to a point on the circle	degree
5. Nonadjacent angles formed by two intersecting straight lines	diameter
6. Two angles which can be made to coincide	complementary angles
7. One ninetieth of a right angle	equal angles
8. The man who wrote the first geometry textbook	Euclid
9. An infinite part of a line	induction
10. Two angles whose sum is a straight angle	intersect
11. Two angles whose sum is a right angle	line segment
12. To divide into two equal parts	minute
13. Two angles with a common vertex and a common side between the angles	oblique lines
14. A line which meets another so as to form equal adjacent angles	perpendicular
15. An instrument used to draw straight lines	protractor
	radius
	ray
	straightedge
	supplementary angles
	vertical angles

CHAPTER 3

Deductive Reasoning

IN THIS CHAPTER we shall begin the study of deductive reasoning and learn the meaning of geometric proof.

30. What Deductive Reasoning Is

In Chapter 2 you performed experiments and arrived at conclusions by inductive reasoning. When making an experiment you observed many specific cases and were able to select a property common to all of them. You stated this common property as a *generalization*, or *general statement*, which fits all cases. In making this generalization you reasoned inductively, proceeding from the specific to the general.

Deductive reasoning is the converse of inductive reasoning. It proceeds from the general to the specific. In it we always start with a general statement and apply the statement to a specific case.

For example, the dictionary states that a fennec is a fox. Then, by the general statement "All foxes have four feet," we know that a fennec has four feet. Do you see that the statement "A fennec is a fox" is a *specific statement* about a particular animal—a fennec? Do you see that the statement "All foxes have four feet" is a general statement because it makes a statement about *all* foxes, or about foxes *in general*? Do you see that by applying the general statement to the specific statement we are led to the conclusion "A fennec has four feet"?

31. How to Reason Deductively

Each of us uses some form of deductive reasoning daily. We constantly make conclusions which depend upon some general principles of conduct. The conclusions reached by deductive reasoning are true only when the general statements upon which they are based are true.

The example above is a very simple and easy case of deductive reasoning. In it there is not much chance of reaching the wrong conclusion. Often, however, when the reasoning is not so simple, we are apt to *jump* at conclusions and err in our reasoning.

Any case in deductive reasoning can be reduced to three steps, which are:

1. A general statement.
2. A specific statement which satisfies all the conditions of the general statement.
3. The conclusion which is the one called for in the general statement.

These three steps in a deduction may be used in any order. In everyday reasoning the general statement usually precedes the other two, while in a geometric proof it usually follows them.

Study the following examples which illustrate good reasoning and faulty reasoning.

Example 1.

General statement. If students intend to graduate from our high school, they must take three years of English.

Specific statement. Martha is a student who expects to graduate from our high school.

Conclusion. Then she must take three years of English.

The specific statement meets all the requirements of the general statement and the conclusion is the one called for in the general statement. The reasoning is valid.

Do you see that the general statement is the *reason* or authority which makes the conclusion true?

Example 2.

Specific statement. $6x = 42$

Conclusion. $x = 7$

General statement. If equals are divided by equals, the quotients are equal.

Note that the general statement is the reason why $x = 7$ if $6x = 42$. The deduction is valid.

Example 3.

Specific statement. $\angle x$ and $\angle y$ have a common vertex and a common side.

Conclusion. $\angle x$ and $\angle y$ are adjacent angles.

General statement. If two angles have a common vertex and a common side between them, they are adjacent.

The reasoning in this example is not valid because the specific statement fails to meet all the requirements of the general statement. It does not state that the two angles have a common side *between* them. The conclusion may be true or it may be false. It is not correct to state that it is true.

Can you draw two angles x and y which have a common vertex and a common side but which are not adjacent?

Example 4.

Specific statement. $\angle x$ and $\angle y$ are nonadjacent and are formed by intersecting straight lines.

Conclusion. Then $\angle x$ and $\angle y$ are complementary angles.
General statement. Two angles which are nonadjacent and are formed by intersecting straight lines are vertical angles.

The reasoning in this example is not valid. The specific statement fills the requirements of the general statement but the conclusion is not the one called for in the general statement. The conclusion should be "then $\angle x$ and $\angle y$ are vertical angles."

EXERCISES

A

See if you can state the conclusion that follows from each of the following pairs of general and specific statements.

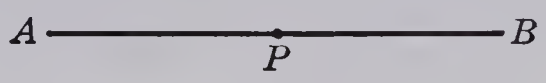
1. *General statement.* All birds have feathers.
   Specific statement. A robin is a bird.
   Conclusion.

2. *General statement.* To be a member of the football team a boy must be passing in three one-credit subjects.
   Specific statement. John is a member of the football team.
   Conclusion.

3. *General statement.* If a dog does not have a temperature of 101.3°, it is not well.
   Specific statement. Our dog's temperature is 105°.
   Conclusion.

4. *General statement.* If a person is a student in our school, he has at least one study period.
   Specific statement. Richard Johnson is a student in our school.
   Conclusion.

5. *General statement.* A line segment is bisected when it is divided into two equal parts.
   Specific statement. Point P divides line segment AB so that $AP = PB$.



   Conclusion.

When possible, draw valid conclusions from each of the following pairs of general and specific statements. If a valid conclusion is not possible, explain why it is not possible.

6. *Specific statement.* $\angle m$ and $\angle n$ are a pair of vertical angles.
   General statement. Any pair of vertical angles are equal.

7. *General statement.* When Elizabeth goes home from school, she rides on the school bus.
   Specific statement. Elizabeth is riding on the bus.

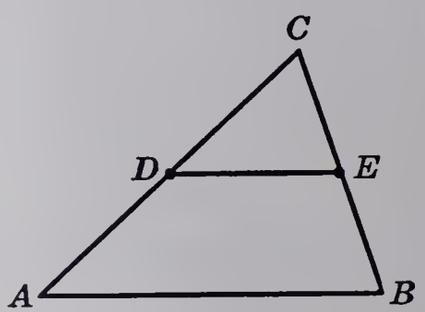
8. *General statement.* When Robert drives his father's car to school, he locks all its doors.
   Specific statement. Robert drove his father's car to school last Thursday.

9. *Specific statement.* Pal is barking.
   General statement. Pal barks whenever a stranger enters the yard.

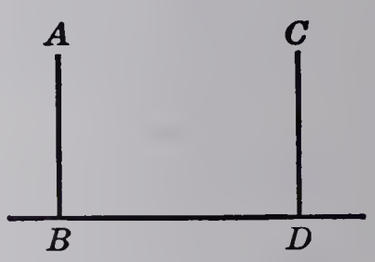
10. *General statement.* When Mr. Black goes fishing on Jasper Lake, he always catches his limit of bass.
    Specific statement. Mr. Black and his son Dale went fishing this morning.

11. *General statement.* Mr. Snider, our basketball coach, uses only boys who are at least six feet tall on his team.
    Specific statement. Philip Duncan is on Mr. Snider's basketball team.

12. *Specific statement.* ABC is a triangle and line segment DE joins the midpoints of the sides AC and BC.
    General statement. If a line segment joins the midpoints of two sides of a triangle, it is one-half the third side.



13. *Specific statement.* Line $AB \perp$ line BD and line $CD \perp BD$.
    General statement. If two lines are perpendicular to a third line, they will not intersect.



32. The Hypothesis and Conclusion

Any general statement can be expressed by a complex sentence which has one clause beginning with the word "if" and a second clause beginning with the word "then." A sentence of this form is called an *if-then* sentence. The if-clause of an if-then sentence is the *hypothesis* and the then-clause is the *conclusion.*

Du Pont—Better Living

The foreman of the jury has just announced its verdict, and the judge, at the request of the defense attorney, is polling the jurors individually. In most states women as well as men are called on for jury duty. Your course in geometry will help you to acquire skill in logical thinking. Why should every citizen try to develop this skill?

Example 1. If equals are added to equals, then the sums are equal.

In this sentence the hypothesis is "If equals are added to equals" and the conclusion is "then the sums are equal."

Example 2. I live in California if I live in Los Angeles.

In this example the hypothesis is "if I live in Los Angeles" and the conclusion is "I live in California."

Note that the hypothesis may precede the conclusion as in Example 1 or follow the conclusion as in Example 2. The word "then," used in Example 1, is usually omitted.

Many general statements do not contain the if-clause. In such cases the complete subject is the hypothesis and the complete predicate is the conclusion.

Example 3. An acute angle is an angle less than a right angle.

In this statement "An acute angle" is the hypothesis and "is an angle less than a right angle" is the conclusion.

As has been stated, any general statement can be expressed by the if-then form of sentence. The above general statement can be changed to "If an angle is acute, then it is less than a right angle."

Does the if-then form of sentence help you understand why the complete subject of the original general statement is the hypothesis? and the complete predicate of the original general statement is the conclusion?

As you studied the examples and exercises of § 31, did you discover any relation between the conclusion and the specific statement? Let us examine Example 1, page 46.

Specific statement. Martha is a student who expects to graduate from our high school.

Conclusion. Then she must take three years of English.

The relation between the conclusion and the specific statement of this deduction can be expressed by the if-then sentence as follows:

If Martha is a student who expects to graduate from our high school, *then* she must take three years of English.

Do you see that "hypothesis," "if-clause," and "specific statement" are different names for the same idea? If you do, then you can understand why in any valid deduction the following is true.

If the hypothesis is true, the conclusion is true.

EXERCISES

A

Name the hypothesis and conclusion in each of the exercises 1–10.

1. If you can run the 100-yard dash in 10 seconds, you are a good sprinter.

2. You must study a subject if you wish to enjoy it.

3. If it rains, I shall not go to the football game.

4. If air is heated, it will expand.

5. Radio reception is best when the weather is clear.

6. An isosceles triangle has two equal sides.

7. The diagonals of a rectangle are equal.

8. If two sides of a triangle are equal, the angles opposite these sides are equal.

9. Two lines perpendicular to a third line are parallel.

10. The opposite sides of a parallelogram are equal.

Change the following general statements into the if-then form:

11. All right angles are equal.

12. Vertical angles are equal.

13. A public official should be honest.

Change to simple sentences:

14. If a line segment joins the midpoints of two sides of a triangle, it is half the third side.

15. If a line intersects one of two parallel lines, it intersects the other.

33. Proof in Geometry

Deductive reasoning carries us from a specific statement to a conclusion (another statement); and it does this logically because the general statement provides proof that the conclusion is valid. For example, if we agree that a fennec is a fox and that all foxes have four feet, we are forced to agree that a fennec has four feet. The only way we can hope to escape the conclusion is to avoid accepting either the specific statement that a fennec is a fox or the general statement that all foxes have four feet.

Having reached a conclusion we can use it as a specific statement to arrive at a second conclusion, and so on until we have built a complete proof. (See Example 1, page 64.)

We must start our system of geometric thought with statements which we are willing to accept as true. Obviously, at the beginning we have no previously proved theorems to use as general statements. We have only the definitions we have accepted, the explanations of the undefined terms, and any statements which we are willing to accept without proof.

34. Assumptions

A statement which we assume to be true is an *assumption* (asmt.). Usually the assumptions we make are statements which we believe are true, but there are times when we make assumptions just for the sake of argument. Some assumptions are called axioms and some are called postulates.

An *axiom* (ax.) is an assumption which applies to all branches of mathematics. You became familiar with many axioms when you solved equations.

A postulate (post.) is less general than an axiom. Postulates apply to particular fields. In geometry the postulates are assumptions about points, lines, and other geometric figures.

To simplify the work in this course in geometry some statements which can be proved are postulated. We simply agree that these statements are true. Any assumption that is given in the text can be used as a reason in the proofs of the text.

35. Assumptions (Axioms)

We shall now list ten assumptions known as axioms. You probably used seven of them in your study of algebra.

Study each assumption and the examples that follow it very carefully. Make sure that you understand the meaning of the assumption and can state it correctly. You will have occasion to quote each of these assumptions many times in your study of geometry.

Assumption 1. If equals are added to equals, the sums are equal.

Example 1.

$$\begin{array}{r} 2 \text{ feet} = 24 \text{ inches} \\ \underline{3 \text{ feet} = 36 \text{ inches}} \\ 5 \text{ feet} = 60 \text{ inches} \end{array}$$

Example 2. If $x = 5$ and $\underline{\quad y = 2}$ then $x + y = 7$

Assumption 2. If equals are subtracted from equals, the remainders are equal.

Example 1. If $2x + 3y = 12$
and $\underline{x + 3y = 9}$
then $x = 3$

Example 2. If $\angle a + \angle b = \angle c$
and $\underline{\angle b = \angle d}$
then $\angle a = \angle c - \angle d$

Assumption 3. If equals are multiplied by equals, the products are equal.

Example 1. If $2x = 7$,
then $6x = 21$.

Example 2. If $\frac{1}{5}\overline{AB} = 6$,
then $\overline{AB} = 30$.

Assumption 4. If equals are divided by equals, the quotients are equal. (Division by zero is not permissible.)

Example 1. If $6x = 30$,
then $x = 5$.

Example 2. If $\angle AOB = 90°$,
then $\frac{1}{3}\angle AOB = 30°$.

Assumption 5. Any quantity equals itself.

Example 1. $3x = 3x$

Example 2. $\angle x = \angle x$

We abbreviate assumption 5 by using the word "identity."

Assumption 6. A quantity may be substituted for its equal.

Example. If $x + y = 6$ and $y = 2$, then $x + 2 = 6$.

Assumption 7. Quantities which are equal to the same quantity or to equal quantities are equal to each other.

Example 1. If $x = m$
and $y = m$,
then $x = y$.

Example 2. If $x = m$,
$y = n$,
and $m = n$,
then $x = y$.

Assumption 8. Like powers or like roots of equals are equal.

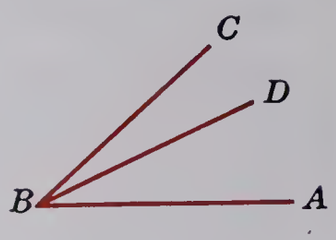
Example 1. If $a = 8$,
then $a^2 = 64$.

Example 2. If $a^2 = 64$,
then $a = 8$.

Example 3. If $m = 3$,
then $m^3 = 27$

Example 4. If $x^3 = 8$,
then $x = 2$.

Assumption 9. The whole is equal to the sum of its parts and is greater than any one of them.



Example 1. $\angle ABC = \angle ABD + \angle DBC$.

Example 2. $\angle ABC > \angle ABD$,
or $\angle ABD < \angle ABC$.

(See page viii at front of book for symbols.)

Assumption 10. Of two quantities of the same kind, the first is greater than, equal to, or less than the second.

Example. Thus, James weighs more than Henry, the same as Henry, or less than Henry.

EXERCISES

A

Answer each of the following questions by quoting an assumption:

1. If $x + 7 = 13$, why does $x = 6$?
2. If $s - 4 = 9$, why does $s = 13$?
3. If $y = 5$, why does $3y = 15$?
4. If $5p = 35$, why does $p = 7$?
5. If $x = 8$ and $y = 3$, why does $x + y = 11$?
6. If $x = 9$ and $y = 1$, why does $x - y = 8$?
7. If $m = 7$ and $n = 8$, why does $mn = 56$?
8. If $\frac{x}{6} = 5$, why does $x = 30$?
9. If $ax = 4a$ and a does not equal zero, why does $x = 4$?
10. If $x + y = m$ and $x = a$, why does $a + y = m$?

In exercises 11–14 give the reasons why the statements are true:

11. If $p = 4$ and $q = 9$, then $pq = 36$.
12. If $x + y = 180$ and $a + b = 180$, then $x + y = a + b$.
13. If $x = 5$, then $x^2 = 25$.
14. If $y^2 = 81$, then $y = 9$.

In exercises 15–19 only the conclusions are written. The figure supplies the specific statements (hypotheses). Give the reasons (general statements).

15. $AD + DB = AB$.

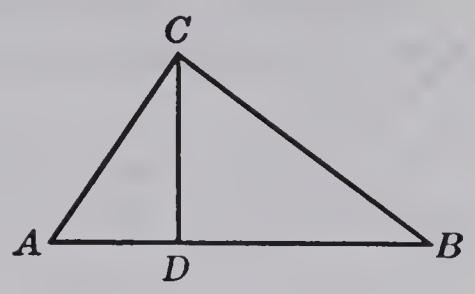
16. $AD < AB$.

(See page viii for the symbol $<$.)

17. $\angle ACD + \angle DCB = \angle ACB$.

18. $\angle ACB > \angle DCB$.

19. $\angle ADC + \angle BDC = \angle ADB$.



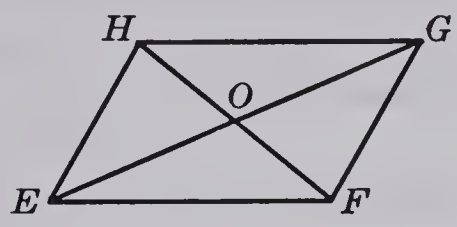
Ex. 15–19

Supply the proper reasons for the conclusions in exercises 20–34.

20. If $EO = OG$ and $HO = OF$, then $EO + HO = OG + OF$.

21. If $\angle HGF = \angle HEF$ and $\angle HGE = \angle GEF$, then $\angle EGF = \angle HEG$.

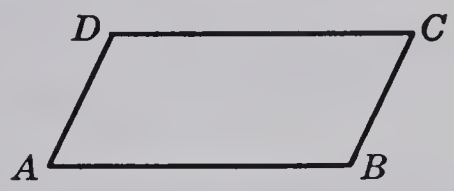
22. If $EH + EF = HG + FG$ and $EH = FG$, then $EF = HG$.



Ex. 20–22

23. If $\angle B = 2 \angle A$ and $\angle D = \angle B$, then $\angle D = 2 \angle A$.

24. If $AD = \frac{1}{2} AB$ and $AB = DC$, then $AD = \frac{1}{2} DC$.



Ex. 23–34

25. If $\angle A = 70°$ and $\angle C = 70°$, then $\angle A = \angle C$.

26. If $AD = 6$ and $AB = 10$, then $AD + AB = 16$.

27. If $\angle B = 110°$ and $\angle A = 70°$, then $\angle B - \angle A = 40°$.

28. If $\angle B = \angle D$ and $\angle A = \angle C$, then $\angle B - \angle A = \angle D - \angle C$.

29. If $AD = BC$ and $AB = DC$, then $AD + AB = BC + DC$.

30. If $\angle A = \angle C$ and $\angle B = \angle D$, then $\angle A + \angle D = \angle B + \angle C$.

31. If $\angle B + \angle C = 180°$ and $\angle A = \angle C$, then $\angle B + \angle A = 180°$.

32. If $\angle B > \angle A$ and $\angle A = \angle C$, then $\angle B > \angle C$.

33. If $AB = DC$, then $3\ AB = 3\ DC$.

34. If $AD = BC$ and $AB = DC$, then $AB - AD = DC - BC$.

36. Assumptions (Postulates)

We shall now make a study of twenty-one assumptions that deal with geometric figures. You will readily agree that most of them are true.

For the sake of argument let us agree that all are true. From now on you may use them as reasons in proofs. Other assumptions will be stated as needed.

In studying these assumptions, read each assumption several times and illustrate it each time with a figure. If any assumption is not clear to you, ask your teacher to explain it. The more familiar you are with the wording of the assumptions, the better you will be able to use them in the exercises that follow. You need not memorize them at this time.

Assumption 11. One straight line, and only one, can be drawn through two points.

Assumption 12. Two straight lines cannot intersect in more than one point.

Assumption 13. A straight line segment can be extended indefinitely in two directions.

Assumption 14. A straight line segment is the shortest line segment that can be drawn between two points.

Assumption 15. One circle, and only one, can be drawn with any given point as a center and any given line segment as a radius.

Assumption 16. Any geometric figure can be moved without changing its size or shape.

Assumption 17. A straight line segment has one midpoint, and only one.

Assumption 18. An angle can be bisected by one line, and only one.

Assumption 19. All right angles are equal and all straight angles are equal.

Assumption 20. At any point in a straight line or from any point outside the line one perpendicular, and only one, can be drawn to the line.

Assumption 21. The perpendicular is the shortest line segment that can be drawn from a given point to a given line.

Assumption 22. The sum of all the angles about a point is two straight angles.

Assumption 23. The sum of all the angles about a point on one side of a straight line is a straight angle.

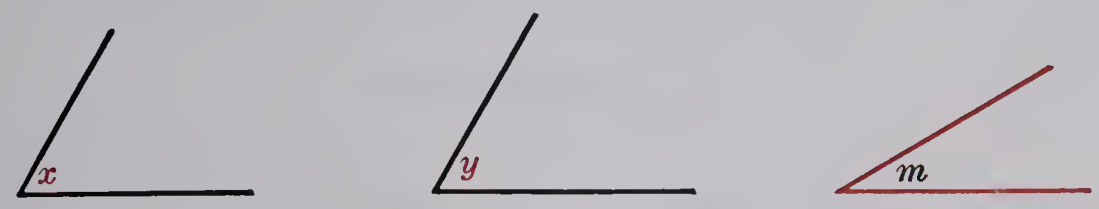
Assumption 24. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

Assumption 25. If two adjacent angles are supplementary, their exterior sides lie in a straight line.

Assumption 26. All radii of a circle or of equal circles are equal.

Assumption 27. A straight line cannot intersect a circle in more than two points.

Assumption 28. Complements of the same angle are equal.



This assumption means that if $\angle x$ is complementary to $\angle m$ and $\angle y$ is complementary to $\angle m$, then $\angle x = \angle y$.

Assumption 29. Complements of equal angles are equal.

This assumption means that if $\angle a = \angle b$, $\angle x$ is complementary to $\angle a$, and $\angle y$ is complementary to $\angle b$, then $\angle x = \angle y$.

Assumption 30. Supplements of the same angle are equal.

This assumption means that if $\angle x$ is supplementary to $\angle a$ and $\angle y$ is supplementary to $\angle a$, then $\angle x = \angle y$.

Assumption 31. Supplements of equal angles are equal.

EXERCISES

A

Answer each of the following questions by quoting a definition or an assumption.

1. If $5c = 40$, why does $c = 8$?
2. If $\angle a$ is comp. $\angle b$, and $\angle c$ is comp. $\angle b$, $\angle a = \angle c$.
3. If $\frac{1}{4}y = 7$, why does $y = 28$?

In this figure AB, CD, and EF are straight lines.

4. $\angle x$ and $\angle m$ are vertical angles. Why?

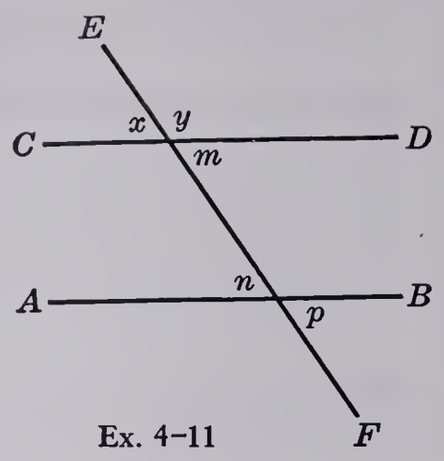
5. $\angle n$ and $\angle p$ are vertical angles. Why?

6. Why are $\angle x$ and $\angle y$ a pair of adjacent angles?

7. Why are $\angle m$ and $\angle y$ supplementary?

8. $\angle x$ is supplementary to $\angle y$ and $\angle m$ is supplementary to $\angle y$. Why is $\angle x = \angle m$?

9. If $\angle x = \angle m$ and $\angle n = \angle m$, why does $\angle x = \angle n$?



Ex. 4-11

10. If $\angle m = \angle n$ and $\angle p = \angle n$, why does $\angle m = \angle p$?

11. If $\angle m + \angle y = 180°$ and $\angle m = \angle p$, why does $\angle p + \angle y = 180°$?

12. If $EG = EF$ and $FG = EF$, why does $EG = FG$?

13. If $\angle G = \angle E$ and $\angle E = \angle F$, why does $\angle G = \angle F$?

14. If $\angle CBE = 25°$ and $\angle EBA = 15°$, why does $\angle CBE + \angle EBA = 40°$?

15. Why does $\angle ABC = \angle ABE + \angle EBC$?

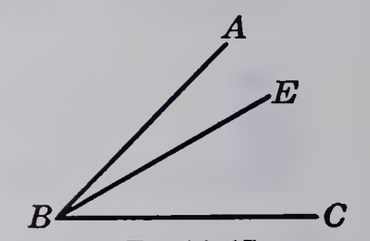
16. If $\angle ABE = \angle EBC$, why does BE bisect $\angle CBA$?

17. $\angle CBA$ has only one bisector. Why?

18. If $NM \perp RS$, why is $\angle RMN = \angle SMN$?

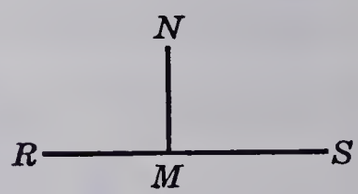
19. If $NM \perp RS$, why are $\measuredangle RMN$ and NMS right angles?

Ex. 12, 13



Ex. 14-17

20. On your paper place two points A and B about two inches apart. Draw straight line AB. Why can you not draw another straight line through A and B?



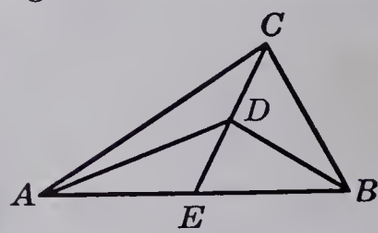
Ex. 18, 19

Figure ABC is a triangle having CE bisecting AB and with line segments AD and BD drawn as shown.

21. Why does $AE + EB = AB$?

22. If $AB = 2$ inches, $AD + DB > 2$ inches. Why?

23. $\angle EAD + \angle DAC = \angle EAC$. Why?



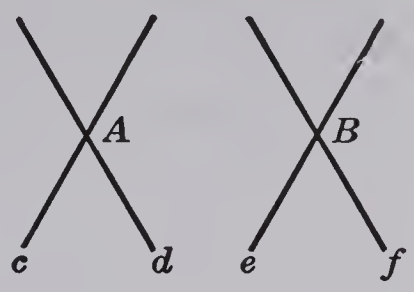
Ex. 21-26

24. AD can be extended to BC. Why?

25. If $\angle EBD + \angle DBC = \angle EBC$, why does $\angle EBD = \angle EBC - \angle DBC$?

26. Why does $AE = EB$?

27. If lines c and d can be moved to coincide with lines e and f respectively, why will point A coincide with point B?



Ex. 27

In the figure at the right below, $ABCD$ is a straight line segment.

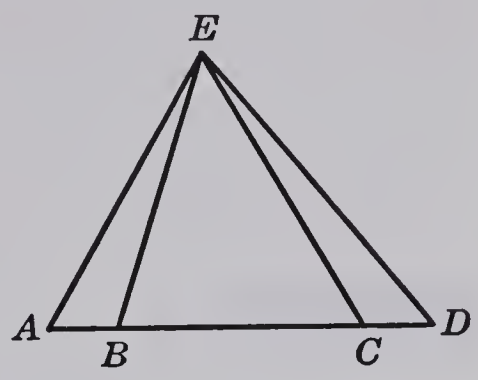
28. Why are $\measuredangle ABE$ and EBC supplementary?

29. Why does $AB + BC = AC$? Why does $BC + CD = BD$?

30. Why is $AE < AB + BE$?

31. Why are $\measuredangle BCE$ and DCE supplementary?

32. If $\angle A = \angle BCE$ and $\angle BCE = 60°$, why does $\angle A = 60°$?



Ex. 28–32

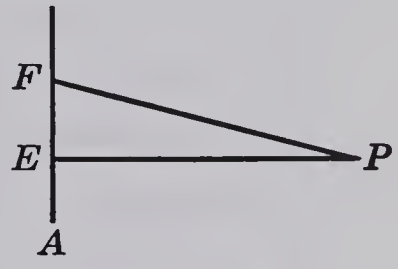
AB is a straight line and $PE \perp AB$.

33. Why is PF not $\perp AB$?

34. Why is $PE < PF$?

35. Why is $\angle PEA = \angle PEB$?

36. Why is $\angle PEA$ a rt. $\angle$?



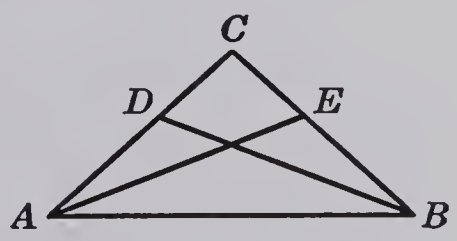
Ex. 33–36

37. If $\angle ABC = \angle BAC$, $\angle ABD = \frac{1}{2} \angle ABC$, and $\angle BAE = \frac{1}{2} \angle BAC$, why is $\angle BAE = \angle ABD$?

38. If $\angle CAB = \angle CBA$ and $\angle EAB = \angle DBA$, why does $\angle CAE = \angle CBD$?

39. If $\angle CDB = \angle CEA$ and $\angle CDA = \angle CEB$, why does $\angle ADB = \angle AEB$?

40. If CEB is a straight line, why is $\angle CEA$ supplementary to $\angle AEB$?



Ex. 37–40

37. Deductive Reasoning Arranged in Column Form

In geometry we find it helpful to arrange the statements of deductive reasoning in column form. In this arrangement we place the hypothesis

(specific statement) and the conclusion in one column with the heading "Statements" and the reason (general statement) in another column with the heading "Reasons." We shall designate an hypothesis by **H**, a conclusion by **C**, and a reason by **R**.

Example. **H.** $\angle a + \angle b = 180°$
C. $\angle a$ and $\angle b$ are supplementary.
R. Two angles are supplementary if their sum is 180°.

In column form this deduction can be written as follows:

STATEMENTS	REASONS
1. If $\angle a + \angle b = 180°$,	
2. then $\angle a$ and $\angle b$ are supplementary.	2. Two angles are supplementary if their sum is 180°.

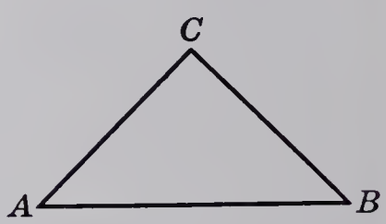
EXERCISES

A

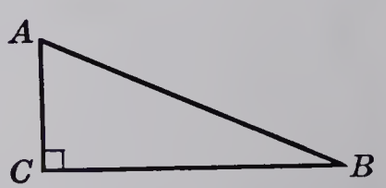
Using the example above as a model, arrange the following deductions in column form:

1. **R.** All fish live in water.
   H. A bass is a fish.
   C. A bass lives in water.

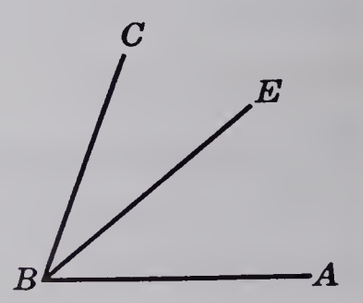
2. **H.** Marcia is a member of the XYZ Club.
   R. All members of the XYZ Club have studied mathematics.
   C. Marcia has studied mathematics.



3. **H.** In triangle ABC, $AC = BC$.
   R. A triangle having two equal sides is isosceles.
   C. Triangle ABC is isosceles.



4. **R.** A triangle having a right angle is a right triangle.
   H. Triangle ABC has $\angle C$ a right angle.
   C. Triangle ABC is a right triangle.



5. **H.** $\angle ABE$ and $\angle EBC$ have a common vertex and a common side between them.
   C. $\angle ABE$ and $\angle EBC$ are adjacent angles.

R. Two angles which have a common vertex and a common side between them are adjacent angles.

38. One-Cycle Proofs

A proof of one cycle of deductive reasoning consists of an hypothesis (specific statement), a conclusion, and a reason (general statement). Study the two examples which follow.

Example 1.

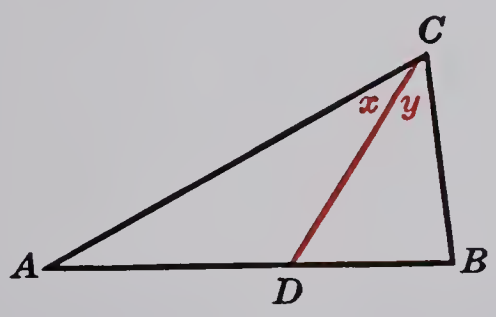
Given $9x = 54$.
Prove that $x = 6$.

Proof

	STATEMENTS	REASONS
1.	$9x = 54$	1. Given.
2.	$x = 6$	2. If equals are divided by equals, the quotients are equal.

This proof corresponds to the solution of the equation $9x = 54$ as carried out in algebra, but it does not prove (or check) that $x = 6$ satisfies the equation.

You should note that the word "given" in the proof means "Given in the hypothesis" and that the words "if" and "then" are omitted in the example.



Given figure $ADBC$ with CD bisecting $\angle ACB$.
To prove that $\angle x = \angle y$.

Proof

	STATEMENTS	REASONS
1.	CD bisects $\angle ACB$.	1. Given.
2.	Then $\angle x = \angle y$.	2. The bisector of an angle is the line which meets the vertex of the angle and separates the angle into two equal angles.

EXERCISES

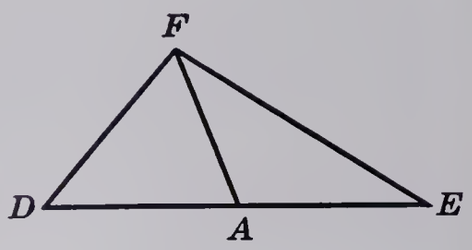
A

Following the form used in the examples on page 61, prove the following. Do not forget to draw the figure if one is needed.

1. *Given* $4x = 40$.
Prove that $x = 10$.

2. *Given* line segment DE bisected by AF.
Prove $DA = AE$.

3. *Given* the line segment DE, and $DA = AE$.
Prove that AF bisects DE.

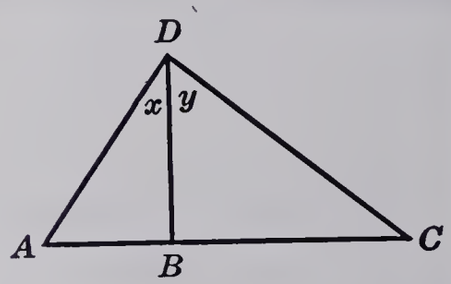


Ex. 2, 3

4. *Given* $x + 9 = 75$.
Prove that $x = 66$.

5. *Given* figure $ABCD$ with $BD \perp$ line ABC.
Prove $\angle ABD = \angle CBD$.

6. *Given* figure $ABCD$ and $\angle x + \angle y = 1$ rt. $\angle$.
Prove that $\angle x$ and $\angle y$ are complementary.

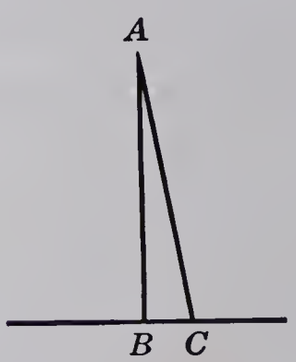


Ex. 5, 6

7. *Given* $y - 3 = 12$.
Prove that $y = 15$.

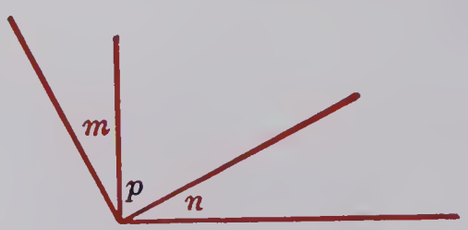
8. *Given* $\frac{y}{4} = 5$.
Prove that $y = 20$.

9. *Given* $AB \perp BC$.
Prove that AC is not $\perp BC$.



Ex. 9

Example 3.



Given $\angle m$ complementary to $\angle p$ and $\angle n$ complementary to $\angle p$.
Prove $\angle m = \angle n$.

Proof

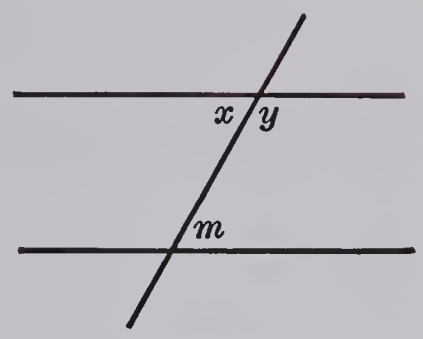
STATEMENTS	REASONS
1. $\angle m$ is complementary to $\angle p$.	1. Given.
2. $\angle n$ is complementary to $\angle p$.	2. Given.
3. $\therefore \angle m = \angle n$.	3. Complements of the same angle are equal.

10. *Given* $\angle x$ supplementary to $\angle y$ and $\angle m$ supplementary to $\angle y$.

Prove $\angle x = \angle m$.

11. *Given* $\angle x + \angle y =$ 1 st. $\angle$, and $\angle x = \angle m$.

Prove that $\angle m + \angle y =$ 1 st. $\angle$.



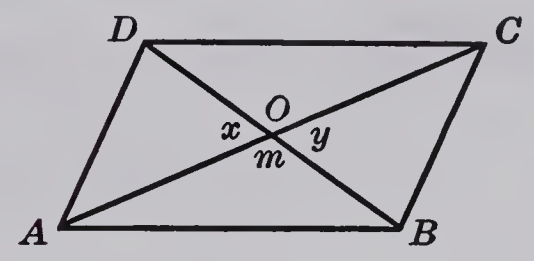
Ex. 10, 11

12. *Given* figure $ABCD$ with AC intersecting BD in O, $AO = OC$, and $BO = OD$.

Prove that $AO + BO = DO + CO$.

13. *Given* figure $ABCD$ with $\angle m$ supplementary to $\angle y$ and $\angle m$ supplementary to $\angle x$.

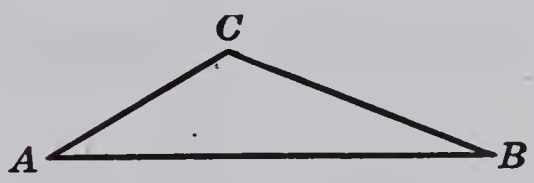
Prove that $\angle x = \angle y$.



Ex. 12, 13

14. *Given* $\angle A = 30°$, $\angle B = 22°$, and $\angle C = 128°$.

Prove that $\angle A + \angle B + \angle C = 180°$.



39. Assumptions (Axioms) Used in Solving Equations

In algebra each time you changed an equation into another equivalent equation, you were thinking "If this equation is true, then the next equation is true."

For example, if you solved the equation $9x = 45$, you thought "If $9x = 45$, then $x = 5$," the reason being "If equals are divided by equals, the quotients are equal."

Study Examples 1 and 2 on the next page.

Example 1. Solve $3x + 14 = 8$.

Solution

STATEMENTS	REASONS
1. $3x + 14 = 8$	1. Given.
2. $3x = -6$	2. If equals are subtracted from equals, the remainders are equal.
3. $x = -2$	3. If equals are divided by equals, the quotients are equal.

In this example there are two deductions. The first deduction is "If $3x + 14 = 8$, then $3x = -6$" and the second is "If $3x = -6$, then $x = -2$." Do you see that the conclusion of the first deduction is the hypothesis of the second deduction? The proof that $x = -2$ consists of two cycles, one cycle for each deduction.

Example 2. Solve $3(x - 1) - (x - 4) = 11$.

Solution

STATEMENTS	REASONS
1. $3(x - 1) - (x - 4) = 11$	1. Given.
2. $3x - 3 - x + 4 = 11$	2. A quantity can be substituted for its equal.
3. $2x + 1 = 11$	3. Same as reason 2.
4. $2x = 10$	4. If equals are subtracted from equals, the remainders are equal.
5. $x = 5$	5. If equals are divided by equals the quotients are equal.

How many cycles of deductive reasoning are there in this proof?

EXERCISES

A

Using the form of Examples 1 and 2, solve the following:

1. $2x - 1 = 15$
2. $7y + 2 = 37$
3. $5x + 4 = -26$
4. $4x = 12 - 2x$
5. $3p = 30 + 5p$
6. $4x - x = 16 + x$
7. $2(y - 1) + (y + 3) = 10$
8. $x - 4(x - 1) = 7$
9. $4x = 10 - (x + 5)$
10. $3(x - 1) = 5(x - 3)$

11. $\frac{m}{4} - \frac{m}{5} = 4$

12. $\frac{5x}{6} + 4 = \frac{x-6}{3}$

13. $\frac{y+6}{4} = 4\frac{1}{2}$

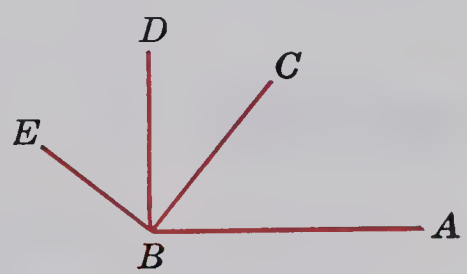
14. $\frac{x-4}{9} - \frac{x}{3} = \frac{x+2}{6}$

40. Definitions and Assumptions Used in Proofs

In the preceding exercises the solutions of the equations required two or more deductions. Each equation after the first was derived from the one preceding it.

We shall now complete some proofs which deal with geometric figures and which have some statements that are derived from preceding statements. In each proof at least one of the reasons is "given."

Example. See if you can complete the following proof without referring to the explanations which follow it. Use paper and pencil.



Given figure $ABCDE$, with $DB \perp AB$ and $EB \perp BC$.

To prove that $\angle ABC = \angle EBD$.

Proof

STATEMENTS	REASONS
1. $AB \perp BD$.	1. Why?
2. $\angle ABD =$ 1 rt. $\angle$.	2. Why?
3. $EB \perp BC$.	3. Why?
4. $\angle CBE =$ 1 rt. $\angle$.	4. Why?
5. $\angle ABC + \angle CBD = \angle ABD$.	5. Why?
6. $\angle ABC + \angle CBD =$ 1 rt. $\angle$.	6. Why?
7. $\therefore \angle ABC$ is complementary to $\angle CBD$.	7. Why?
8. $\angle EBD + \angle DBC = \angle CBE$.	8. Why?
9. $\angle EBD + \angle DBC =$ 1 rt. $\angle$.	9. Why?
10. $\angle EBD$ is complementary to $\angle CBD$.	10. Why?
11. From (7) and (10), $\angle ABC = \angle EBD$.	11. Why?

Now check your work with the following:

The reason for Statement 1 is "Given." The word "Given" is used as the reason for a statement which repeats given data or information.

Statement 2 follows from Statement 1 and its reason is "An angle that is formed by a line and its perpendicular is a right angle."

The reason for Statement 3 is "Given."

The reason for Statement 4 is the same as Reason 2.

The reason for Statement 5 is "The whole is equal to the sum of its parts."

The reason for Statement 6, which is derived from Statements 2 and 5, is "A quantity may be substituted for its equal" or "Quantities which are equal to the same quantity are equal to each other."

Statement 7 follows from Statement 6 and its reason is "Two angles whose sum is a right angle are complementary."

The reason for Statement 8 is the same as Reason 5.

The reason for Statement 9 is the same as Reason 6.

The reason for Statement 10 is the same as Reason 7.

The reason for Statement 11 is "Complements of the same angle are equal."

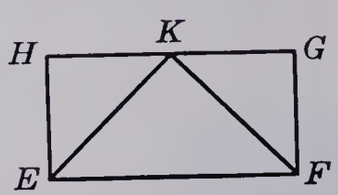
EXERCISES

A

Complete the following proofs:

1. *Given* figure $EFGH$ with K a point on HG, $FG \perp KG$, and $EH \perp HK$.

To prove that $\angle H$ is supplementary to $\angle G$.



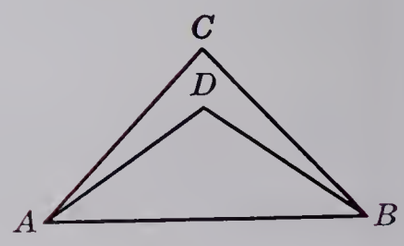
Proof

STATEMENTS	REASONS
1. $EH \perp HK$.	1. Why?
2. $FG \perp KG$.	2. Why?
3. $\angle H$ is a rt. $\angle$.	3. Why?
4. $\angle G$ is a rt. $\angle$.	4. Why?
5. $\angle H + \angle G = 2$ rt. $\measuredangle$s.	5. Why?
6. $\therefore \angle H + \angle G = 1$ st. $\angle$.	6. Why?
7. $\angle H$ is supp. to $\angle G$.	7. Why?

2. Complete the following:

Given triangles ABC and ABD having $\angle CAB = \angle CBA$ and $\angle DAB = \angle DBA$.

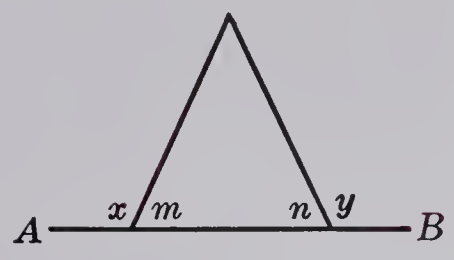
To prove that $\angle CAD = \angle CBD$.



Proof

STATEMENTS	REASONS
1. $\angle CAB = __?__$.	1. Why?
2. $__?__ = \angle DBA$.	2. Why?
3. $\angle CAB - __?__ = __?__ - \angle DBA$.	3. Why?
4. $\angle CAD = \angle CBD$.	4. Why?

3. In the figure at the right, $\angle m = \angle n$ and AB is a straight line. We are to prove that $\angle x = \angle y$. Complete the following proof.

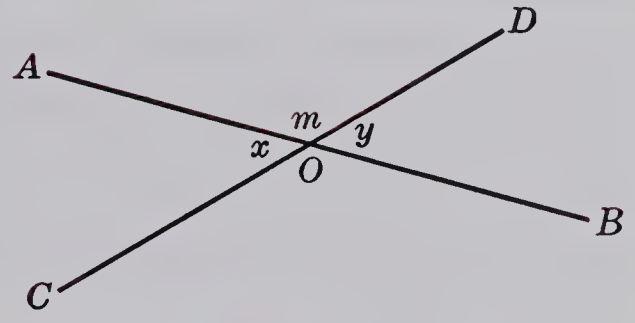


Proof

STATEMENTS	REASONS
1. AB is a st. line.	1. Why?
2. $\angle x$ is the supp. of $\angle m$.	2. Why?
3. $\angle y$ is the supp. of $\angle n$.	3. Why?
4. $\angle m = \angle n$.	4. Why?
5. $\therefore \angle x = \angle __?__$.	5. Why?

4. *Given* st. lines AB and CD intersecting in point O and forming $\measuredangle x$, m, and y.

To prove that $\angle x = \angle y$.



Proof

STATEMENTS	REASONS
1. CD is a st. line.	1. Why?
2. $\angle x$ is the supp. of $\angle m$.	2. Why?
3. AB is a st. line.	3. Why?
4. $\angle y$ is the supp. of $\angle __?__$.	4. Why?
5. $\therefore \angle x = \angle y$.	5. Why?

41. Theorems

A *theorem* is a statement to be proved. We accept a theorem as true only after we have proved it. How does a theorem differ from an assumption? After a theorem has been proved, it can be used along with definitions and assumptions as a general statement (reason) in other

proofs. Theorems are stated by simple sentences or by if-then complex sentences.

42. Parts of a Formal Proof

When a proof is arranged in a definite order and in a certain form, it is called a *formal proof*. A proof which does not follow a model form is *informal*. Most of the proofs in this text are formal. When we write a proof or give it orally we say that we are demonstrating it.

Each formal proof consists of the following five parts in the order given:

I. A statement of the theorem.
II. The figure.
III. A statement of what is given.
IV. A statement of what is to be proved.
V. An orderly proof.

43. The Theorem

The *theorem* is always stated as a complete sentence. Since the theorem may be used later to prove other theorems, it should be thoroughly understood and memorized.

44. The Figure

The *figure* is a lettered drawing of the points, lines, angles, and other geometric figures which are described in the theorem.

The figure should be a general one. For example, when drawing an angle we should not draw a right angle unless the hypothesis of the theorem states that the angle is a right angle.

45. The "Given"

Under this title is a description of the figure as described by the hypothesis. This description should include all important data of the hypothesis.

46. The "To prove"

Under this title is the conclusion of the theorem stated in terms of the figure.

47. The Proof

The proof consists of a series of deductions beginning with the facts given in the hypothesis and ending with the conclusion. The statements consist of all the items listed under the word "given" and one or more applications of definitions, assumptions, and theorems previously proved.

For each statement of the proof there must be a reason. This reason must be one of the following: given; definition; assumption; theorem previously proved; identity; or construction. As was previously stated, the word "identity," abbreviated "Iden.," is used for the assumption "any quantity equals itself." For example, the reason for $\angle m = \angle m$ is "identity" or "Iden."

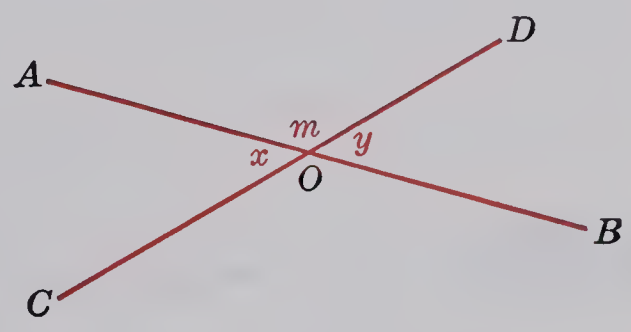
48. How to Make a Formal Proof

If you know what we mean by the five parts of a formal proof, you are prepared to learn how a proof is discovered and how it is written. We shall prove the following theorem.

Theorem. If two lines intersect, the vertical angles are equal.

I. We first state the theorem:

"If two lines intersect, the vertical angles are equal."



II. We draw the figure to represent the theorem. The hypothesis of the theorem states that two lines intersect, so we draw two intersecting lines and letter them with capital letters denoting points on the lines. We use small letters to name the angles. Why do we use $\angle x$ instead of $\angle AOC$?

III. Under the title "Given" we describe the figure as given by the hypothesis. Thus, we state:

Given the lines AB and CD intersecting in O and forming the vertical angles x and y.

IV. By referring to the conclusion of the theorem, we state in terms of the figure what is to be proved. Thus, we write

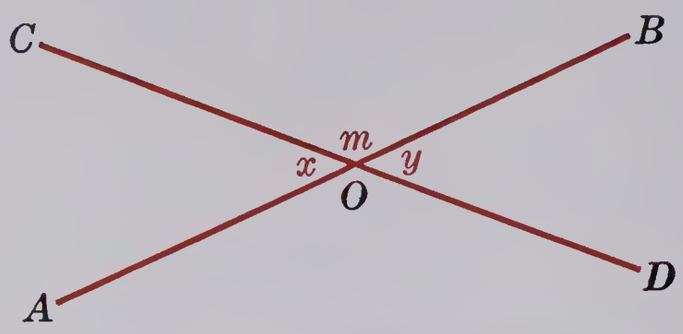
To prove that $\angle x = \angle y$.

V. Before we attempt to write the proof of the theorem, we should think through the proof. We are to prove two angles equal. We should recall all the ways we have had to prove angles equal. These ways are: the definition of equal angles (§11) and assumptions 1–8, 19, 28–31. By studying the figure we discover that $\angle x$ is the supplement of $\angle m$ and that $\angle y$ is the supplement of $\angle m$. Then we know that $\angle x = \angle y$ because "Supplements of the same angle are equal" (Assumption 30).

We are now prepared to write the proof.

★ Theorem 1

49. If two lines intersect, the vertical angles are equal.



Given the lines AB and CD intersecting in O, forming the angles x, y, and m.

To prove that $\angle x = \angle y$.

*Planning the Proof**: 1. We can prove $\measuredangle s = by$ Asmts. 1–8, 19, 28–31.
2. We shall use Asmt. 30.

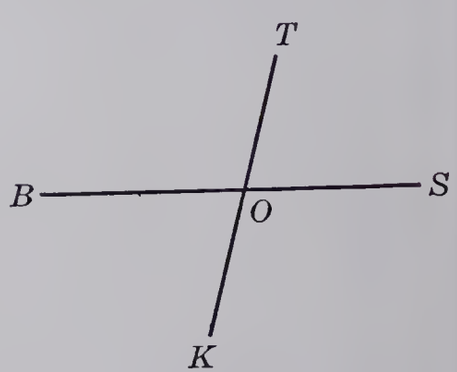
Proof

STATEMENTS	REASONS
1. AB is a st. line.	1. Given.
2. $\angle x$ is the supp. of $\angle m$.	2. Asmt. 24 (If two adjacent angles have their exterior sides in a straight line, they are supplementary).
3. CD is a st. line.	3. Given.
4. $\angle y$ is the supp. of $\angle m$.	4. Reason 2.
5. $\therefore \angle x = \angle y$.	5. Asmt. 30 (Supplements of the same angle are equal).

This theorem gives us another way of proving angles equal.

Example. If $\angle BOT = 105°$, how large is $\angle KOS$?

Solution. $\angle KOS = 105°$, because "If two lines intersect, the vertical angles are equal."

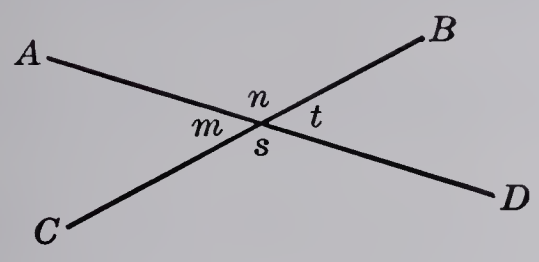


*"Planning the Proof" is not part of the proof. It shows how the proof is discovered.

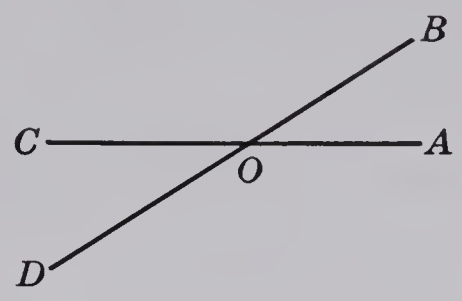
EXERCISES

A

Give a reason for each answer in the exercises.



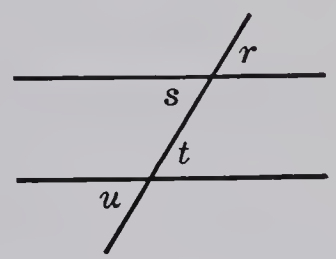
Ex. 1–4



Ex. 5

1. If $\angle n = 100°$, how many degrees are there in $\angle s$? in $\angle m$? in $\angle t$?
2. If $\angle t = 55°$, how many degrees are there in $\angle m$? in $\angle s$? in $\angle n$?
3. If $\angle m = 30° 45'$, how large is $\angle t$? $\angle n$?
4. If $\angle t = x°$, how many degrees are there in $\angle m$? in $\angle s$?
5. $\angle AOB = (x - 10)°$. Find the size of $\angle COD$; of $\angle AOD$.

6. The adjoining figure consists of three straight lines.



Given $\angle s = \angle t$.

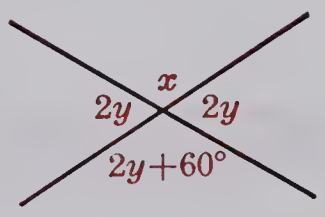
To prove that $\angle r = \angle u$.

Proof

STATEMENTS	REASONS
1. $\angle r = \angle s$.	1. Why?
2. $\angle u = \angle t$.	2. Why?
3. $\angle s = \angle t$.	3. Why?
4. $\therefore \angle r = \angle u$.	4. Why?

B

7. Find the number of degrees in each of the four angles.



Ex. 7

Oral Exercises in Reasoning [Optional]

In everyday conversations people make statements without stating the assumptions upon which the statements are based; i.e., they use

implied assumptions. In the following exercises tell what assumptions are implied.

1. Teachers should be retired when they become 65 years of age.

2. Ella makes good grades in high school, so she will make good grades in college.

3. I know I can make the basketball team because I am tall, fast, and have a good shooting average.

4. Mr. Frye should have his house painted this year. It was painted four years ago.

5. Mr. Fleming, the sales manager of the Richland Men's Shop, tells his salesmen to avoid arguments with the customers.

6. Dick's car failed to start this morning. He should buy new spark plugs.

7. Jeff should have plenty of money. He never spends any.

8. You should see how Mrs. Raleigh dresses. She must be wealthy.

9. The barometer reading is falling. It will probably rain soon.

10. Frank Locke will soon be in prison. He was arrested yesterday for stealing a bicycle.

11. If Helen really loved her mother, she would help with the housework.

12. Frieda cannot enter college because she never studied geometry.

13. You should eat more apples and live longer.

14. Ben is 15 years old. Next year he can get a driver's license.

15. I am going to reduce my weight. I am too short of breath.

16. Mr. Wilson is 58 years old and Mr. Reddick is only 48 years old. I will follow Mr. Wilson's advice in preference to the advice of Mr. Reddick.

REVIEW QUESTIONS — Chapter 3

1. How many bisectors may an angle have?

2. What is the sum of all the angles about a point on one side of a straight line?

3. What is the "if" clause of a theorem called?

4. What is the "then" clause of a theorem called?

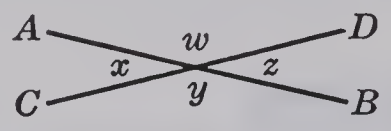
Given straight lines AB and CD in the diagram below.

5. Are angles x and y supplementary?

6. Are angles y and w vertical?

7. Are angles x and z complementary?

8. What kind of angles are w and z?

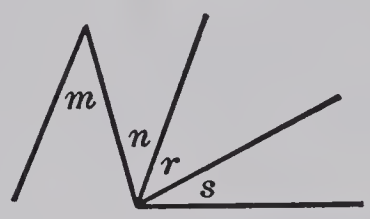


Ex. 5–8

9. Are angles m and n adjacent angles?

10. Are angles n and s vertical angles?

11. What should follow each statement in a proof?



Ex. 9, 10

12. Do equal circles have equal radii?

13. Are supplementary angles equal?

14. Do we prove axioms? postulates? definitions? theorems?

15. How many degrees does an angle 9 times its supplement contain?

16. How many midpoints does a line segment have?

17. How many perpendiculars may be drawn to a line at a point in the line?

18. What is the sum of all the angles about a point?

19. What relation do two adjacent angles have if their exterior sides form a straight line?

20. What name is given to the shortest line segment that can be drawn from a point to a line?

21. What do we know about the exterior sides of two supplementary adjacent angles?

State the hypothesis and conclusion of each sentence:

22. If it rains this morning, I shall wear a raincoat.

23. We shall not have school Monday if we win this football game.

Complete the following sentences by supplying the missing words:

24. The three steps of a deduction are the general statement, the __?__ statement, and the __?__.

25. We prove theorems but we do not prove __?__ and __?__.

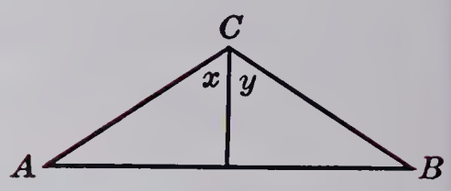
Supply the reasons in exercises 26 and 27.

26. *Given* $x - 9 = 12$
Prove $x = 21$

Proof

STATEMENTS	REASONS
1. $x - 9 = 12$	1. Why?
2. $x = 21$	2. Why?

27. *Given* $\angle x = \angle y$, $\angle A$ comp. $\angle x$, and $\angle B$ comp. $\angle y$.
Prove $\angle A = \angle B$.



Proof

STATEMENTS	REASONS
1. $\angle x = \angle y$.	1. Why?
2. $\angle A$ is comp. $\angle x$.	2. Why?
3. $\angle B$ is comp. $\angle y$.	3. Why?
4. $\angle A = \angle B$.	4. Why?

SUMMARY of Principal Bases for Proof

50. Two angles are equal

a. If they are right angles.
b. If they are complements of the same angle or of equal angles.
c. If they are supplements of the same angle or of equal angles.
d. If they are vertical angles.
e. By the use of assumptions 1–8.

51. Two lines are perpendicular

a. If they form a right angle, or equal adjacent angles, with each other.
b. If one is the shortest line segment that can be drawn from a given point to the other line.

52. Two angles are complementary

a. If their sum is a right angle.

53. Two angles are supplementary

a. If their sum is a straight angle.
b. If they are adjacent angles having their exterior sides in a straight line.

WORD LIST

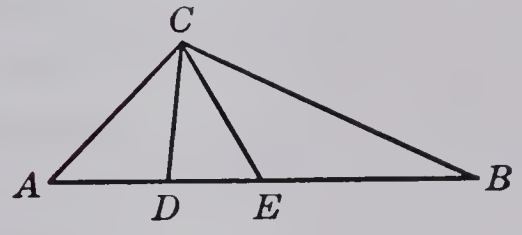
At this time you should be able to spell each of the following words and know their meaning:

assumption	deductive	identity	proof
axiom	definition	inductive	quotient
bisector	exterior	intersect	theorem
complement	figure	perpendicular	triangle
conclusion	hypothesis	postulate	vertical

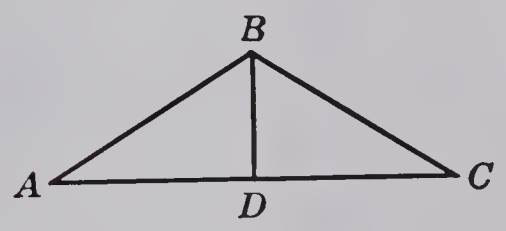
TEST 3

Supply the definitions, assumptions, or theorem for the following conclusions.

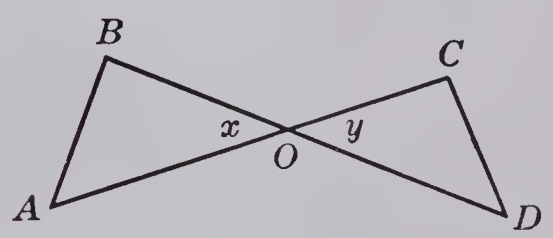
1. If $3m = 21$, $m = 7$.
2. If $x + 6 = 14$, $x = 8$.
3. If $x + y = 18$ and $x - y = 2$, then $2x = 20$.
4. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.
5. If $x = p$ and $y = p$, then $x = y$.
6. If $x + y = 8$ and $y = 2$, then $x + 2 = 8$.
7. If CD bisects $\angle ACB$ then CE does not bisect $\angle ACB$.
8. If E is the midpoint of AB, then D is not the midpoint of AB.
9. If $\angle ADB = \angle CDB$, $BD \perp AC$.
10. If $x^2 = 36$, then $x = 6$.
11. If $\angle x = 36°$ and $\angle y = 54°$, then $\angle x$ and $\angle y$ are complementary.
12. If $\angle x = 62°$, then $\angle y = 62°$.
13. $AB < AO + OB$.
14. $AO + OC = AC$.



Ex. 7, 8



Ex. 9



Ex. 12–14

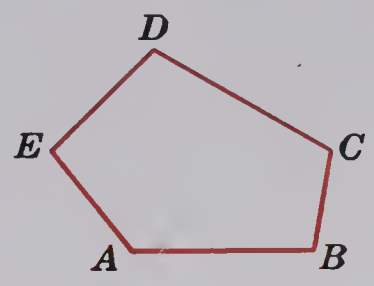
CHAPTER 4

Triangles

Because of its wide application
in art, architecture, and engineering,
the triangle is probably
the most important figure
studied in plane geometry.
Since the shape of a triangle
cannot be altered
without changing the length of a side,
it is used to give rigidity
in nearly all constructions.

54. Polygon

A *polygon* is a closed broken line. The line segments AB, BC, CD, DE, and EA are the *sides* of the polygon $ABCDE$. The points A, B, C, D, and E are the *vertices* of the polygon, and $\measuredangle A$, B, C, D, and E are the angles of the polygon. $\angle A$ and $\angle B$ are *consecutive angles*; $\angle B$ and $\angle C$ are consecutive angles; etc. Sides AB and BC are *adjacent* sides; sides BC and CD are adjacent sides; etc.



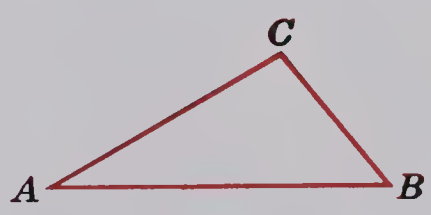
Although a polygon is defined as a broken line, it sometimes means the portion of plane enclosed by a broken line.

55. Triangles

A *triangle* ($\triangle$) is a polygon having three sides. Every triangle has six parts,—three sides and three angles. The three vertices are not considered parts of the triangle.

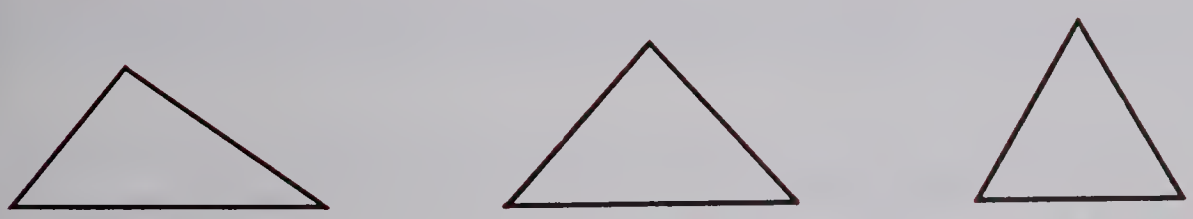
The *perimeter* of a triangle is the sum of the lengths of the three sides.

The *base* of a triangle is the side upon which the triangle is supposed to stand. Any side of a triangle may be taken as the base. The *vertex angle* is the angle opposite the base, and the vertex of this angle is often called the *vertex* of the triangle. Thus in the figure, AB may be considered the base, ACB the vertex angle (opposite AB), and C the vertex of the triangle ABC.



56. Triangles Classified as to Sides

A *scalene triangle* is a triangle having *no two* sides equal.
An *isosceles triangle* is a triangle having *two* sides equal.
An *equilateral triangle* is a triangle having *all three* sides equal.



Scalene Isosceles Equilateral

57. Triangles Classified as to Angles

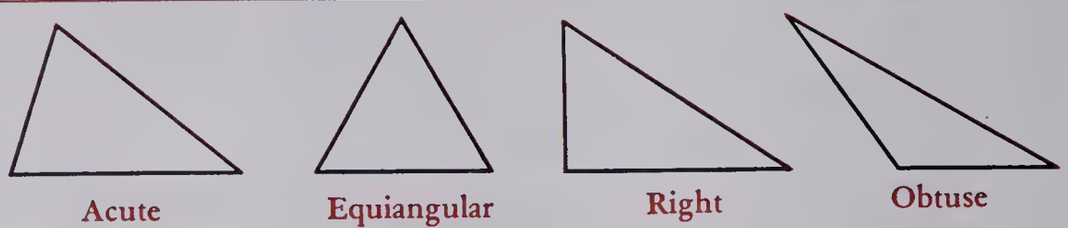
An *acute triangle* is a triangle having *all three* angles acute.

An *equiangular triangle* is a triangle having *all three* angles equal.

A *right triangle* is a triangle having *one* right angle.

An *obtuse triangle* is a triangle having *one* obtuse angle.

An *oblique triangle* is a triangle that does not contain a right angle.

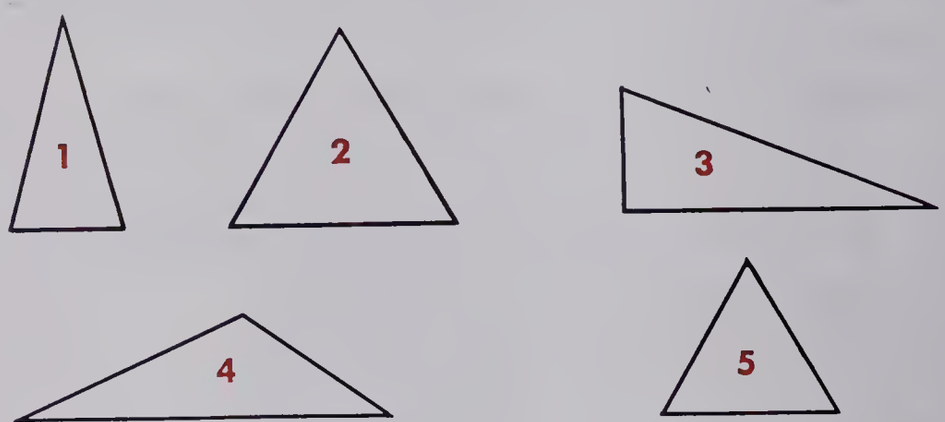


Acute Equiangular Right Obtuse

EXERCISES

A

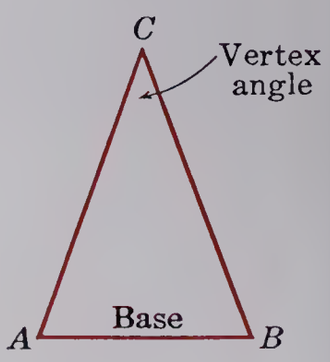
1. Which, if any, of the triangles shown below appear to be scalene? isosceles? equilateral? right? acute? oblique? equiangular?



2. Draw a right scalene triangle; an isosceles right triangle.

58. Parts of an Isosceles Triangle

In an isosceles triangle the angle formed by the two equal sides is usually considered as the *vertex angle*. In the figure at the right $\angle C$ is the vertex angle.



The side opposite the vertex angle in an isosceles triangle is usually called the *base* of the triangle. In this figure AB is the base.

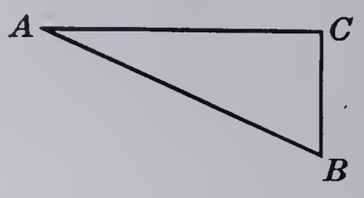
The angles opposite the equal sides are called the *base angles* of the triangle. $\angle A$ and $\angle B$ are the base angles of $\triangle ABC$.

Fred Lyon from Rapho-Guillumette

What important property of the triangle is illustrated in the construction of this bridge—the San Francisco–Oakland Bay Bridge?

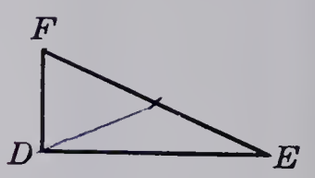
59. Included Sides and Included Angles

In $\triangle ABC$, $\angle A$ is included by the sides AB and AC; $\angle B$ is included by the sides AB and BC; and $\angle C$ is included by the sides AC and BC. The side AB is included by $\angle A$ and $\angle B$; the side BC is included by $\angle B$ and $\angle C$; and the side AC is included by $\angle A$ and $\angle C$. The word *included* means "confined" or "enclosed."



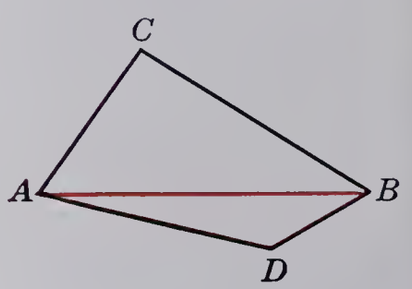
60. Parts of a Right Triangle

In a right triangle the sides including the right angle are often called the *legs* and the side opposite the right angle is called the *hypotenuse*. In $\triangle DEF$, EF is the hypotenuse and DF and DE are the legs.

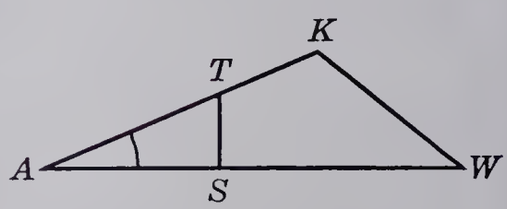


61. Common Sides and Common Angles

When a side of one triangle is a side of another triangle, we say that the two triangles have a *common side*. In this figure AB is a common side of $\triangle ADB$ and $\triangle ABC$.

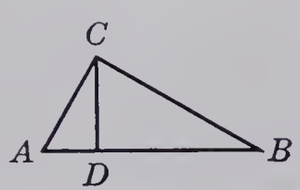


When an angle of one triangle is an angle of another triangle, we say that the triangles have a *common angle*. In the figure $\angle A$ is the common angle of $\triangle AST$ and $\triangle AWK$.

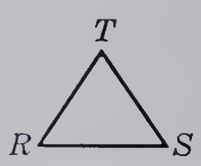


EXERCISES Oral Exercises

1. In $\triangle ABC$, $\angle ACB$, $\angle CDB$, and $\angle ADC$ are right angles. Name the legs and the hypotenuse of $\triangle ABC$; of $\triangle ADC$; of $\triangle DBC$.



Ex. 1

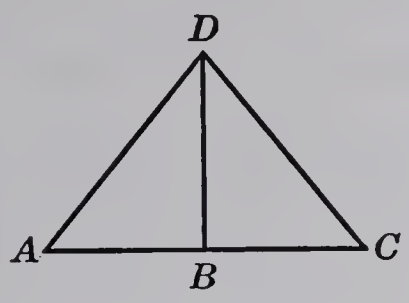


Ex. 2

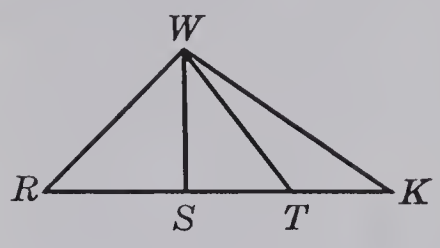
2. In $\triangle RST$, what sides include $\angle R$? $\angle S$? $\angle T$?
What angles include side RS? side ST? side RT?

3. In $\triangle ABD$ what angles include side AB? side AD? side BD?

4. In $\triangle ABD$ what sides include $\angle A$? $\angle ABD$? $\angle ADB$?



Ex. 3–6



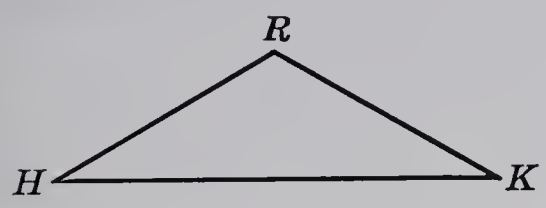
Ex. 7, 8

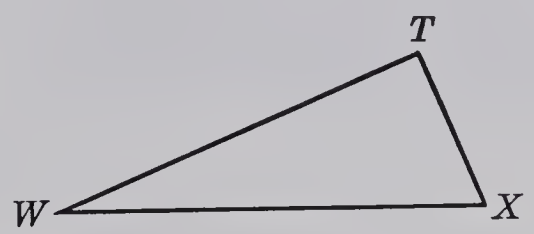
5. In $\triangle BCD$ what side is included by $\angle DBC$ and $\angle BCD$?

6. In $\triangle ACD$ what angles include side AC? side AD?

7. In $\triangle SWK$ what side is included by $\angle WSK$ and $\angle K$?

8. In $\triangle RTW$ what sides include $\angle R$? $\angle RWT$?



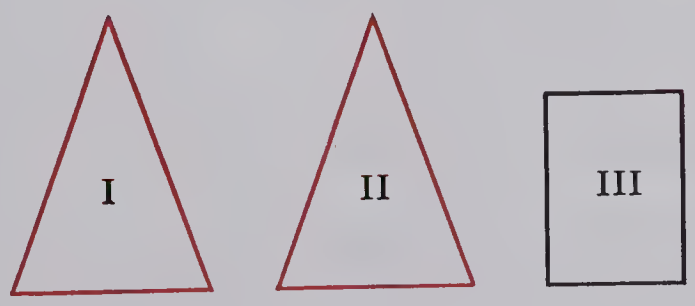


9. In $\triangle HKR$, $HR = KR$. Classify the triangle. What is the base? What is the vertex angle?

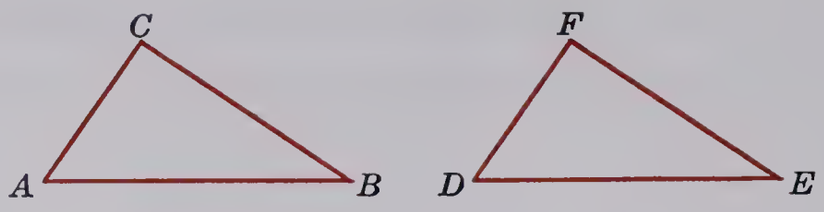
10. In $\triangle WXT$, $\angle T = 90°$. Classify the triangle. What is the name of WX?

62. Congruent Figures

Any two geometric figures are said to be *congruent* ($\cong$) if they can be made to coincide (fit exactly on each other). Since a definition is reversible, if two figures are congruent, they can be made to coincide. The word "congruent" means more than the word "equal." Figures I, II, and III are all equal in area, but only Figs. I and II are congruent.



If two figures coincide, each side of one coincides with the corresponding side of the other and each angle of one coincides with the corresponding angle of the other. Thus, if triangle ABC coincides with triangle



DEF, then AB, BC, and AC coincide respectively with DE, EF, and DF; and $\measuredangle A$, B, and C coincide respectively with $\measuredangle D$, E, and F.

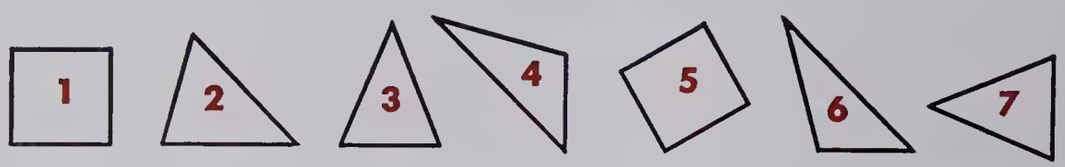
If two triangles are congruent, do you see that all six parts of one triangle are equal respectively to the six parts of the other?

EXERCISES

A

1. If two line segments are equal, are they necessarily congruent? If two line segments are congruent, are they necessarily equal? If two angles are equal, are they necessarily congruent? Are two line segments congruent if their end points can be made to coincide?

2. Name the pairs of figures below which seem to be congruent. Test each conclusion by tracing one of the figures on thin paper and applying the tracing to the other figure.



3. Draw two triangles which have the same shape but are not congruent. Can you draw two triangles which have about the same area but not the same shape?

4. If two circles have equal radii, are they congruent?

5. Are two squares congruent if their sides have the same length?

6. If two automobile parts are made from the same mold, are they congruent?

7. Name some articles in your home that seem to be congruent.

63. Proof by Superposition

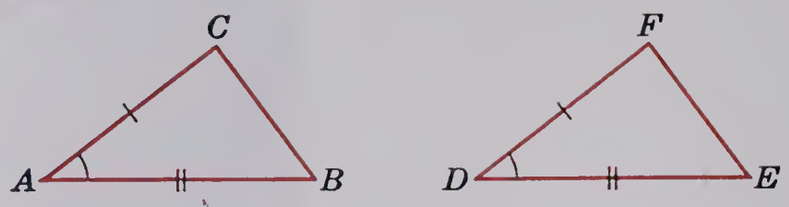
A few theorems of geometry are proved by imagining one figure placed on another and reasoning that the figures coincide. This imaginary placing of one figure on another is called *superposition*. We shall use superposition in the following article.

Illustrations of congruence are found in many occupations and industries. At the right you see a student in a dressmaking class. Below are men inspecting burner cans used in the combustion chamber of a jet engine; turbine discs and blades are also shown. Can you bring to class pictures of further examples of congruence?

Simplicity Pattern Co.

Pratt & Whitney Aircraft

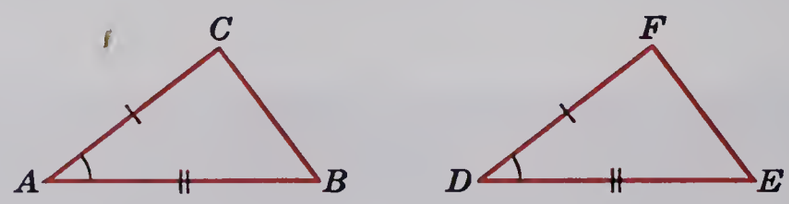
64. An Investigation



In $\triangle ABC$ and DEF, $AB = DE$, $AC = DF$, and $\angle A = \angle D$. Trace each triangle on thin paper. Then cut out your tracings and place one on the other so that the point A falls on the point D and AC falls along DF. Can AB be made to fall on DE? Will C fall on F? Will B fall on E? Will the triangles coincide? Would they coincide if $\angle A$ were less than $\angle D$? Is $\angle B = \angle E$? Why? Is $CB = FE$? Why? Is $\angle C = \angle F$?

If you have answered the questions above, you are ready to accept the following assumption.

65. Assumption 32. **If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent. (S.A.S.)**

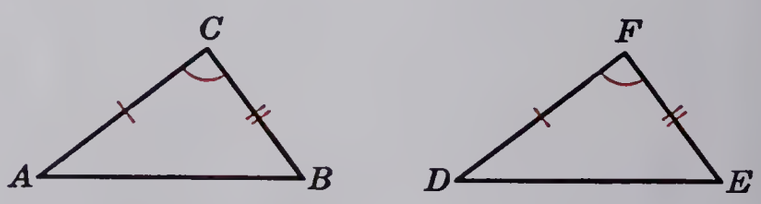


Note that we shall use S.A.S. to designate this assumption.

66. How to Use Assumption 32

By accepting this assumption we can now state that any two triangles are congruent when we have shown that two sides and the included angle of one triangle are equal respectively to two sides and the included angle of the other. We quote Asmt. 32 as our reason or authority.

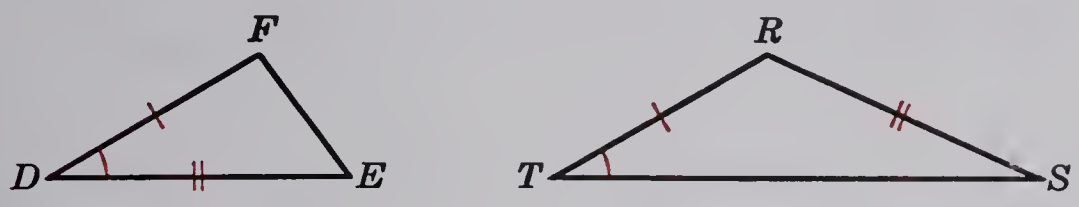
Example 1. *Given* $\triangle ABC$ and $\triangle DEF$ with $AC = DF$, $BC = EF$, and $\angle C = \angle F$. Can Asmt. 32 be used to prove that $\triangle ABC \cong \triangle DEF$?



Solution. Since it is given that $AC = DF$, $BC = EF$, $\angle C = \angle F$, $\angle C$ is included by AC and BC, and $\angle F$ is included by DF and EF, Asmt. 32 tells us that the triangles are congruent.

Note that we have indicated the parts that are given equal by like tick marks. Do you see why we use these marks?

Example 2. *Given* $\triangle EDF$ and $\triangle RST$, with $ED = RS$, $DF = TR$, and $\angle D = \angle T$. Is $\triangle DEF \cong \triangle TRS$ by Asmt. 32?



Solution. The two triangles have two sides and an angle of one equal to two sides and an angle of the other but $\angle T$ is not included by RT and RS. Then Asmt. 32 does not tell us that $\triangle DEF \cong \triangle TRS$.

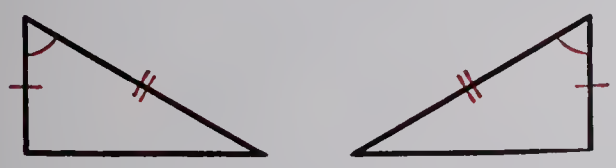
EXERCISES

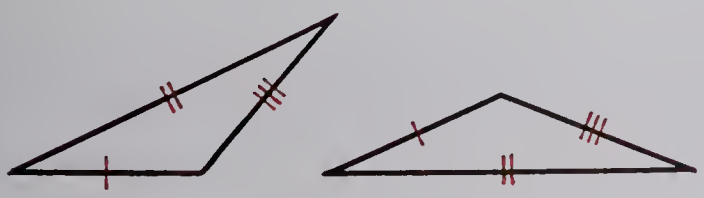
A

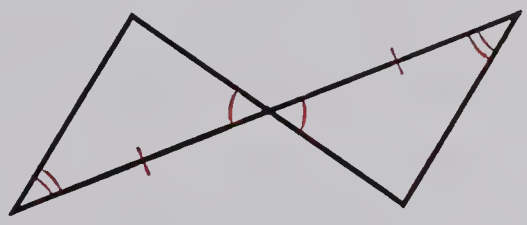
In each of the exercises 1–10 is a pair of triangles whose equal parts have like tick marks. After observing which parts are equal, tell which triangles are congruent by Assumption 32. Remember that you are to decide whether Assumption 32 tells us that the pairs of triangles are congruent.

1.

2.

3. 

4. 

5. 

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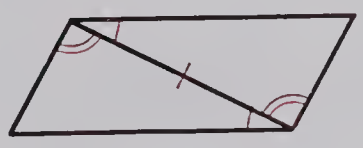
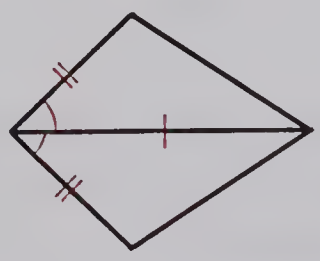
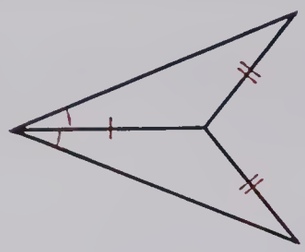
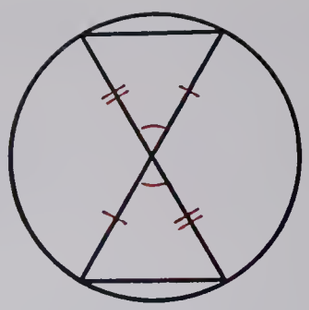
7. 

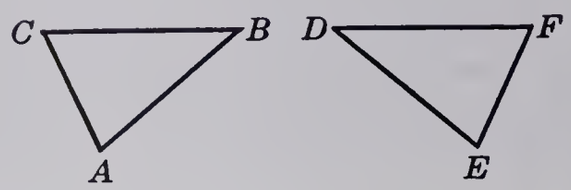
8. 

9. 

10. 

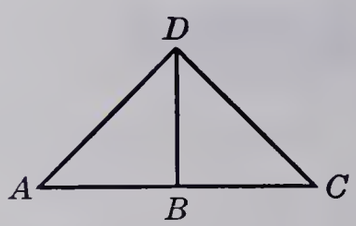
In Exs. 11–14 copy the triangles and mark the equal parts as was done in Examples 1 and 2.

11. *Given* $\triangle ABC$ and $\triangle DEF$, $AC = EF$, $CB = DF$, and $\angle C = \angle F$. Does Asmt. 32 tell us that $\triangle ABC \cong \triangle EDF$?



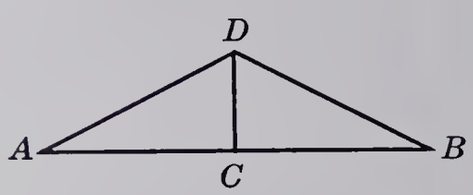
12. *Given* st. line ABC, $AB = BC$, and $BD \perp AC$. Is $\triangle ABD \cong \triangle BCD$ by Asmt. 32?

SUGGESTIONS. Does $\angle ABD = \angle DBC$? Does BD of $\triangle ABD$ equal BD of $\triangle DBC$?

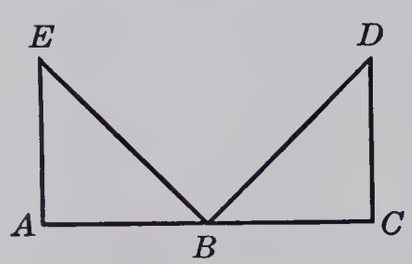


13. In this figure $DC \perp AB$ and $AD = BD$.

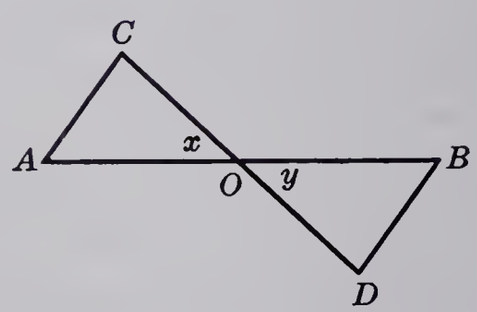
Is $\angle ACD = \angle BCD$? Is CD a common side of $\triangle ACD$ and $\triangle BCD$? Does Asmt. 32 tell us that $\triangle ACD \cong \triangle BCD$?



14. *Given* figure $ABCDE$, $AB = BC$, and $\angle E = \angle D$. By Asmt. 32, is $\triangle ABE \cong \triangle BCD$?



Ex. 14



Ex. 15

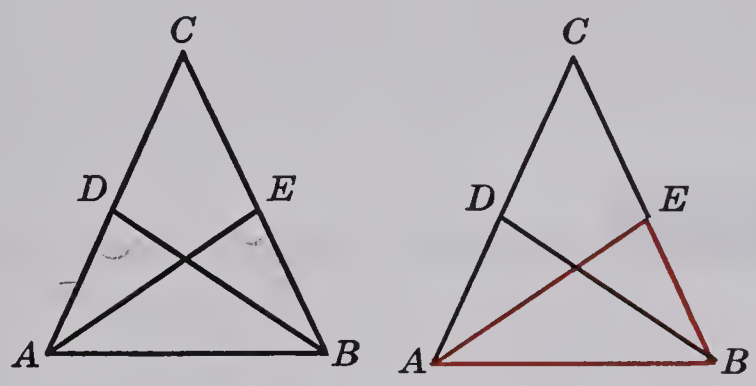
15. *Given* st. lines AB and CD, $AO = OB$, and $CO = OD$. What do you know about $\angle x$ and $\angle y$? By Asmt. 32 is $\triangle AOC \cong \triangle BOD$?

67. Overlapping Triangles

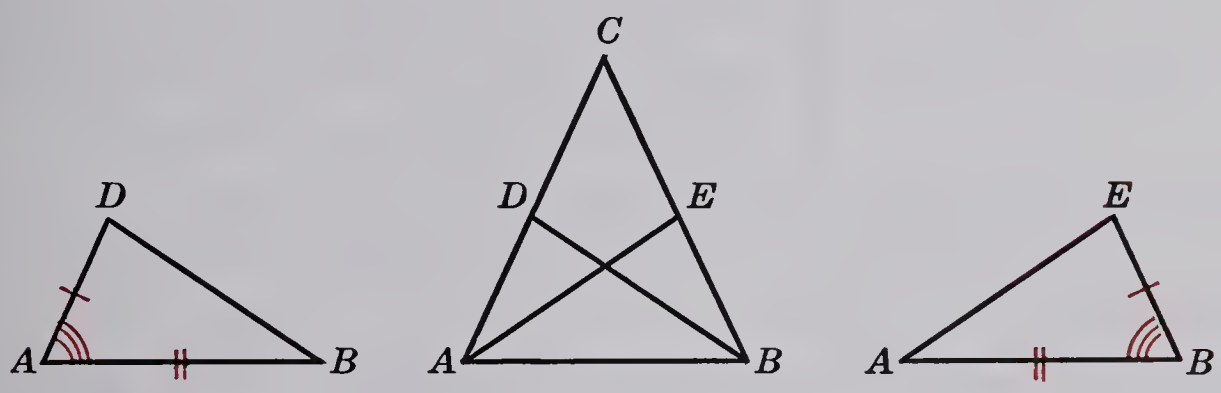
Overlapping triangles are often confusing to the eye and as a result the relations between the triangles are obscured. Let us consider an example.

Given $\triangle ABC$ with $AD = BE$ and $\angle DAB = \angle EBA$. Is $\triangle ABD \cong \triangle ABE$?

The triangles ABD and ABE overlap each other. They may be made to stand out by tracing them with colored crayon. For example, $\triangle ABD$ could be heavy black and $\triangle ABE$ could be red; or you could make AD and BE red, AE and BD blue, and AB yellow.



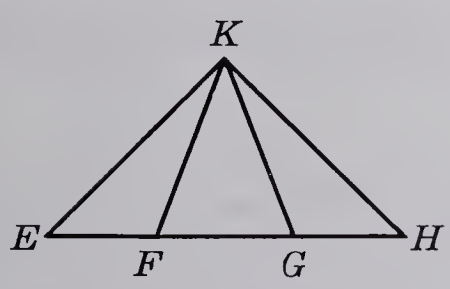
Another method of aiding the eye to visualize the triangles is to draw the triangles separately as shown below.



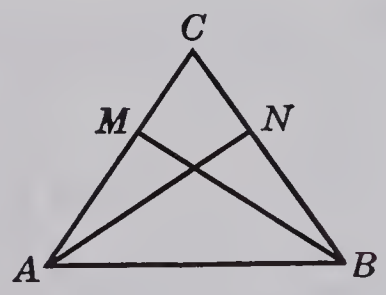
EXERCISES — Oral Exercises

A

1. *Given* the figure $EFGHK$, st. line EH, $EK = HK$, $\angle E = \angle H$, and $EF = GH$. Does $EG = FH$? Does Asmt. 32 tell us $\triangle EGK \cong \triangle FHK$?



Ex. 1



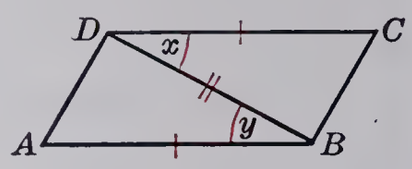
Ex. 2

2. *Given* $\triangle ABC$ with $AC = BC$ and $CM = CN$. Does Asmt. 32 tell us that $\triangle ACN \cong \triangle BCM$?

68. Using Assumption 32 in Proofs

The exercises on pages 85–87 were intended to give you practice in applying Asmt. 32. We shall now use this assumption in proofs.

Example 1.



Given figure $ABCD$ with $AB = CD$ and $\angle x = \angle y$.

To prove that $\triangle ABD \cong \triangle BCD$.

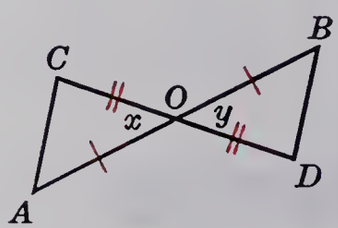
Planning the Proof: 1. We can prove triangles $\cong$ by § 65.
2. We shall use § 65.

Proof

STATEMENTS	REASONS
1. $AB = CD$.	1. Given.
2. $BD = BD$.	2. Iden.
3. $\angle x = \angle y$.	3. Given.
4. $\triangle ABD \cong \triangle BCD$.	4. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

Note that Part I of "Parts of a Formal Proof," page 68, is omitted.

Example 2.



Given the st. lines AB and CD with $CO = OD$ and $AO = OB$.

To prove that $\triangle AOC \cong \triangle BOD$.

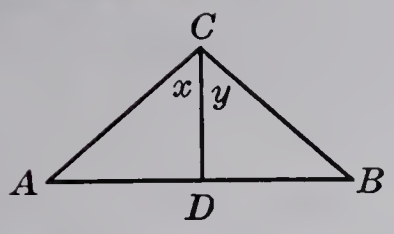
Planning the Proof: 1. We can prove $\triangle \cong$ by § 65.
2. We shall use § 65.

Proof

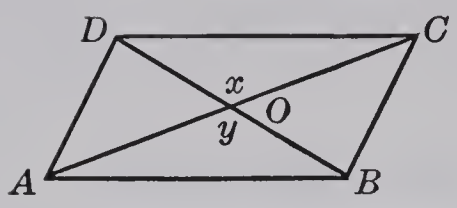
STATEMENTS	REASONS
1. $CO = OD$.	1. Given.
2. $AO = OB$.	2. Given.
3. $\angle x = \angle y$.	3. If two lines intersect, the vertical $\angle$s are $=$.
4. $\therefore \triangle AOC \cong \triangle BOD$.	4. If two $\triangle$ have two sides and the included $\angle$ of one $=$ respectively to two sides and the included $\angle$ of the other, the $\triangle$ are $\cong$.

EXERCISES — Triangles Congruent

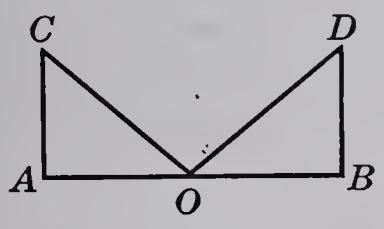
A



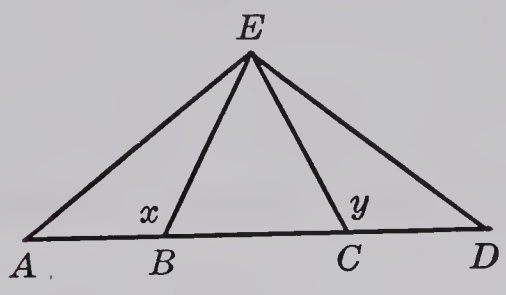
1. *Given* the isos. $\triangle ABC$ with $AC=BC$ and CD bisecting $\angle ACB$.
Prove $\triangle ADC \cong \triangle DBC$.



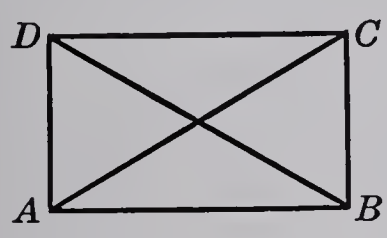
2. *Given* the st. lines AC and DB with $AO = OC$ and $DO = OB$.
Prove $\triangle AOB \cong \triangle DOC$.



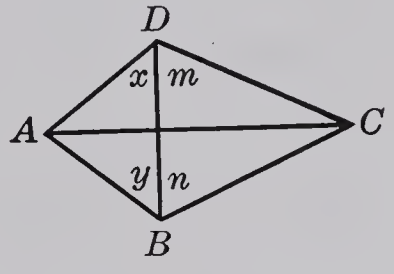
3. *Given* $CA \perp AB$, $DB \perp AB$, $CA = DB$ and $AO = OB$.
Prove $\triangle AOC \cong \triangle OBD$.



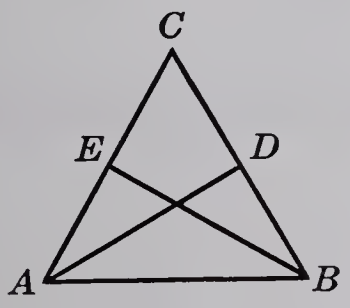
4. *Given* figure $ABCDE$ with st. line $ABCD$, $AB = CD$, $BE = CE$, and $\angle EBA = \angle ECD$.
Prove $\triangle ABE \cong \triangle CED$.



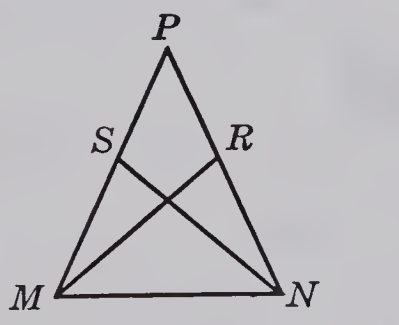
5. *Given* $AD \perp AB$, $CB \perp AB$, and $AD = CB$.
Prove $\triangle ABD \cong \triangle ABC$.



6. *Given* $AB = AD$, $BC = DC$, $\angle x = \angle y$, and $\angle m = \angle n$.
Prove $\triangle ADC \cong \triangle ABC$.



7. *Given* $AC = CB$, E the midpoint of AC, and D the midpoint of CB.
Prove $\triangle ADC \cong \triangle BEC$.



8. *Given* $\triangle MNP$ with $MR = NS$, $\angle MNP = \angle NMP$, and $\angle RMP = \angle SNP$.
Prove $\triangle MNR \cong \triangle NMS$.

69. Corollaries

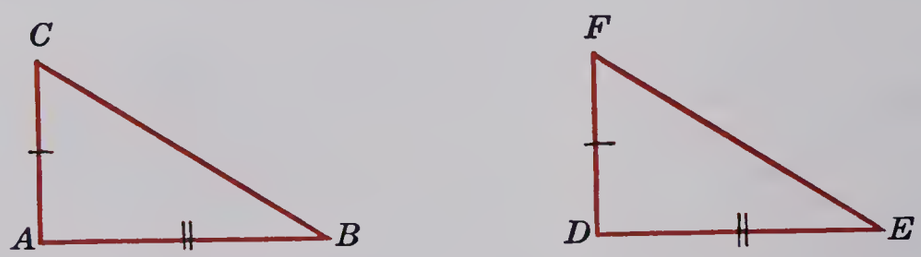
A *corollary* is a theorem that is easily derived from a recently proved theorem or corollary or from a recently stated assumption.

70. Proof of a Corollary

A corollary is proved by showing that the hypothesis of the corollary is a special case of the hypothesis of the theorem, assumption, or other corollary which precedes it. The steps needed to show that the theorem applies to the corollary are usually evident.

It is not necessary to give "Planning the Proof" when proving a corollary because we know that its proof is based upon an assumption or a recently proved theorem or corollary.

71. Corollary to Assumption 32. If two right triangles have the two legs of one equal respectively to the two legs of the other, the triangles are congruent.



Given the two rt. $\triangle ABC$ and DEF with $AC = DF$, $AB = DE$, and $\angle A$ and D the rt. $\angle$.

To prove that $\triangle ABC \cong \triangle DEF$.

Proof

STATEMENTS	REASONS
1. $AC = DF$.	1. Given.
2. $AB = DE$.	2. Given.
3. $\angle A$ and $\angle D$ are rt. $\angle$.	3. Given.
4. $\angle A = \angle D$.	4. All rt. $\angle$ are $=$.
5. $\therefore \triangle ABC \cong \triangle DEF$.	5. If two $\triangle$ have two sides and the included $\angle$ of one $=$ respectively to two sides and the included $\angle$ of the other, the $\triangle$ are $\cong$.

EXERCISES

A

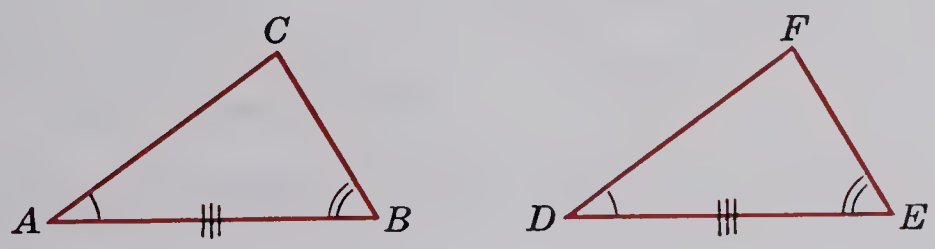
1. Prove exercise 3, page 89, by the use of § 71.

2. Prove exercise 5, page 89, by the use of § 71.

3. *Given* $\triangle ABC$ with H a point on AC, F a point on BC, $BH \perp AC$, $AF \perp BC$, $CF = CH$, and $AF = BH$.
Prove $\triangle AFC \cong \triangle BHC$.

4. *Given* $\triangle ABC$, having P the midpoint of AB and $CP \perp AB$.
Prove $\triangle APC \cong \triangle BPC$.

72. An Investigation

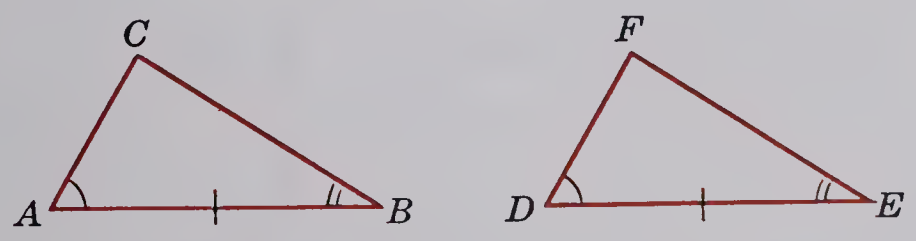


In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$, and $AB = DE$. Trace each triangle on thin paper and cut out your tracings. Place the tracing of $\triangle ABC$ on the tracing of $\triangle DEF$ so that AB falls along DE. If point A falls on point D, will B fall on E? Can $\angle A$ be made to coincide with $\angle D$? Why? Can $\angle B$ be made to coincide with $\angle E$? Will the point C fall on the point F?

Do you think the following assumption is true?

73. Assumption 33.

If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent. (A.S.A.)

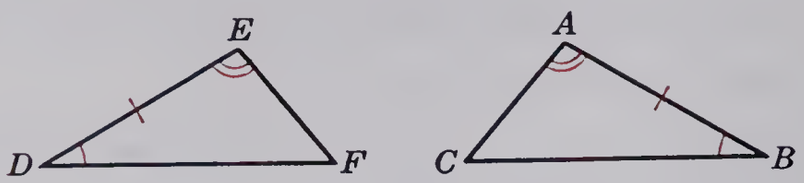


74. How to Use Assumption 33

After we have shown that two triangles have two angles and the included side of one triangle equal respectively to two angles and the

included side of the other, we can say that the triangles are congruent. We quote Assumption 33 for the reason.

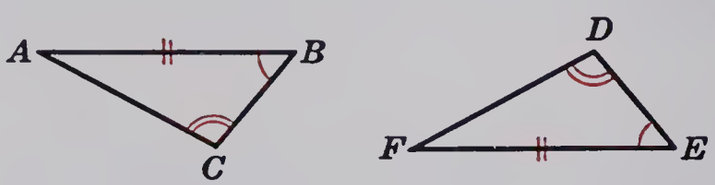
Example 1.



Given $\triangle DEF$, $\triangle ABC$, $DE = BA$, $\angle D = \angle B$, and $\angle E = \angle A$.

Two angles of $\triangle DEF$ are equal respectively to two angles of $\triangle ABC$ and side $DE =$ side BA. The side DE is included by $\angle D$ and $\angle E$, and the side BA is included by $\angle B$ and $\angle A$. Assumption 33 tells us the triangles are congruent.

Example 2.



Given $\triangle ABC$ and $\triangle DEF$ with $AB = FE$, $\angle B = \angle E$, and $\angle C = \angle D$.

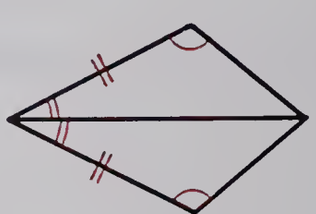
These two triangles have two angles and a side of one equal to two angles and a side of the other. Since we do not know that the included side CB of $\triangle ACB$ is equal to the included side ED of $\triangle FED$, we cannot state that the triangles are congruent by Asmt. 33.

EXERCISES — Oral Exercises

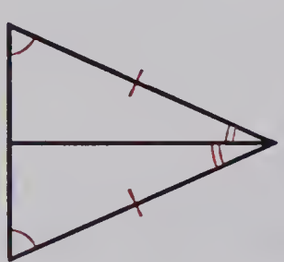
In each of the exercises 1–6 there are two triangles. In each exercise, like tick marks designate equal parts. Tell which triangles are congruent by Assumption 33.

1.

2.

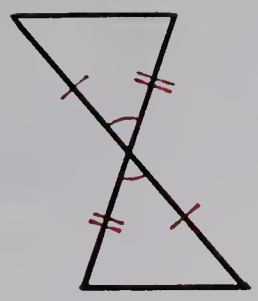


3.

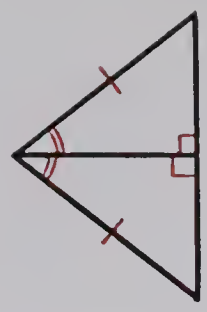


4.

5.



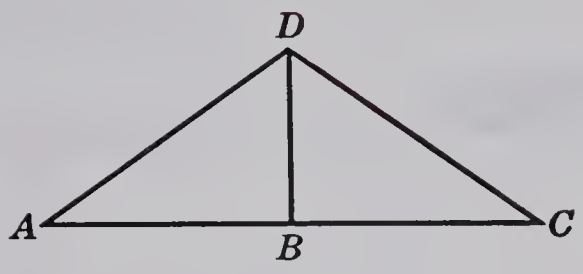
6.



7. *Given* st. line ABC, $BD \perp AC$, and $\angle ADB = \angle CDB$.

Do $\triangle ABD$ and BCD have a common side?

Does $\angle ABD = \angle CBD$? Does Assumption 33 tell us the triangles are congruent?



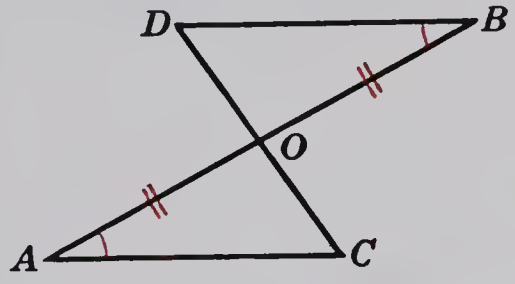
75. Using Assumption 33 in Proofs

Study Examples 1 and 2 to learn how Assumption 33 is used in proofs.

Example 1.

Given line CD bisecting line AB in O, and $\angle A = \angle B$.

To prove that $\triangle AOC \cong \triangle DOB$.

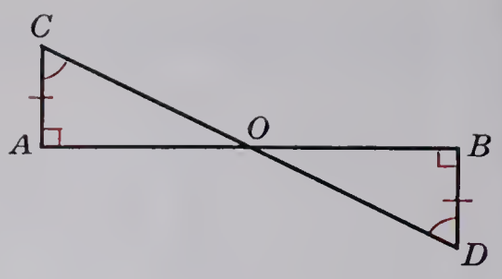


Planning the Proof: 1. We can prove triangles congruent by §§ 65, 71, 73.
2. We shall use § 73.

Proof

STATEMENTS	REASONS
1. AB is bisected by CD.	1. Given.
2. Then $AO = OB$.	2. A line segment is bisected when it is divided into two equal parts.
3. $\angle A = \angle B$.	3. Given.
4. $\angle AOC = \angle BOD$.	4. If two st. lines intersect, the vertical angles are equal.
5. $\therefore \triangle AOC \cong \triangle DOB$.	5. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

Example 2.



Given st. lines AB and CD intersecting in O, $AC = BD$, $\angle C = \angle D$, $AC \perp AB$, and $BD \perp AB$.

To prove that $\triangle AOC \cong \triangle BOD$.

Planning the Proof: 1. We can prove $\triangle \cong$ by §§ 65, 71, 73.
2. We shall use § 73.

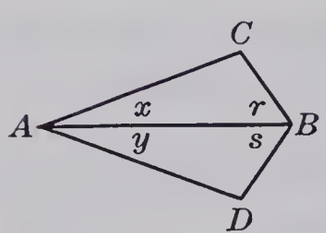
Proof

STATEMENTS	REASONS
1. $\angle C = \angle D$.	1. Given.
2. $AC = BD$.	2. Given.
3. $AC \perp AB$.	3. Given.
4. $BD \perp AB$.	4. Given.
5. From (3), $\angle A$ is a rt. $\angle$.	5. An angle formed by a line and its perpendicular is a rt. $\angle$.
6. From (4), $\angle B$ is a rt. $\angle$.	6. Same as Reason 5.
7. $\angle A = \angle B$.	7. All right angles are equal.
8. $\triangle AOC \cong \triangle BOD$.	8. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

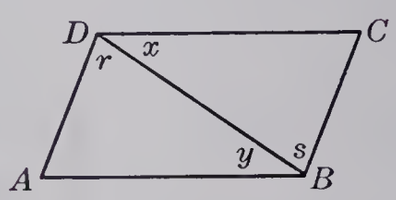
EXERCISES Triangles Congruent

A

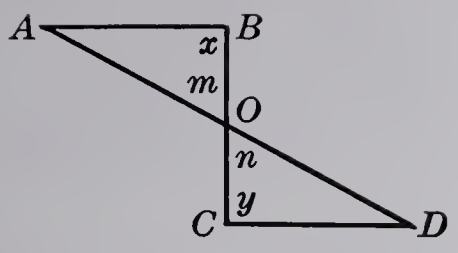
Use § 73 in proving the following exercises:



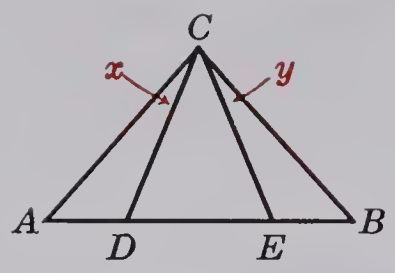
1. *Given* figure $ADBC$ with $\angle x = \angle y$ and $\angle r = \angle s$.
Prove $\triangle ABC \cong \triangle ADB$.



2. *Given* figure $ABCD$ with $\angle x = \angle y$ and $\angle r = \angle s$.
Prove $\triangle ABD \cong \triangle DBC$.

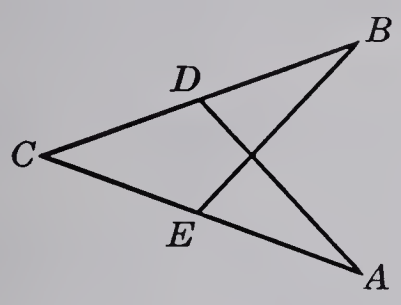


3. *Given* $AB \perp BC$, $CD \perp BC$, and O the midpoint of BC and AD.
Prove $\triangle AOB \cong \triangle CDO$.

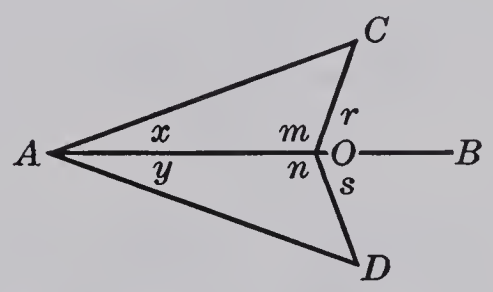


4. *Given* $\angle A = \angle B$, $AC = BC$, and $\angle x = \angle y$.
Prove $\triangle ADC \cong \triangle EBC$.

C



5. *Given* $\angle A = \angle B$, $AC = BC$.
Prove $\triangle CEB \cong \triangle CAD$.



6. *Given* AOB a st. line, $\angle x = \angle y$, and $\angle r = \angle s$.
Prove $\triangle AOC \cong \triangle AOD$.

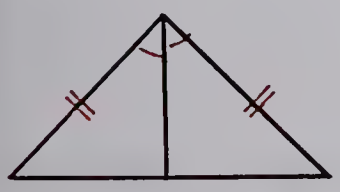
76. Using Assumptions 32 and 33

Thus far we have had two ways of proving any two triangles congruent,—Asmt. 32 and Asmt. 33. If the triangles are right triangles, we have, in addition, § 71. We never use superposition unless it is necessary.

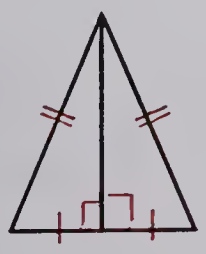
EXERCISES — Oral Exercises

In each of the following exercises there are two triangles having their equal parts, except a common side, indicated by like tick marks. Tell which triangles are congruent by Asmt. 32; by Asmt. 33.

1.



2.



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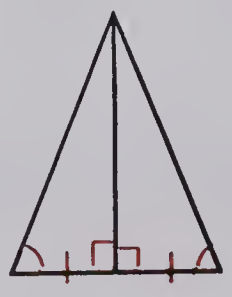

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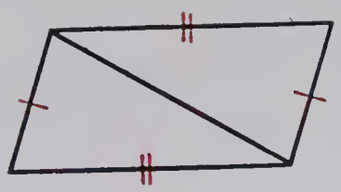
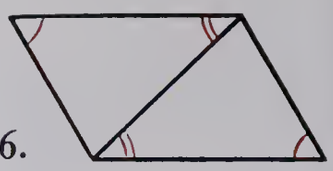
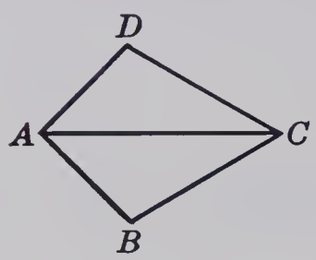
5. 

6. 

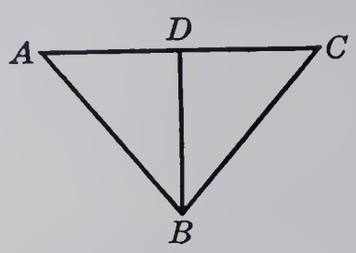
EXERCISES — Additional Exercises on Congruent Triangles

A

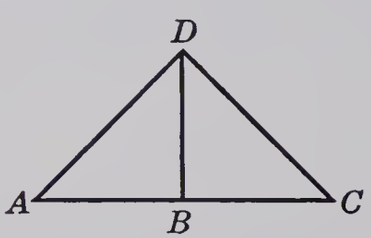
Use either § 65 or § 73.



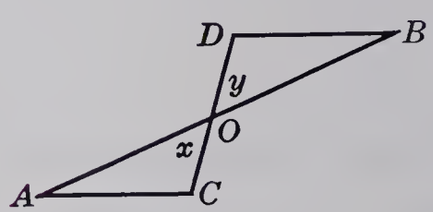
1. *Given* figure $ABCD$ with AC bisecting $\angle BAD$ and $AB = AD$.
Prove $\triangle ABC \cong \triangle ADC$.



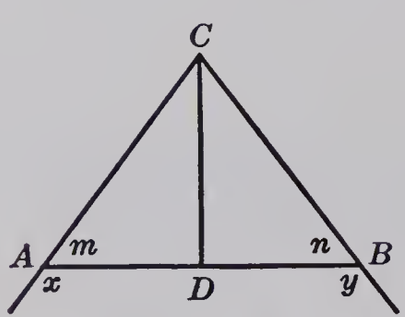
2. *Given* $\triangle ABC$ with $BD \perp AC$ and $AD = DC$.
Prove $\triangle ADB \cong \triangle CDB$.



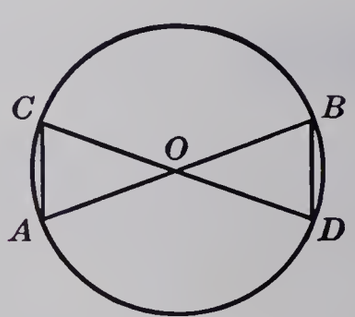
3. *Given* $\triangle ACD$ with $DB \perp AC$, $\angle A = \angle C$, and $AB = BC$.
Prove $\triangle ABD \cong \triangle CBD$.



4. *Given* st. lines AB and CD with $\angle A = \angle B$ and $AO = OB$.
Prove $\triangle AOC \cong \triangle BOD$.

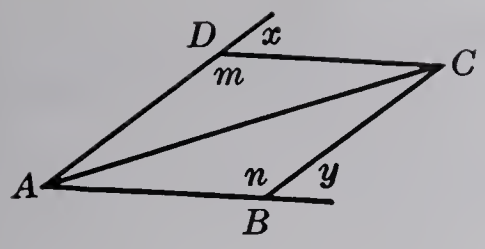


5. *Given* figure ABC with $AC = BC$, $AD = DB$, and $\angle x = \angle y$.
Prove $\triangle ADC \cong \triangle BDC$.

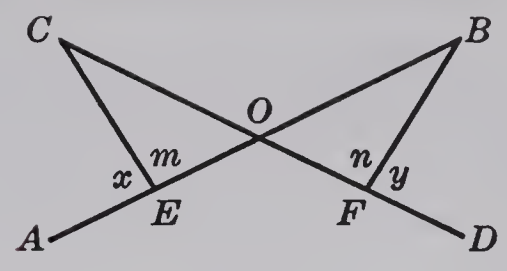


6. *Given* diameters AB and CD of circle O, with line segments AC and BD.
Prove $\triangle AOC \cong \triangle BOD$.

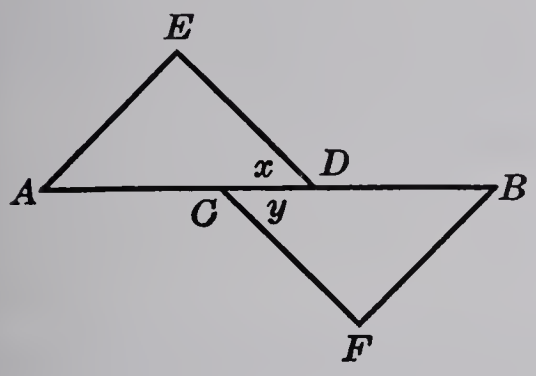
B



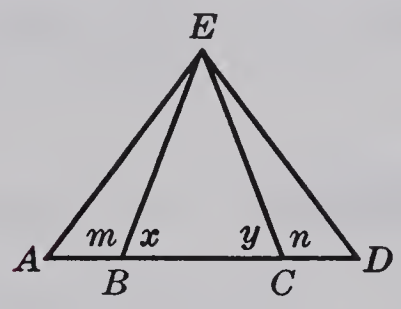
7. *Given* figure $ABCD$, $AB = BC = CD = DA$, and $\angle x = \angle y$.
Prove $\triangle ABC \cong \triangle ADC$.



8. *Given* st. lines AB and CD intersecting at O, $EO = OF$, and $\angle x = \angle y$.
Prove $\triangle CEO \cong \triangle BFO$.

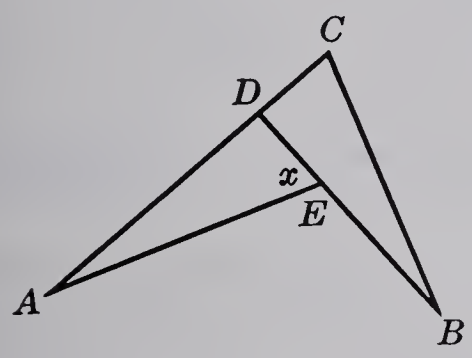


9. *Given* $ACDB$ a st. line, $AC = DB$, $\angle A = \angle B$, and $\angle x = \angle y$.
Prove $\triangle ADE \cong \triangle BCF$.

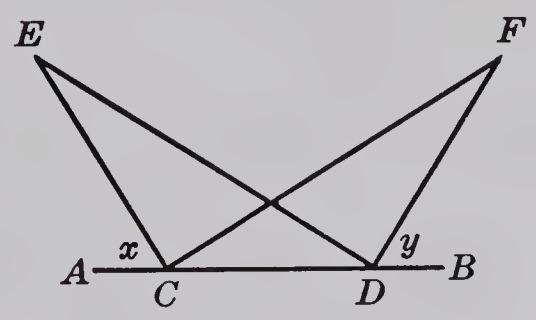


10. *Given* $\triangle ADE$, $AC = BD$, $BE = EC$, and $\angle x = \angle y$.
Prove $\triangle ABE \cong \triangle DCE$.

C

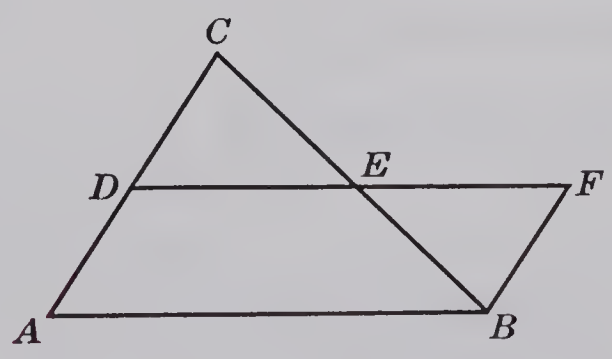


11. *Given* DEB a st. line, $BD \perp AC$, $CD = DE$, and $\angle x = \angle C$.
Prove $\triangle ADE \cong \triangle BDC$.



12. *Given* $ACDB$ a st. line, $CE = DF$, and $\angle x = \angle y$.
Prove $\triangle CDE \cong \triangle DCF$.

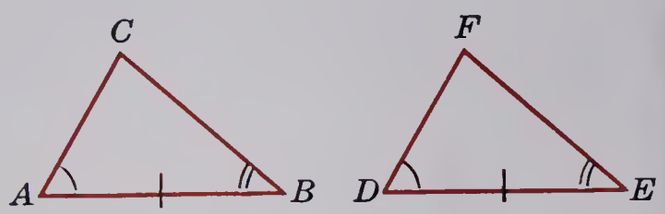
13. *Given* $\triangle ABC$, DF bisecting AC and BC, $AD = BF$, and $\angle C = \angle EBF$.
Prove $\triangle CDE \cong \triangle EBF$.



77. Corresponding Parts of Congruent Polygons

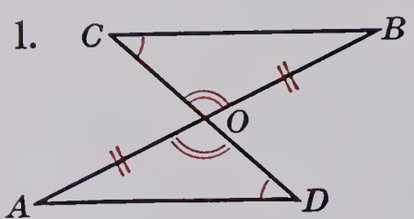
Corresponding (corr.) *parts* of two congruent triangles are the parts which coincide if the two polygons are made to coincide. In two congruent triangles the parts that are known to be equal are always corresponding parts, and the parts lying opposite these parts are also corresponding parts.

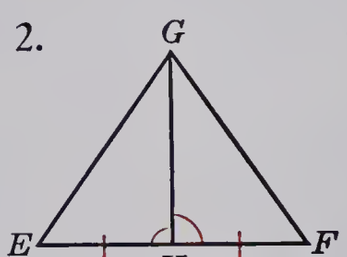
Thus if $AB = DE$, $\angle A = \angle D$, and $\angle B = \angle E$, the two triangles are congruent. Then $\angle C$, lying opposite AB, corresponds to $\angle F$, lying opposite DE; also AC, lying opposite $\angle B$, corresponds to DF, lying opposite $\angle E$, and, similarly, BC corresponds to EF. Does $\angle A$ correspond to $\angle D$?



EXERCISES Oral Exercises

In each of the following exercises there is a pair of congruent triangles. Tick marks are used to designate some of the equal corresponding parts. Name the remaining corresponding parts.

1.


2.


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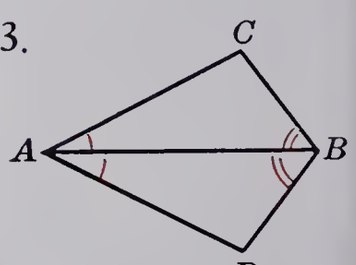

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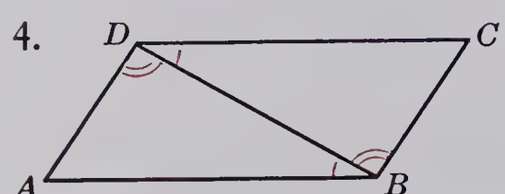

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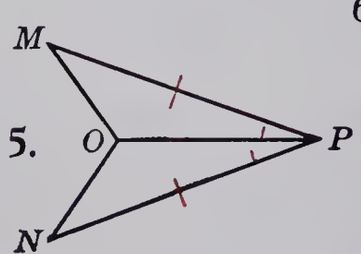

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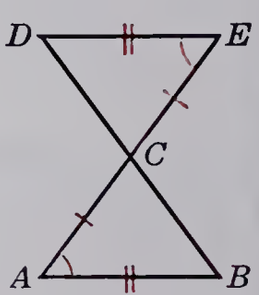


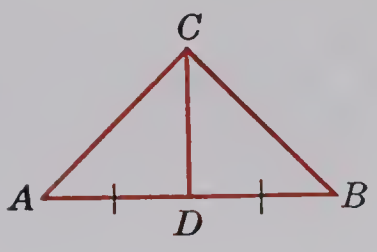
78. Assumption 34. Corresponding parts of congruent polygons are equal; polygons are congruent if their corresponding parts are equal.

This assumption is often used in proving two line segments equal or two angles equal.

Example.

Given D, the midpoint of AB, and $CD \perp AB$.

To prove that $\angle A = \angle B$.



Planning the Proof: 1. We can prove $\triangle =$ by §§ 50, 78.

2. We shall use § 78.

Proof

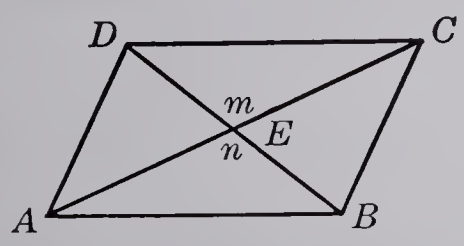
STATEMENTS	REASONS
1. $CD \perp AB$.	1. Why?
2. $\angle ADC = \angle BDC$.	2. § 14. If two lines are $\perp$ to each other, either forms two equal adjacent angles with the other.
3. D is the midpoint of AB.	3. Why?
4. $AD = DB$.	4. Why?
5. $CD = CD$.	5. Asmt. 5 (Identity).
6. $\triangle ADC \cong \triangle CDB$.	6. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.
7. $\therefore \angle A = \angle B$.	7. Corr. parts of $\cong \triangle$ are $=$.

EXERCISES — Line Segments, Angles

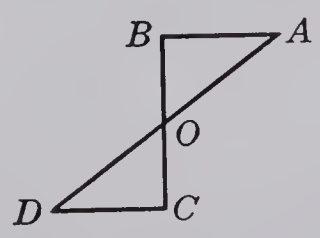
A

1. State the two ways we have had of proving triangles congruent. State three ways of proving right triangles congruent.

2. Name all the ways we have had of proving angles equal; of proving line segments equal.

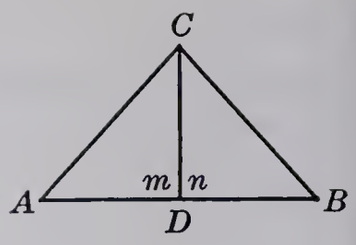


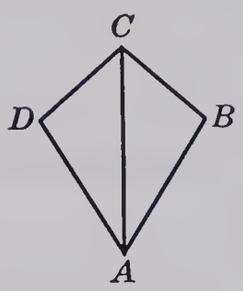
3. *Given* the st. lines AC and DB with $DE = EB$ and $AE = EC$.
Prove $DC = AB$.



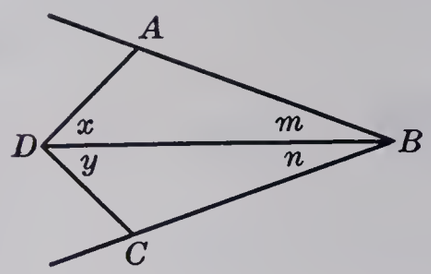
4. *Given* the st. lines BC and DA with $DO = OA$ and $BO = OC$.
Prove $\angle D = \angle A$.

5. *Given* $\triangle ABC$ with CD bisecting AB at D and forming $= \measuredangle$ m and n.
Prove $AC = BC$.

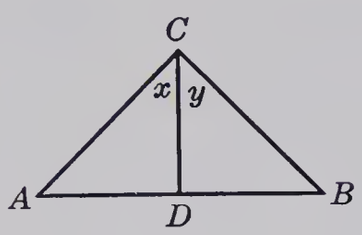




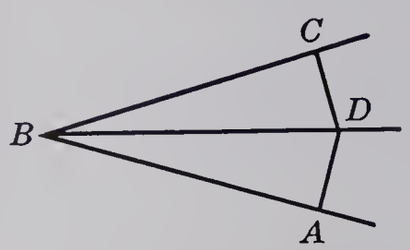
6. *Given* AC bisecting $\measuredangle BCD$ and BAD.
Prove $\angle B = \angle D$.



7. *Given* figure $ADCB$ with $AD = DC$ and $\angle x = \angle y$.
Prove $\angle m = \angle n$.

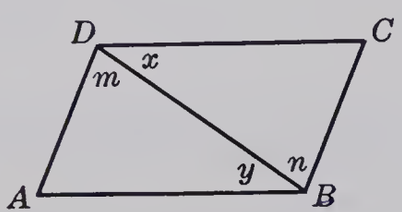


8. *Given* $\triangle ABC$ with CD bisecting $\angle ACB$ and $AC = BC$.
Prove $CD \perp AB$.

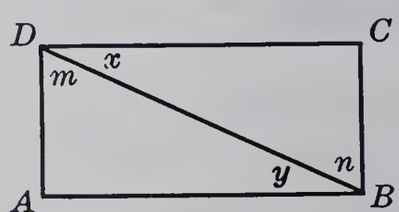


9. *Given* BD bisecting $\angle ABC$, $CD \perp BC$, $DA \perp BA$, and $BA = BC$.
Prove $AD = CD$.

SUGGESTION. Why is $\angle ADC = \angle BDC$?

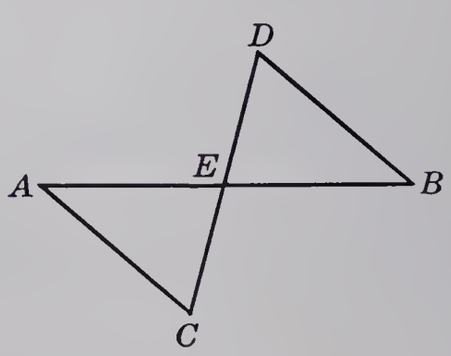


10. *Given* figure $ABCD$ with $\angle x = \angle y$ and $\angle m = \angle n$.
Prove $\angle A = \angle C$.

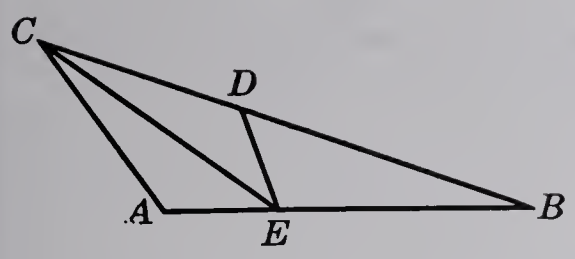


11. *Given* figure $ABCD$ with $AD \perp DC$, $CB \perp AB$, and $\angle x = \angle y$.
Prove $AD = BC$.

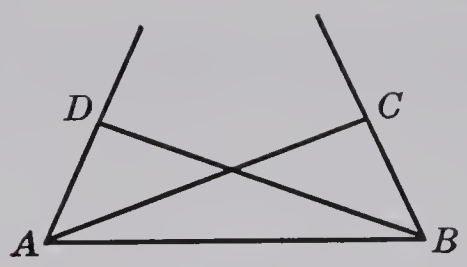
12. *Given* st lines AB and CD bisecting each other in E.
Prove $AC = BD$.



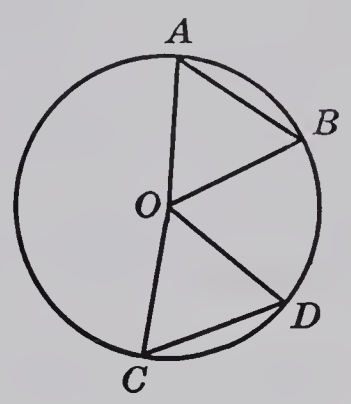
B



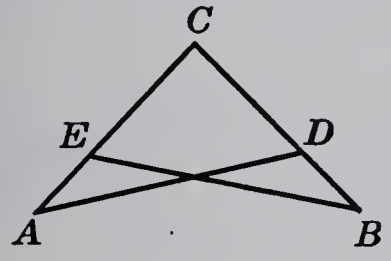
13. *Given* $\triangle ABC$ with $CA = CD$, and CE bisecting $\angle ACB$.
Prove $AE = DE$.



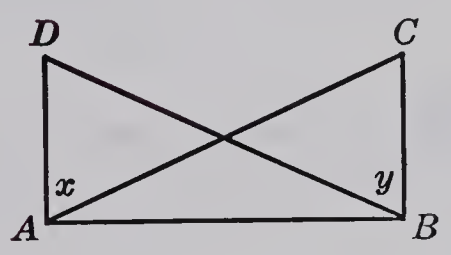
14. *Given* $\angle DAB = \angle ABC$ and $AD = BC$.
Prove $AC = BD$.



15. *Given* circle O, radii OA, OB, OC, OD, and $\angle AOB = \angle COD$.
Prove $AB = CD$.

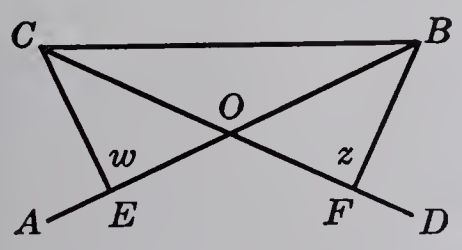


16. *Given* $AE = BD$, and $AC = BC$.
Prove $\angle A = \angle B$.

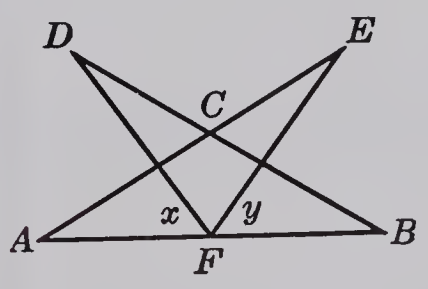


17. *Given* st. lines AC and BD, $AD \perp AB$, $BC \perp AB$, and $\angle x = \angle y$.
Prove $\angle D = \angle C$.

C



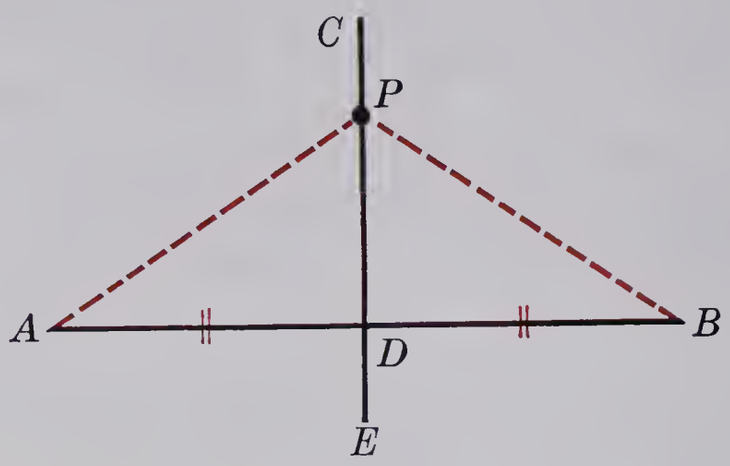
18. *Given* $CE \perp AB$, $BF \perp CD$, $BE = CF$, and $CE = BF$.
Prove $\angle OCB = \angle OBC$.



19. *Given* st. lines ACE and BCD, F the midpoint of AB, $\angle A = \angle B$, and $\angle x = \angle y$.
Prove $AE = BD$.

★ Theorem 2

79. Any point in the perpendicular bisector of a line segment is equidistant from the end points of the segment.



Given P any point in CE, the $\perp$ bisector of line segment AB.
To prove that P is equidistant from A and B.

Planning the Proof: 1. We can prove line segments = by Asmt. 1–8, 26, and § 78.
2. We shall use § 78.

Proof

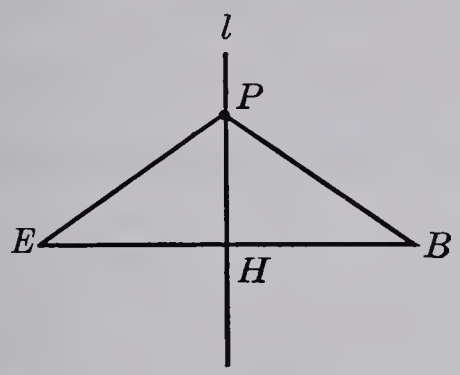
STATEMENTS	REASONS
1. Draw PA and PB.	1. One straight line, and only one, can be drawn through two points.
2. $CE \perp AB$.	2. Given.
3. $\angle ADP = \angle BDP$.	3. If two lines are perpendicular to each other, either forms two equal adjacent angles with the other.
4. PD bisects AB.	4. Given.
5. $AD = DB$.	5. If a line segment is bisected, it is divided into two equal parts.
6. $DP = DP$.	6. Asmt. 5 (Iden.)
7. $\triangle ADP \cong \triangle BDP$.	7. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.
8. $AP = PB$.	8. Corresponding parts of congruent polygons are equal.
9. $\therefore P$ is equidistant from A and B.	9. The distance between two points is the length of the line segment joining them.

80. Meaning of the Word "Any"

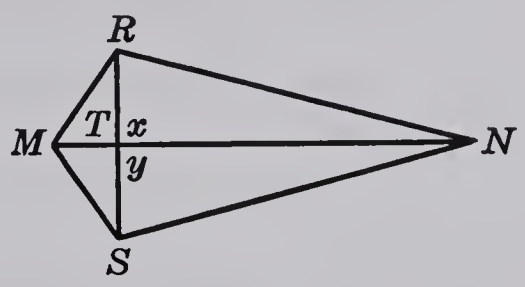
Were you disturbed by the word "any" in the proof above? In the proof, P represents *any* point (i. e. all points) in CE. When we have proved that P is equidistant from A and B, we have proved that *every* point in CE is equidistant from A and B. The word "any" has a like meaning in other expressions. For example, "any triangle" means a triangle which represents *every* triangle.

EXERCISES — Oral Exercises

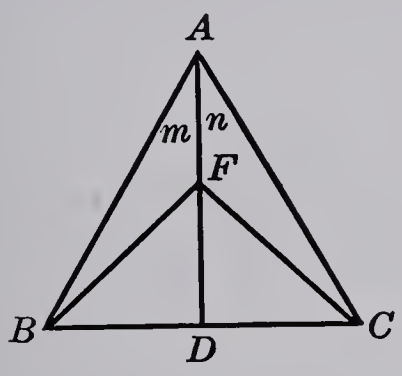
Without drawing any other lines, give as many conclusions as you can in the following exercises. Give a reason for each conclusion.



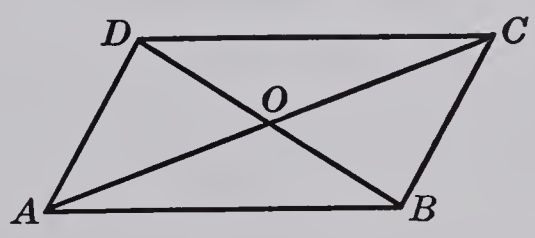
1. *Given* line $l \perp EB$ and $EH = HB$.



2. *Given* $RT = TS$, $\angle x = \angle y$, $MS = 3$, and $RN = 10$.



3. *Given* $\triangle ABC$, $\angle m = \angle n$, and $BA = CA$.



4. *Given* polygon $ABCD$ with AC and BD bisecting each other at O.

81. Directions for Proving Theorems

The demonstration of Theorem 2 was given in full. No specific rules for proving theorems can be given. However, the following general direc-

tions will be useful to you and should be referred to as a model in the development of proofs. The directions will be applied to the following theorem:

If two sides of a triangle are equal, the angles opposite these sides are equal.

1. Read the theorem carefully, being certain that you know the meaning of the words used.

If you do not understand every word, refer to the index at the back of the book, where you will be directed to a definition or an explanation of the word.

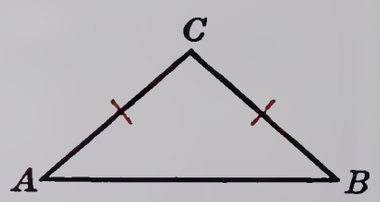
2. Determine the hypothesis and the conclusion.

The hypothesis of the theorem is If two sides of a triangle are equal. The conclusion is the angles opposite these sides are equal. (See § 32.)

3. Draw a figure which illustrates each point, angle, and line described in the hypothesis.

Letter the figure, marking the parts which are given equal.

$\triangle ABC$ is the figure for the theorem.



4. From the hypothesis write what is *given* in terms of the figure.

Be sure that you make use of each relation given in the hypothesis, as it will be needed in the proof.

Then write "*Given* $\triangle ABC$ with $AC = BC$."

5. From the conclusion write what there is *to prove* in terms of the figure.

Then write "*To prove* that $\angle A = \angle B$."

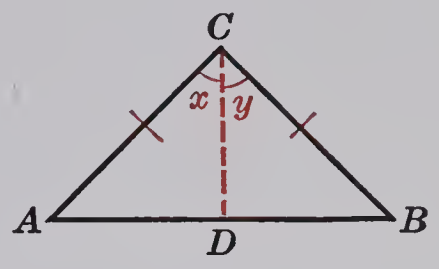
6. Plan the proof of the theorem.

In planning the proof you should first search the assumptions, definitions, and the theorems and corollaries already proved to find all known ways which might be used to prove the theorem. Then select the one most suitable to your proof.

In this theorem you wish to prove that two angles are equal. So you recall all the ways you have had of proving angles equal. These are: Assumptions 1–8, 19, 28–31, and 34, and Theorem 1. You then study these assumptions to determine if one of them can be used to prove

$\angle A = \angle B$. You do not find any assumptions that can be directly applied to the two angles. Theorem 1 does not apply because $\angle A$ and $\angle B$ are not vertical angles. Assumption 34 states that "corresponding parts of congruent triangles are equal."

Possibly you may be able to prove that $\angle A = \angle B$ if you can draw a line dividing $\triangle ABC$ into two congruent triangles. If you draw a line from C to some point D in AB, you will have two triangles with $AC = BC$ by hypothesis and $CD = CD$ by identity. You now recall the three ways of proving triangles congruent (§§ 65, 71, and 73). From a study of these three sections you see that if you draw CD to bisect $\angle C$, you can prove the triangles congruent by § 65. You then draw CD bisecting $\angle C$.



7. Write the proof, giving a reason for each statement used.

The demonstration of this theorem is given in § 83.

82. Auxiliary Lines

The line CD in the figure above is an auxiliary line. Lines that are introduced in a figure to assist in establishing a proof are called *auxiliary lines* (helping lines). Auxiliary lines should be dotted lines to distinguish them from the lines which are given in the hypothesis.

When drawing an auxiliary line be sure that you can give a reason why it can be drawn.

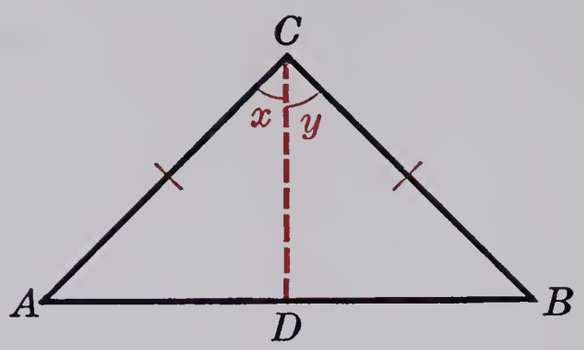
So far we have had three authorities for drawing auxiliary lines. They are:

1. One straight line, and only one, can be drawn through two points. (Asmt. 11)
2. An angle can be bisected by one line, and only one. (Asmt. 18)
3. At any point on a straight line or from any point outside the line one perpendicular, and only one, can be drawn to the line. (Asmt. 20)

These assumptions do not state how the lines can be drawn but state that it is possible to draw them.

★★ Theorem 3

83. If two sides of a triangle are equal, the angles opposite these sides are equal.



Given the $\triangle ABC$ with $AC = BC$.
To prove that $\angle A = \angle B$.

Planning the Proof: 1. We can prove $\measuredangle s =$ by §§ 50, 78.
2. We shall use § 78.

Proof

STATEMENTS	REASONS
1. Draw CD bisecting $\angle C$ and meeting AB at D.	1. Asmt. 18 (An $\angle$ can be bisected by one line, and only one).
In $\triangle ADC$ and BDC,	
2. $AC = BC$.	2. Given.
3. $CD = CD$.	3. Asmt. 5 (Iden.).
4. $\angle x = \angle y$.	4. Const. (Statement 1).
5. $\therefore \triangle ADC \cong \triangle BDC$.	5. § 65 (S.A.S.*).
6. $\therefore \angle A = \angle B$.	6. Corr. $\measuredangle s$ of $\cong \triangle$ are $=$.

A test of your understanding of a theorem is to close the book and write out the complete demonstration.

84. Corollary. An equilateral triangle is equiangular.

The theorem and corollary above give us two new ways of proving angles equal.

On page 564, § 568, there is a list of bases for proving angles equal. Find on it the methods which you have already learned. Hereafter, as you learn a new way of proving angles equal, locate it on this list. When planning a proof you will often find it helpful to refer to this list.

*See Abbreviations, p. viii.

EXERCISES

A

1. State two assumptions for proving triangles congruent.

2. State the ways of proving line segments equal.

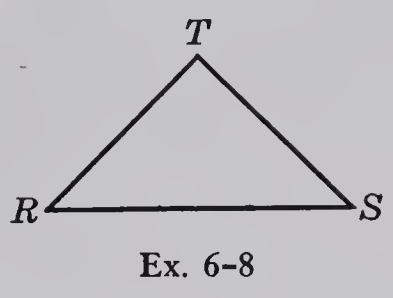
3. State the ways of proving angles equal.

4. State the definition for proving lines perpendicular (§ 14).

5. State the definition for proving a triangle isosceles (§ 56).

6. If $RT = ST$ and $\angle R = 36°$, find the number of degrees in $\angle S$.

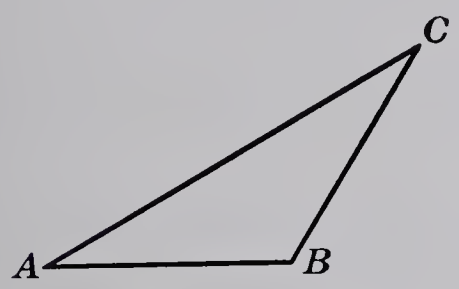
7. If $RT = ST$ and $\angle R + \angle S = 78°$, find the number of degrees in $\angle R$ and in $\angle S$.

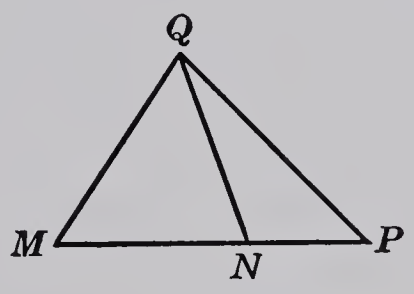


Ex. 6-8

8. If $RT = ST$, $\angle R = (3x + 6)°$, and $\angle S = (x + 46)°$, find the number of degrees in $\angle R$ and in $\angle S$.

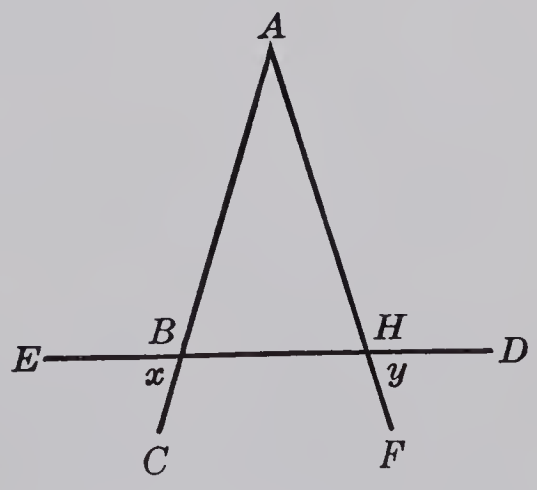
9. In $\triangle ABC$ below, which side is opposite $\angle A$? $\angle B$? $\angle C$? Which angle is opposite side AB? side BC? side AC?

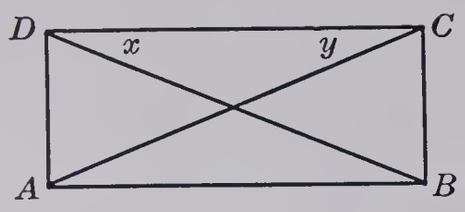




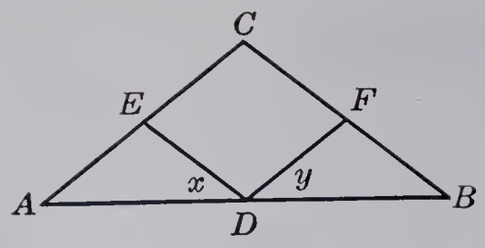
10. In $\triangle MNQ$ which side is opposite $\angle M$? $\angle MQN$? $\angle MNQ$?

11. *Given* st. lines AC, AF, and ED, making $AB = AH$.
Prove $\angle x = \angle y$.

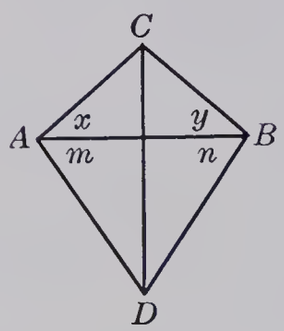


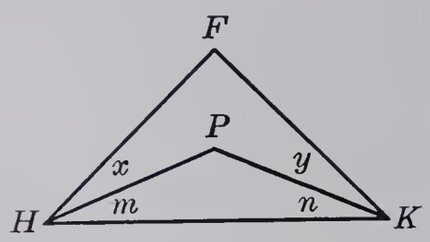


12. *Given* polygon $ABCD$, $\angle x = \angle y$, $AD \perp DC$, and $BC \perp DC$.
Prove $AD = BC$.



13. *Given* $\triangle ABC$ having $AC = BC$, D the midpoint of base AB, and $\angle x = \angle y$.
Prove $DE = DF$.





14. *Given* $AC = BC$; $AD = BD$.
a. *Prove* $\angle x = \angle y$.
b. *Prove* $\angle m = \angle n$.
c. *Prove* $\angle CAD = \angle CBD$.
d. *Prove* $\triangle ACD \cong \triangle BCD$.

15. *Given* the isosceles $\triangle HKF$ and HKP with $HF = KF$ and $HP = KP$.
Prove $\angle x = \angle y$.

16. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base.

17. Prove that the bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

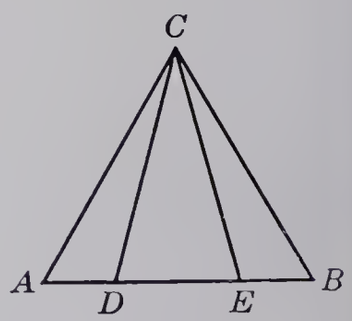
SUGGESTIONS. Prove that it makes equal angles with the base. Read the definition of a perpendicular to a line.

*18. Two isosceles triangles are congruent if the vertex angle and one of the equal sides of one are equal respectively to the vertex angle and one of the equal sides of the other.

19. The perimeter of an equilateral triangle is 168 feet. Find the length of each side.

20. *Given* $AC = BC$ and $AD = EB$.
Prove $CD = CE$.

21. *Given* $CD = CE$ and $AD = EB$.
Prove as many facts as you can.



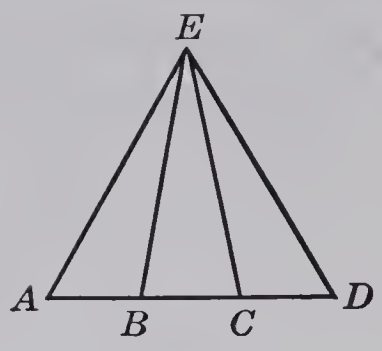
Ex. 20, 21

*Since this is a theorem, you are to prove it without being told to do so.

B

22. If a line is perpendicular to the bisector of an angle, it forms an isosceles triangle with the sides of the angle.

23. If the base of an isosceles triangle is trisected (divided into three equal parts), line segments drawn from the vertex to the points of division are equal.



Ex. 23

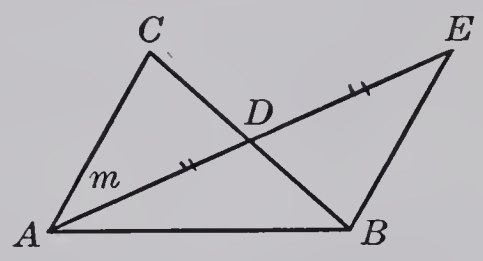
24. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

25. *Given* $\triangle ABC$ and ABE, D the midpoint of BC, and $AD = DE$.

a. *Prove* $\triangle ADC \cong \triangle DBE$.

b. *Prove* $\angle E = \angle m$.

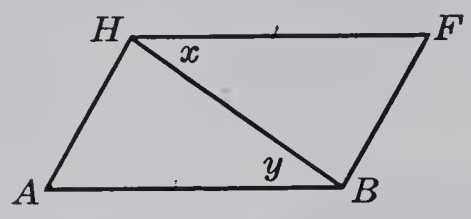
c. *Prove* $\angle BAC > \angle E$. (See Asmts. 9, 6.)



26. *Given* figure $ABFH$ with $AB = HF$ and $\angle x = \angle y$.

a. *Prove* $AH = BF$.

b. *Prove* $\angle AHF = \angle FBA$.

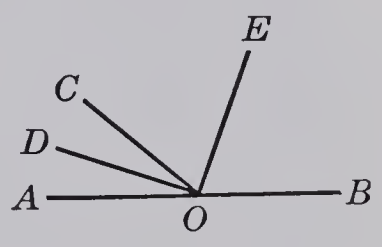


27. If two right triangles have a leg and the adjacent acute angle of one equal respectively to a leg and the adjacent acute angle of the other, the triangles are congruent.

C

28. The bisectors of two supplementary adjacent angles are perpendicular to each other.

29. The line segments joining the midpoints of the sides of an equilateral triangle form another equilateral triangle.

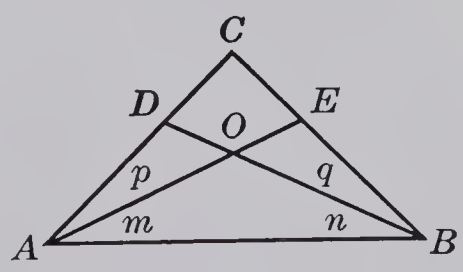


30. *Given* $\triangle ABC$ with $AC = BC$, lines AE and BD, and $\angle m = \angle n$.

Prove $AE = BD$.

31. *Given* $\triangle ABC$, lines AE and BD, with $AO = BO$ and $\angle p = \angle q$.

Prove $AD = BE$.



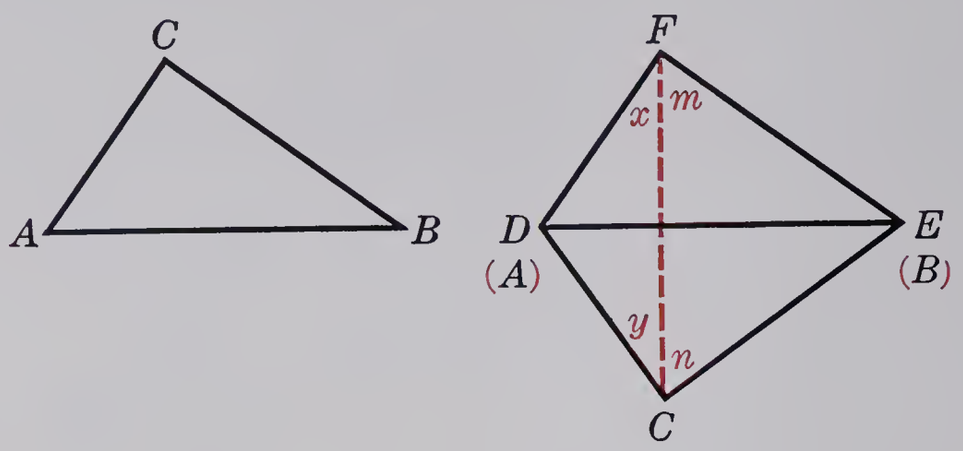
Ex. 30–32

32. *Given* $\triangle ABC$ with $AC = BC$, lines AE and BD, and $CD = CE$.

Prove $AE = BD$.

★★ Theorem 4

85. If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent. (S.S.S.)



Given the △ ABC and DEF with $AB = DE$, $AC = DF$, and $CB = FE$.
To prove that $\triangle ABC \cong \triangle DEF$.

Planning the Proof: 1. We can prove △ ≅ by §§ 65, 71, 73.
2. We shall use § 65.

Proof

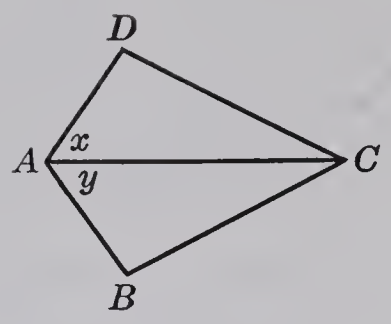
STATEMENTS	REASONS
1. $AB = DE$.	1. Given.
2. Place $\triangle ABC$ next to $\triangle DEF$ so that AB coincides with DE, A falls on D, B on E, and C opposite F.	2. Asmt. 16, § 6.
3. Draw FC.	3. Asmt. 11.
4. In $\triangle DFC$, $DF = AC$, and	4. Given.
5. $\angle x = \angle y$.	5. § 83.
6. In $\triangle FEC$, $FE = BC$, and	6. Given.
7. $\angle m = \angle n$.	7. § 83.
8. $\therefore \angle x + \angle m = \angle y + \angle n$.	8. Asmt. 1.
9. $\angle x + \angle m = \angle DFE$.	9. Asmt. 9.
10. $\angle y + \angle n = \angle ACB$.	10. Asmt. 9.
11. $\therefore \angle DFE = \angle ACB$.	11. Asmt. 6.
12. $\therefore \triangle DEF \cong \triangle ABC$.	12. § 65.

This theorem gives us a new base for proving triangles congruent. On page 563, § 566, note the ways you have learned.

NOTE. In demonstrating a theorem you should always give the reasons in full.

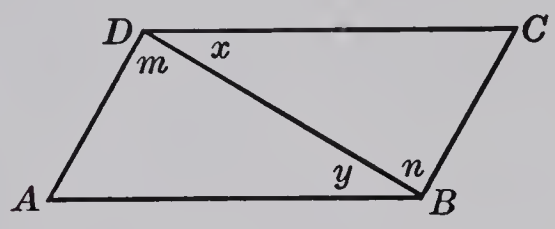
EXERCISES*

A

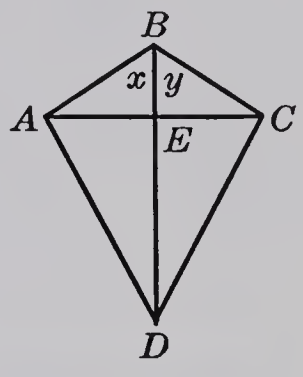


1. *Given* $AD = AB$ and $DC = BC$.
   a. *Prove* $\angle x = \angle y$.
   b. *Prove* that AC bisects $\angle BCD$.

2. The line joining the vertex of an isosceles triangle to the midpoint of the base divides the triangle into two congruent triangles.



3. *Given* $AD = BC$ and $AB = DC$.
   a. *Prove* $\triangle ABD \cong \triangle BCD$.
   b. *Prove* $\angle A = \angle C$.
   c. *Prove* $\angle x = \angle y$.
   d. *Prove* $\angle m = \angle n$.
   e. *Prove* $\angle ADC = \angle ABC$.



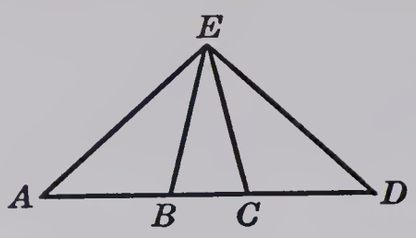
4. *Given* $AB = BC$, $AD = CD$, and AC intersecting BD in E.
   a. *Prove* $\angle x = \angle y$.
   b. *Prove* $\triangle ABE \cong \triangle BEC$.
   c. *Prove* $AE = EC$.
   d. *Prove* $BE \perp AC$.

Raleigh Bicycles

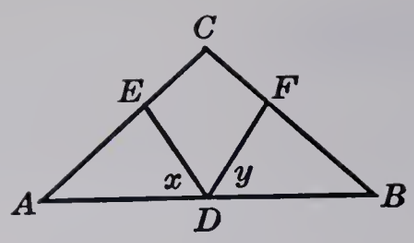
5. Why are braces needed on the gate? Which theorem states that the bicycle frame is rigid?

6. *Given* two circles having centers O and O' and intersecting in points A and B.
   Prove that $\angle OAO' = \angle OBO'$.

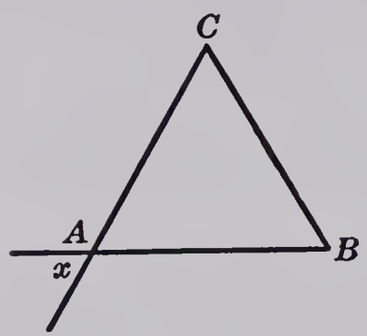
*If desired, pages 185-189 inclusive may be studied at this time.



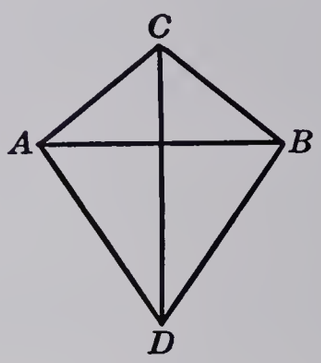
7. *Given* $\triangle ADE$ with $AE = DE$ and $\angle AEB = \angle CED$.
Prove $BE = CE$.



8. *Given* $\triangle ABC$ with $AC = BC$ $AD = DB$, and $\angle x = \angle y$.
Prove as many facts as you can.

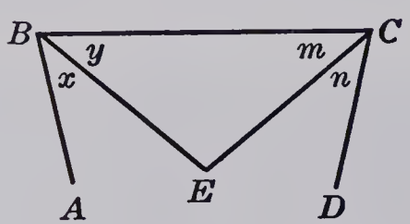


9. *Given* $\triangle ABC$, $AB = BC = AC$, and $\angle x$.
Prove $\angle x = \angle C$.

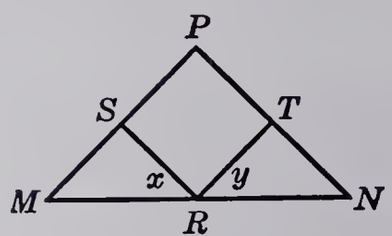


10. *Given* CD the $\perp$ bisector of AB and line segments AC, BC, AD, and BD.
Prove $\angle CAD = \angle CBD$.

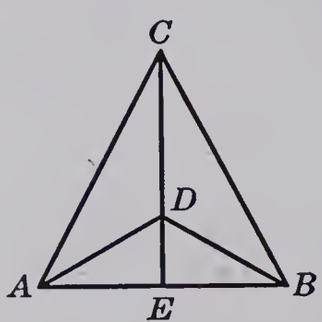
B



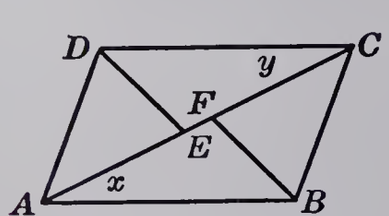
11. *Given* figure $ABCDE$ with EB bisecting $\angle ABC$, EC bisecting $\angle BCD$, and $EB = EC$.
Prove $\angle x = \angle n$.



12. *Given* $\triangle MNP$, $MP = NP$, RS bisecting MP, RT bisecting NP, and R the midpoint of MN.
Prove $\angle x = \angle y$.



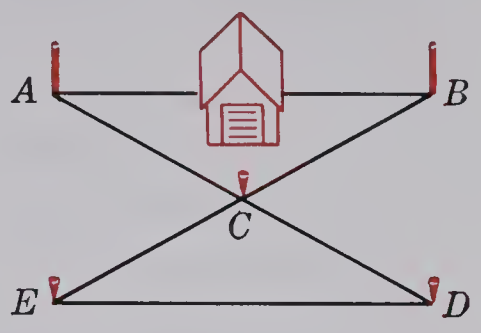
13. *Given* line segments AB and CE, $AC = BC$, and $AD = BD$.
Prove $CE \perp AB$.



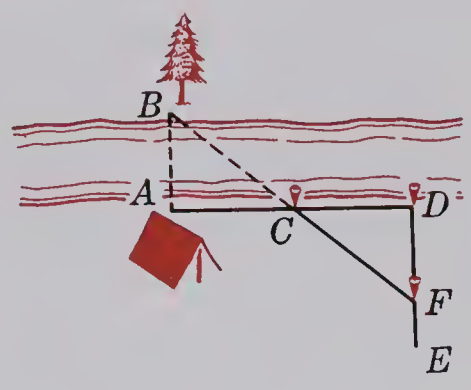
14. *Given* polygon $ABCD$ with $AE = FC$, $DE = BF$, and $AB = DC$.
Prove $\angle x = \angle y$.

EXERCISES — Miscellaneous Exercises

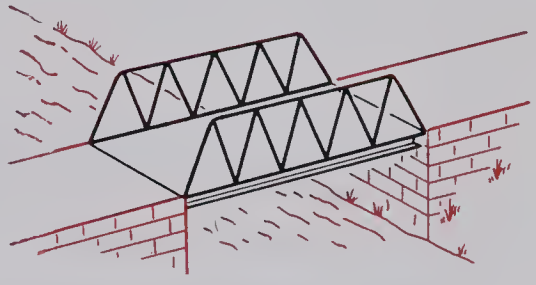
1. Two boys wished to know the distance between two posts, A and B, which were located on opposite sides of a garage. They first set a stake at C. Then they set another stake at D in line with AC, making $CD = CA$. In like manner they set a stake at E in line with CB, making $CE = CB$. By measurement they found that ED was 48 feet. Prove that the distance between the two posts A and B is 48 feet.



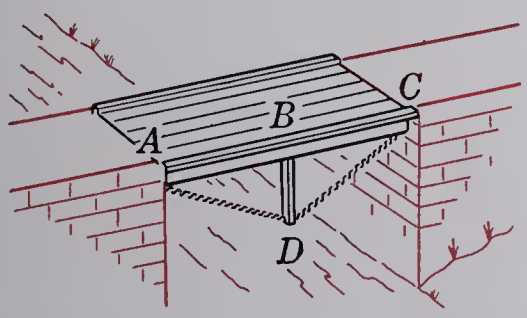
2. Some boys wished to find the distance from their camp at A to a tree at B on the opposite side of a river. They set a stake at C so that $\angle BAC = 90°$. They placed another stake at D in line with AC, making $CD = CA$. Then they stretched a tape from D, making $DE \perp AD$ and placed a stake at F, the intersection of BC and DE. They measured DF to find the length of AB. Prove that $DF = AB$.

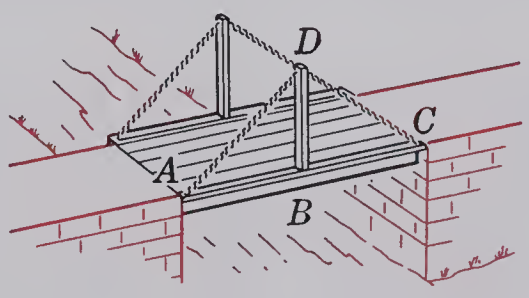


3. This is a diagram of an iron bridge. Why do you think it is composed of triangles instead of quadrilaterals (figures having four sides)? Can the bridge change its shape without breaking?



4. ABC in the pictures below is a wood girder for a footbridge. ADC is an iron-wire cable to make the bridge rigid, and BD is a steel beam which can withstand a compression or a tension. Which of the two diagrams below shows the correct method of bracing the bridge? Give reasons for your answer.

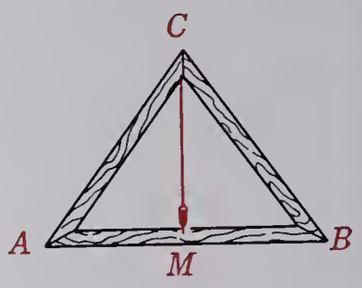




5. A contractor stood at a point P on an edge of an excavation and sighted a point Q on the opposite edge. Then, without raising or lowering

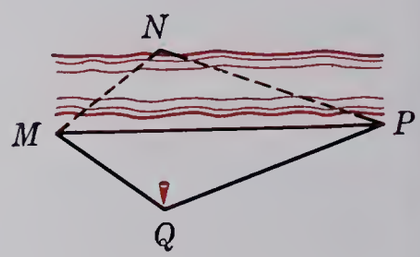
his eyes, he faced about and sighted a point R on the ground. Then he measured the distance PR, which was equal to the distance across the excavation. Prove that his method is correct.

6. Before the spirit level was invented, the *plumb level* was often used. This consisted of a frame in the form of an isosceles triangle (ABC) and a plumb line suspended from the vertex angle (C). In use, the instrument was held upright with the edge AB resting on the surface to be leveled. Prove that when the plumb line hung directly over the midpoint M of AB, the surface upon which the edge AB rested was level. ("Level" means perpendicular to vertical.)



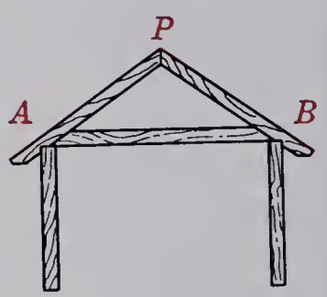
7. Make a plumb level and bring it to class.

8. A geometry class wishing to know the distance between M and the inaccessible point N, set a stake at Q so that $\angle QMP = \angle PMN$ and $\angle QPM = \angle MPN$. What line should they measure to find MN?



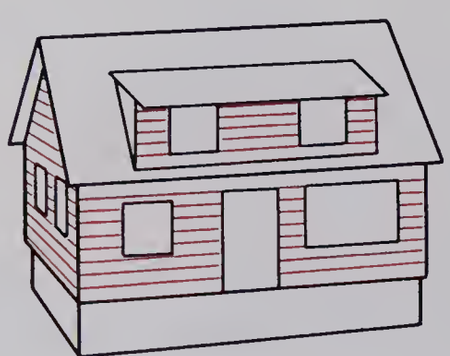
9. If the base of an isosceles triangle is extended in both directions, the angles so formed are equal.

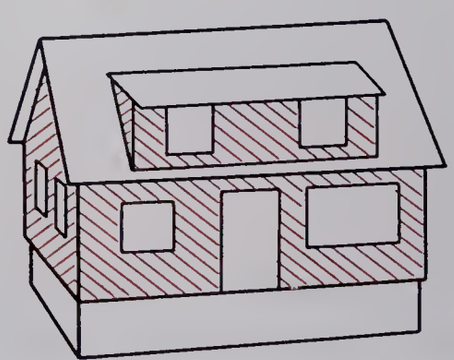
10. Can you show that the rafters PA and PB are of equal length if the end of the ridgepole P is directly above the middle of the tie beam AB?



11. Prove that the line segments drawn from the midpoint of the base of an isosceles triangle to the midpoints of the equal sides make equal angles with the base.

12. These two houses will be brick veneer and identical except that the sheathing boards of one are horizontal and those of the other are slanting. Which method of applying the sheathing is superior? Why?





13. Two oblique line segments drawn from a point on the perpendicular to a line so that they cut off equal line segments from the foot of the perpendicular are equal.

14. The bisectors of the base angles A and B of the isosceles $\triangle ABC$ cut off equal segments from C on AC and BC.

86. Syllogisms [Optional]

In Chapter 3 you learned that any deduction in logic can be reduced to three steps, which are:

1. A general statement.
2. A specific statement which satisfies all the conditions of the general statement.
3. The conclusion, which agrees with the conclusion of the general statement.

The above arrangement of a deduction is called a *syllogism*. In a syllogism, the general statement is the *major premise* and the specific statement is the *minor premise*.

Example 1. 1. (If) complements of the same angle are equal.
2. (If) $\angle A$ and $\angle B$ are complements of $\angle C$.
3. (Then) $\angle A = \angle B$.

In this example, statement 1 is the major premise, statement 2 is the minor premise, and statement 3 is the conclusion. Note that the minor premise states that the major premise applies to $\angle A$ and $\angle B$.

Example 2. 1. All A is B.
2. All C is A.
3. Then all C is B.

Any geometric proof may be arranged so as to consist of one or more syllogisms. The proof in Example 2, § 68, can be arranged in two syllogisms as follows:

1. Vertical angles are equal.
2. $\angle x$ and $\angle y$ are vertical angles.
3. $\angle x = \angle y$.

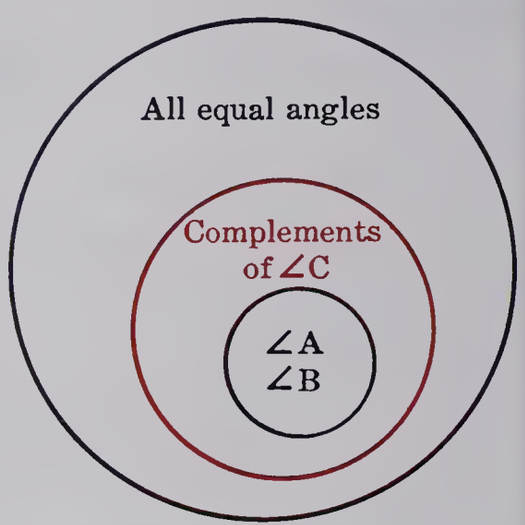
1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.
2. $CO = OD$, $AO = OB$, and $\angle x = \angle y$.
3. $\triangle AOC \cong \triangle BOD$.

87. Euler's Circles [Optional]

Euler (1707–1783), a Swedish mathematician, used circles to represent deductive reasoning. Let us see how we can represent a syllogism by circles.

Example 1

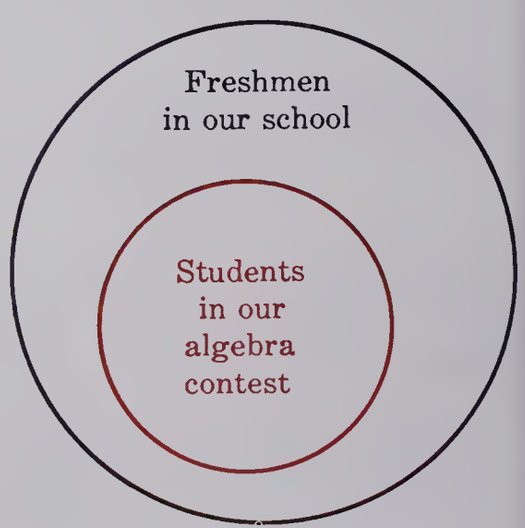
Major premise. Complements of the same angle are equal.
Minor premise. $\angle A$ and $\angle B$ are complements of $\angle C$.
Conclusion. Then $\angle A = \angle B$.



Let a large circle contain all equal angles. Then we may have a smaller circle within it containing all complements of $\angle C$. Within this smaller circle we may have a third circle containing $\angle A$ and $\angle B$. Since the smallest circle is within the next larger circle and the latter is within the largest circle, we know that the smallest circle is within the largest.

Example 2

Major premise. If a student enters our algebra contest, that student is a freshman in our school.
Minor premise. Dick is a freshman in our school.
Conclusion. Dick is entered in our algebra contest.



The conclusion is not valid. Dick's circle is inside the larger circle but we cannot say that it is inside the smaller circle.

Example 3

Major premise. Same as in Example 2.
Minor premise. Helen is in the algebra contest.
Conclusion. Helen is a freshman in our school.

This conclusion is valid. If Helen's circle is within the smaller of two other circles, it is within the larger.

EXERCISES — Optional

Draw circles to show:

1. If a point is within circle A, it is within circle B.
2. If a point is within circle B, it is within circle A.
3. If a point is within either of circles A and B, it is within the other.
4. If a point is not within circle B, it is not within circle A.

Test the conclusions in Exercises 5–11:

5. *Major premise.* All girls in the gymnasium class II have doctors' certificates.
   Minor premise. Lucille is in the gymnasium class II.
   Conclusion. Lucille has a doctor's certificate.
6. *Major premise.* Same as in Ex. 5.
   Minor premise. Frances is not in the gymnasium class II.
   Conclusion. Frances does not have a doctor's certificate.
7. *Major premise.* Same as in Ex. 5.
   Minor premise. Caroline has a doctor's certificate.
   Conclusion. Caroline is in the gymnasium class II.
8. *Major premise.* Same as in Ex. 5.
   Minor premise. Sue does not have a doctor's certificate.
   Conclusion. Sue is not in the gymnasium class II.
9. *Major premise.* All dogs eat Ration Biscuits.
   Minor premise. Susie eats Ration Biscuits.
   Conclusion. Susie is a dog.

10. *Major premise.* Only dogs eat Ration Biscuits.
Minor premise. Fido eats Ration Biscuits.
Conclusion. Fido is a dog.

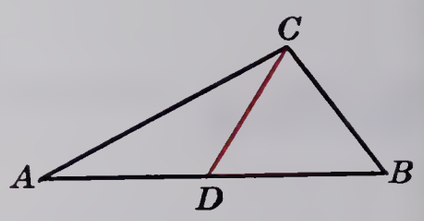
11. *Major premise.* Some dogs eat Ration Biscuits.
Minor premise. Blimpy is a cat.
Conclusion. Blimpy does not eat Ration Biscuits.

88. Line Segments and Angles Associated with Triangles

Every triangle can have three medians, three altitudes, three angle bisectors, and six exterior angles.

89. Medians of a Triangle

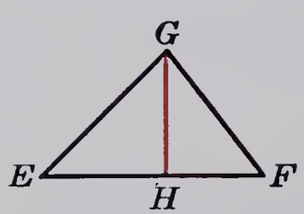
A *median* of a triangle is a line segment joining any vertex to the midpoint of the opposite side. CD is a median of $\triangle ABC$. How many medians has a triangle?

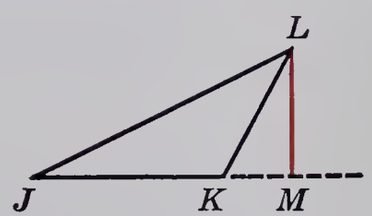


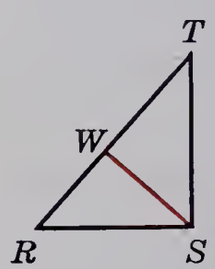
90. Altitudes of a Triangle

An *altitude* of a triangle is a perpendicular line segment from any vertex to the opposite side.

In $\triangle EFG$ below, GH is the altitude from G to EF; in $\triangle JKL$, LM is the altitude from the vertex L to the side JK; and in $\triangle RST$, SW is the altitude from S to RT. Notice that the altitude LM meets JK produced. Remember that any triangle has three altitudes.

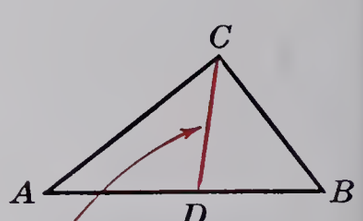


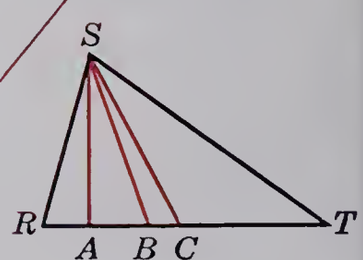




91. Angle Bisectors of a Triangle

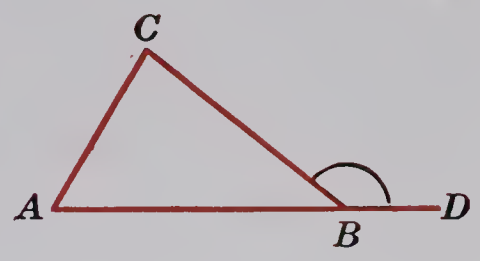
A *bisector* of an angle of a triangle is a line which bisects that angle. The bisector is usually regarded as being terminated by the side opposite the angle. In $\triangle ABC$, CD is the bisector of $\angle C$. In $\triangle RST$, the altitude, the median, and the angle bisector are drawn from S to RT. Which is SA? Which is SB? Which is SC?





92. Exterior Angles of a Triangle

An *exterior angle* (ext. $\angle$) of a triangle is the angle formed by one side and an extension of an adjacent side. $\angle DBC$ is an exterior angle of $\triangle ABC$. $\measuredangle A$ and C are called nonadjacent interior angles to $\angle CBD$.

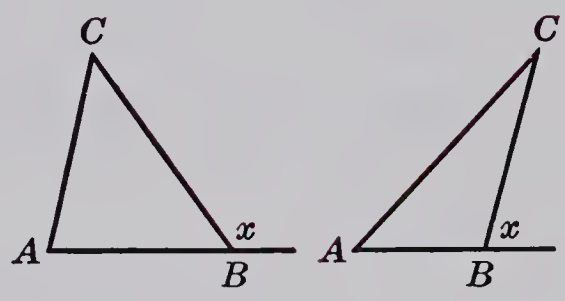


By extending all three sides of the triangle, six exterior angles can be formed.

EXERCISES

A

1. In the adjoining figures, measure $\measuredangle x$, A, and C with a protractor. How does the size of $\angle x$ compare with that of $\angle A$? with that of $\angle C$? with the sum of $\angle A$ and $\angle C$?

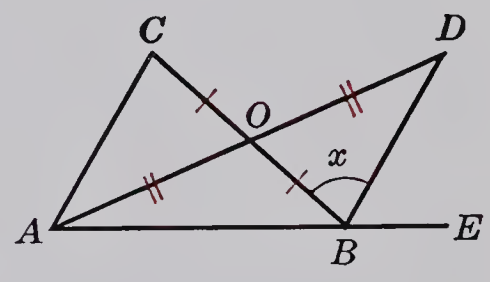


2. *Given* st. line ABE and lines AD and CB bisecting each other.

a. *Prove* that $\angle x = \angle C$.

b. *Prove* that $\angle CBE > \angle x$. (See Asmt. 9.)

c. *Prove* that $\angle CBE > \angle C$.



3. If the median of a triangle is also an altitude, the triangle is isosceles.

4. The median to the base of an isosceles triangle bisects the vertex angle.

5. The median to the base of an isosceles triangle is also an altitude of the triangle.

6. The bisectors of two corresponding angles of two congruent triangles are equal.

7. The corresponding medians of two congruent triangles are equal.

B

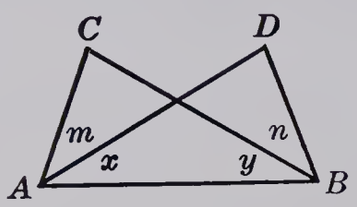
8. The vertex of an isosceles triangle lies on the perpendicular bisector of the base. (Prove the median an altitude.)

9. Line segments drawn from any point in the bisector of the vertex angle of an isosceles triangle to the extremities of the base are equal.

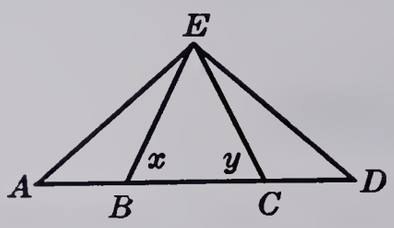
10. The line segments connecting the midpoints of the equal sides of an isosceles triangle with the midpoint of the base are equal.

11. A point P lies outside $\triangle ABC$ and is equidistant from BC and AB produced through B. If the perpendiculars from P to BC and AB produced form equal angles with the line PB, prove that P lies on the bisector of the exterior angle at B.

12. AE and BD, the bisectors of the base angles of isosceles $\triangle ABC$, intersect in P. Prove that $\angle CDP = \angle CEP$.



13. *Given* figure $ABDC$ with $\angle x = \angle y$ and $\angle m = \angle n$.
Prove $\angle C = \angle D$.

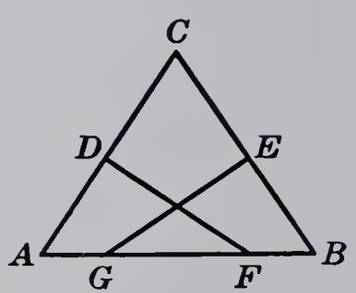


14. *Given* $\triangle ADE$, $BE = CE$, and $AC = BD$.
Prove $AE = DE$.

C

15. If a point in the base of an isosceles triangle is equally distant from the midpoints of the equal sides, the point bisects the base.

16. Lines drawn perpendicular to the equal sides of an isosceles triangle at their midpoints and terminated by the base are equal.

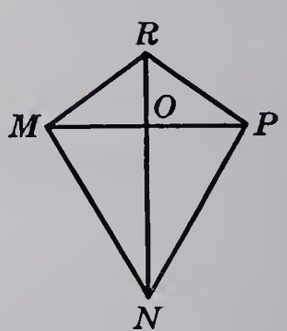


SUGGESTION. Prove that $DF = EG$.

17. The medians to the equal sides of an isosceles triangle are equal.

18. The point of intersection of the perpendicular bisectors of two sides of a triangle is equidistant from the three vertices.

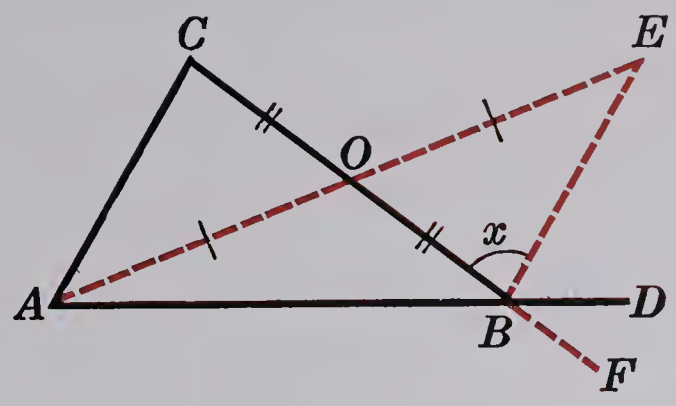
19. The line segments bisecting the angles at the base of an isosceles triangle and terminated by the equal sides are equal.



20. *Given* line segments RN and MP intersecting in O, $MR = PR$, and $MN = PN$.
Prove that RN is the perpendicular bisector of MP.

★ Theorem 5

93. An exterior angle of a triangle is greater than either nonadjacent interior angle.



Given the $\triangle ABC$ with the ext. $\angle DBC$.

To prove that $\angle DBC > \angle C$ and $\angle DBC > \angle A$.

Planning the Proof: 1. We can prove $\measuredangle$s unequal by Asmt. 9.
2. We shall use Asmt. 9.

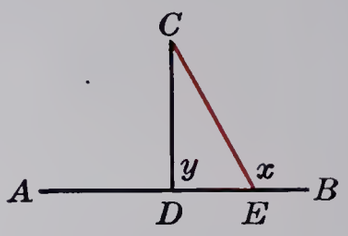
Proof

STATEMENTS	REASONS
1. Let O be the midpoint of CB.	1. Asmt. 17.
2. Join A to O.	2. Asmt. 11.
3. Extend AO to E, making $OE = AO$.	3. Asmt. 13.
4. Join B to E.	4. Asmt. 11.
5. Point E lies within $\angle DBC$. Otherwise AE would intersect BC or AD in another point, which is impossible.	5. Asmt. 12.
In $\triangle AOC$ and OBE,	
6. $AO = OE$ and $CO = OB$.	6. Const.
7. $\angle COA = \angle BOE$.	7. Why?
8. $\triangle AOC \cong \triangle OBE$.	8. Why?
9. $\therefore \angle C = \angle x$.	9. Why?
10. But $\angle DBC > \angle x$.	10. Asmt. 9.
11. $\therefore \angle DBC > \angle C$.	11. Asmt. 6.

In like manner, by drawing a line through C and the midpoint of AB it can be proved that $\angle ABF > \angle CAB$, and hence $\angle CBD > \angle CAB$.

NOTE. If desired, pages 504-507 may be studied at this time.

94. Corollary. There cannot be more than one perpendicular to a line from an external point.



Given $CD \perp AB$ and CE any other line from C to AB.

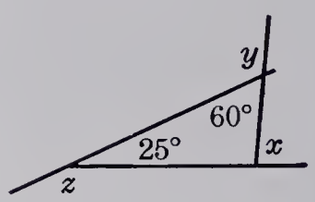
To prove that CE is not $\perp AB$.

SUGGESTIONS. $\angle y$ is a rt. $\angle$. Why? $\angle x > \angle y$. Why? $\therefore \angle x$ is an obtuse $\angle$. Why?

EXERCISES — Angles Unequal

A

1. Is $\angle x > 25°$? Is $\angle x > 60°$? Why? How many degrees are in $\angle y$? in $\angle z$?



2. Is $\angle x + \angle y = 180°$ or $> 180°$? Why?

3. Show that $\angle x + \angle y + \angle z > 335°$.

REVIEW QUESTIONS — Chapter 4

1. What is the name of a statement to be proved?

2. What is the name of a geometric statement which is accepted as true without proof?

3. What is the name of a triangle having all sides equal?

4. What is the name of a triangle having two sides equal?

5. What is the name of the side opposite a right angle in a triangle?

6. What is the name given to the two sides that include the right angle of a right triangle?

7. What do you know about an exterior angle of a triangle?

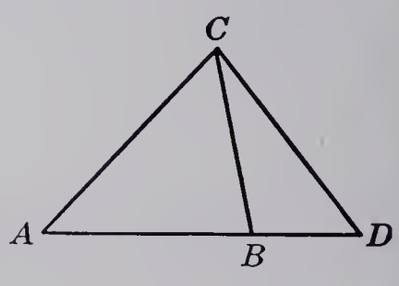
8. How many altitudes does a triangle have? How many medians? How many angle bisectors?

9. What are the five parts of a demonstration?

10. Are two triangles congruent if their corresponding parts are equal?

11. What are auxiliary lines?

12. Have we proved "If two angles of a triangle are equal, the sides opposite these angles are equal"?

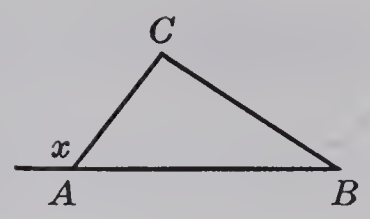


13. If $\triangle ABC$ is isosceles, is it equilateral?

14. Name the common side and common angle of $\triangle ABC$ and $\triangle ADC$ in the diagram.

15. Have we proved that corresponding parts of congruent triangles are equal?

16. What do you know about $\angle x$ in this figure?



SUMMARY — of Principal Bases for Proof

95. Two triangles are congruent

a. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

b. If two angles and the included side of one are equal respectively to two angles and the included side of the other.

c. If the three sides of one are equal respectively to the three sides of the other.

96. Two right triangles are congruent

a, b, c. Same as § 95.

d. If two legs of one are equal respectively to two legs of the other.

97. Two line segments are equal

a. If they are corresponding sides of congruent triangles.

b. If they are drawn from any point in the perpendicular bisector of a line segment to the end points of the segment.

c. By Assumptions 1-8, 26.

98. Two angles are equal

a. If they are angles opposite equal sides in a triangle.

b. If they are corresponding angles of congruent triangles.

WORD LIST

Do you know the meaning of all these words? Can you spell them?

acute
altitude
auxiliary
complementary
corollary
corresponding
equiangular
equilateral
exterior
hypotenuse
identity
isosceles
median
miscellaneous
perimeter
polygon
prove
right
scalene
superposition
triangle
vertices

TEST 4 — Determining the Hypothesis and Conclusion

[20 Minutes]

Copy these statements and underscore each hypothesis with one line and each conclusion with two lines:

1. If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.

2. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

3. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

4. If a point lies on the perpendicular bisector of a line segment, it is equidistant from the ends of the segment.

5. If the median of a triangle is also an altitude, the triangle is isosceles.

6. Two triangles are congruent if two sides and the median to one of these sides are equal respectively to two sides and the corresponding median of the other.

7. If two medians of a triangle are equal, the triangle is isosceles.

8. The bisectors of two corresponding angles of two congruent triangles are equal.

9. If two oblique line segments are drawn from a point on the perpendicular to a line so that they cut off equal line segments from the foot of the perpendicular, they are equal.

10. The base angles of an isosceles triangle are equal.

TEST 5 — True-False Statements

[6 minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number.

1. A theorem that is derived from another theorem with little or no proof is called an assumption.

2. A triangle is a polygon.

3. A median is a line joining the midpoints of two sides of a triangle.

4. Two triangles are congruent if two sides and an angle of one are equal respectively to two sides and an angle of the other.

5. The sum of two sides of a triangle is greater than the third side.

6. Two right triangles are congruent if the two legs of one are equal respectively to the two legs of the other.

7. The exterior angles of a triangle are obtuse.

8. An angle less than a right angle is acute.

9. A theorem is a statement to be proved.

10. Any two altitudes of a triangle are perpendicular to each other.

TEST 6 — Applications

[25 Minutes]

1. Find the complement of an angle of 38°.

2. An angle contains 37°. Find the size of its vertical angle.

3. $\triangle ABC \cong \triangle DEF$. If $AC = 8$, $BC = 9$, $AB = 10$, $DF = 8$, and $EF = 9$, how long is DE?

4. How many degrees are there in the smaller angle formed by the hands of a clock when it is 4 P.M. by the clock?

5. Through how many degrees does the minute hand of a watch pass in one hour?

6. An angle exceeds its supplement by 10°. How many degrees does it contain?

7. In polygon $ABCD$, $AB = DC = 10$ inches and $AD = BC = 8$ inches. The line BD makes $\angle ADB = 72°$. How many degrees are there in $\angle DBC$?

8. In $\triangle ABC$ the median CD is an altitude. If $\angle A = 66°$, how large is $\angle B$?

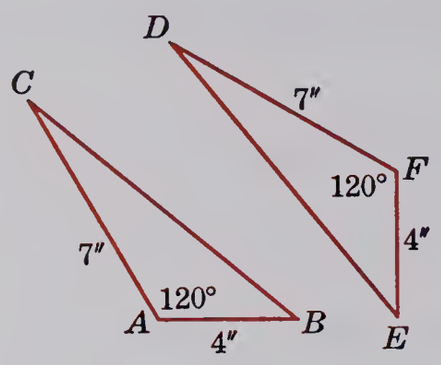
9. If two adjacent angles whose sum is 96° are bisected, how many degrees are there in the angle formed by their bisectors?

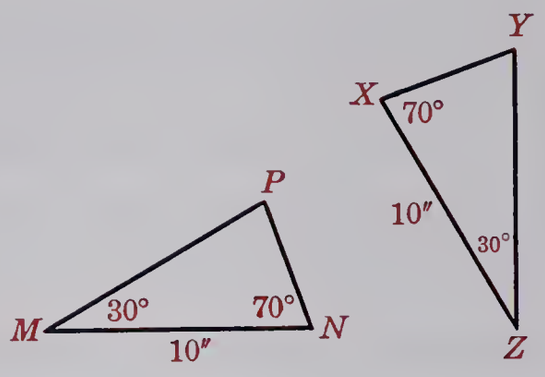
10. In $\triangle ABC$, $AC = BC$, $\angle A = \left(\frac{x+24}{5}\right)°$, and $\angle B = \left(\frac{x-6}{2}\right)°$. How large is each angle?

TEST 7 — Supplying Reasons

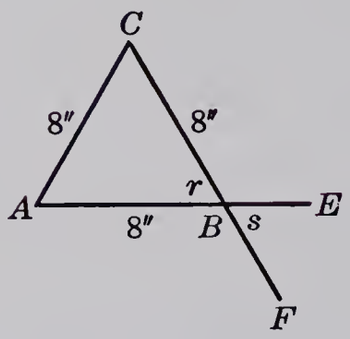
[22 Minutes]

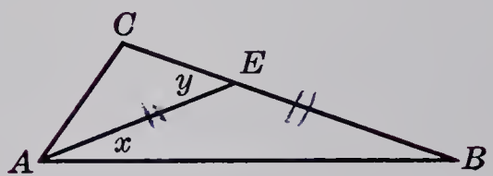
Supply assumptions, definitions, or theorems for the following statements:





1. $\triangle ABC \cong \triangle DEF$.
2. $\angle B = \angle E$.
3. $\triangle MPN \cong \triangle XYZ$.
4. $PN = XY$.





If AC, ABE, CBF are st. lines,

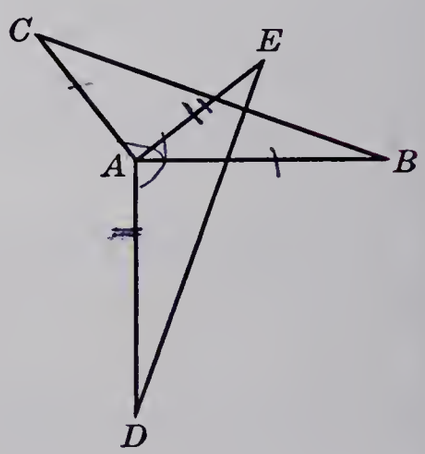
5. $\triangle ABC$ is equiangular.
6. $\angle r = \angle s$.

If $AE = EB$,

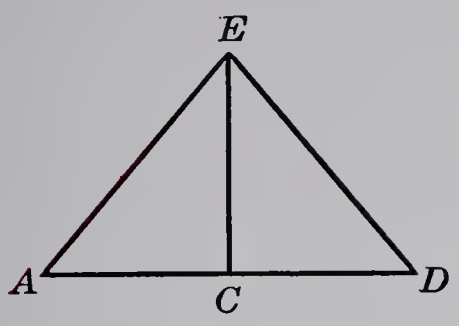
7. $\angle x = \angle B$.
8. $\angle y > \angle B$.

If $AC = AE$, $AB = AD$, and $\angle CAB = \angle EAD$,

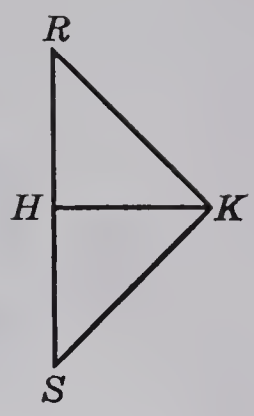
9. $\triangle CAB \cong \triangle EAD$.
10. $CB = ED$.



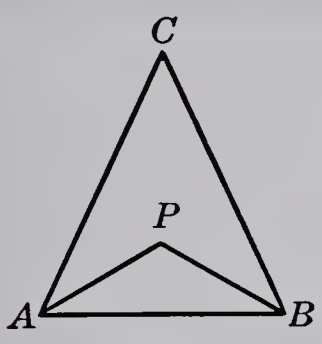
TEST 8 — Proofs



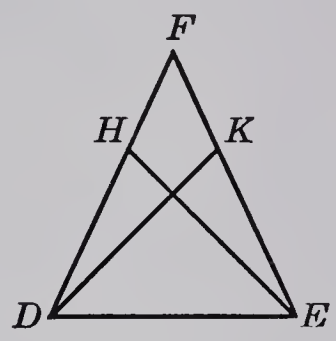
1. *Given* $\triangle ADE$, CE bisecting $\angle AED$, and $CE \perp AD$.
Prove that $\triangle ADE$ is isosceles.



2. *Given* $HK \perp RS$ and $HR = SH$.
Prove that $\angle RKH = \angle HKS$.



3. *Given* $\triangle ABC$, $AC = BC$, and $AP = BP$.
Prove that $\angle PAC = \angle PBC$.



4. *Given* $\triangle DEF$, $DF = EF$, and $\angle DEH = \angle EDK$.
Prove $DK = HE$.

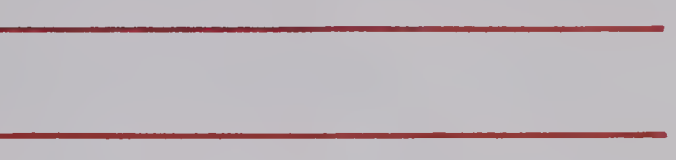
CHAPTER 5

Parallel and Perpendicular Lines

In this chapter
we shall discover and prove
properties of parallel
and perpendicular lines.
We see examples of parallel lines
and perpendicular lines all about us.
We see them in our homes,
in the countryside,
in public buildings,
and in our parks.

99. Parallel Lines

Two lines are *parallel* (∥) if they lie in the same plane and do not intersect even if extended. Thus l and l' are parallel lines.



Since in plane geometry all points and lines lie in the same plane, the words "in the same plane" are usually omitted in courses in this subject.

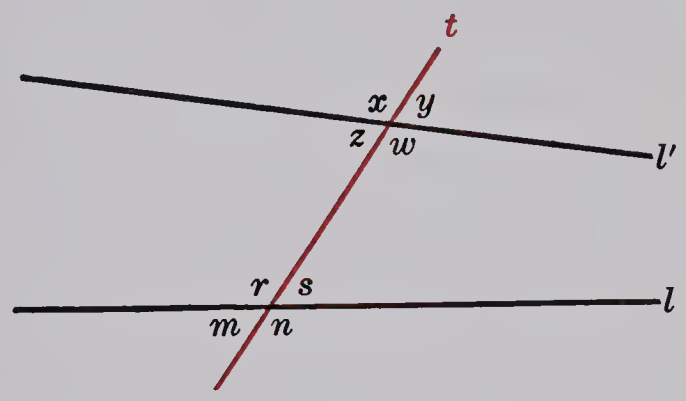
In projective geometry we think of parallel lines as lines that meet at an infinite distance. In this course we shall assume that parallel lines never meet.

100. Assumption 35. Two straight lines in the same plane are either parallel lines or intersecting lines.

101. Angles Formed by a Transversal

A *transversal* is a line which intersects two or more lines in different points.

The line t is a transversal of the lines l and l'.



In the figure there are four pairs of vertical angles. Can you name them?

There are eight pairs of supplementary angles. Can you name them?

The angles x, y, m, and n are *exterior angles* because they are outside l and l'. The angles z, w, r, and s are *interior angles* because they are inside l and l'. Besides the pairs of vertical angles and supplementary angles we have the following pairs:

The $\measuredangle z$ and s are *alternate interior* angles; also $\measuredangle w$ and r.

The $\measuredangle y$ and m are *alternate exterior angles*; also $\measuredangle x$ and n.

The $\measuredangle x$ and r are *corresponding angles*; also $\measuredangle y$ and s, $\measuredangle n$ and w, and $\measuredangle m$ and z. Corresponding angles are often called *exterior-interior* angles. Why?

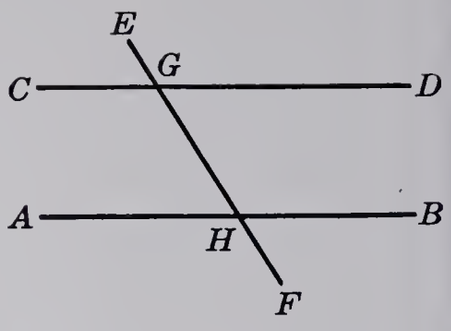
Do you see that when one line is a transversal of two other lines, there are two pairs of alternate interior angles, two pairs of alternate exterior angles, and four pairs of corresponding angles?

EXERCISES

A

1. Line l intersects lines a and b in the point P. Is l a transversal of a and b?

2. On your paper draw two parallel lines AB and CD. Then draw any transversal EF intersecting AB in H and CD in G. Name the two pairs of alternate interior angles. Do you think a pair of alternate interior angles are equal? Test your conclusion by measuring the angles with a protractor.



Ex. 2

102. Indirect Method of Proof

So far we have used the *direct method* in proving theorems and exercises. The direct method consists in putting together known truths (definitions, assumptions, theorems, the hypothesis), step by step, to form a proof. If the known facts are insufficient to prove a theorem directly, we can often prove it by a method known as the *indirect method.*

Let us see how Dale used indirect reasoning. One afternoon he listened to the broadcast of the football game between the Eagles and the Ravens until the end of the third quarter, when the game was tied 21 to 21. Then he delivered papers on his paper route and returned home. When his brother told him that the Eagles won the game 24 to 21, Dale said that the Eagles won the game by a place kick. How did Dale reason that the Eagles won by a place kick?

He knew that a touchdown counts 6 points, a place kick 3 points, a safety 2 points, and a goal after touchdown 1 point. He knew that the 3 points made by the Eagles in the fourth quarter were not made by a touchdown, by a safety and a point after touchdown, or by three points after touchdowns. Then he knew that a place kick was the only possibility of making the three points.

The following is another example of indirect reasoning:

Detective Jones used indirect reasoning when he investigated the death of an elderly woman on Linden Avenue. He learned from the coroner that the woman's death was caused by a bullet which entered her body below the right shoulder blade and emerged below the right collar bone.

Mr. Jones knew this was a murder, a suicide, or an accidental death. After a thorough examination of the body and surroundings, he knew that the death did not result from suicide or an accident. Therefore he knew this was a murder.

There are two forms of the indirect method—*proof by coincidence*, which will be studied on page 159, and *proof by exclusion*.

The exclusion method consists in examining all the different possible conclusions and showing that none but the one we wish to prove can be true. If there are only two conclusions possible from the facts given in the hypothesis, it is only necessary to prove that one conclusion leads to a contradiction of some known fact.

For example, if we wish to prove two lines parallel, the possibilities are:

(1) the lines are $\parallel$;
(2) the lines are not $\parallel$.

If we assume that the lines are not parallel and then show that this assumption leads to a contradiction of some known truth, we have established the fact that the lines are parallel.

Also, if we wish to prove that $\angle A = \angle B$, there are three possibilities:

(1) $\angle A = \angle B$;
(2) $\angle A > \angle B$;
(3) $\angle A < \angle B$.

If we prove that $\angle A$ is not greater than $\angle B$ and that $\angle A$ is not less than $\angle B$, then we know by Assumption 10 that $\angle A = \angle B$.

To prove a theorem by the exclusion method:

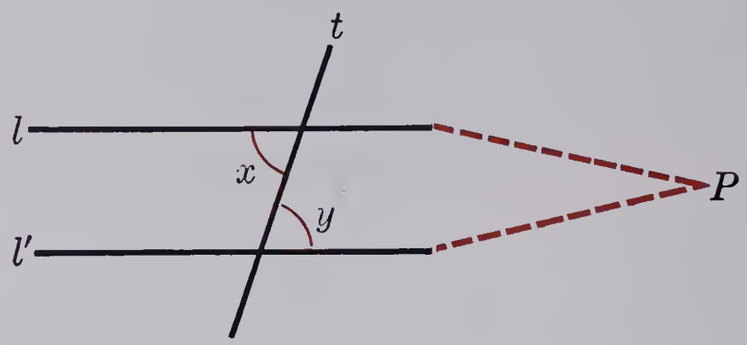
1. State the conclusion and the other possibilities.
2. Show that the other possibilities lead to a contradiction of the hypothesis or of a truth previously learned.
3. The conclusion is the only remaining possibility.

In proof by exclusion we shall use the following assumption.

103. Assumption 36. If a hypothetical statement leads to a contradiction of a known fact or hypothesis, the statement is false.

★★ Theorem 6

104. If two lines form equal alternate interior angles with a transversal, the lines are parallel.



Given the lines l and l' cut by the transversal t so that $\angle x = \angle y$.

To prove that $l \parallel l'$.

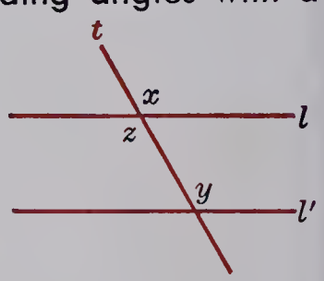
Planning the Proof: 1. We can prove lines ∥ by § 99.
2. We shall use § 99 and the indirect method.

Proof

STATEMENTS	REASONS
1. l and l' are either ∥ lines or intersecting lines.	1. Asmt. 35.
2. If l and l' intersect at some point P, a △ is formed, and $\angle x > \angle y$.	2. § 93.
3. But $\angle x = \angle y$.	3. Given.
4. Since Statement 2 contradicts Statement 3, l does not intersect l'.	4. Asmt. 36.
5. $\therefore l \parallel l'$.	5. § 99. (Def. of ∥s.)

NOTE. Statement 2 is a hypothetical statement.

★ **105. Corollary I.** If two lines form equal corresponding angles with a transversal, the lines are parallel.



SUGGESTIONS. $\angle x = \angle y$. Why? $\angle x = \angle z$. Why? $\therefore \angle z = \angle y$. Why? Use § 104.

NOTE. Corollaries following a theorem proved by the *indirect method* are usually proved by the *direct method*.

★ **106. Corollary II.** If two lines form supplementary interior angles on the same side of a transversal, the lines are parallel.

SUGGESTION. Show that two ∡s are supp. of the same ∠ and are therefore =.

★ **107. Corollary III.** Two lines perpendicular to a third line are parallel.

General Motors

Modern industrial structures offer beautiful examples of geometric form. Notice how the parallel lines, and the perpendicular ones as well, contribute to the pleasing appearance of these buildings near Detroit.

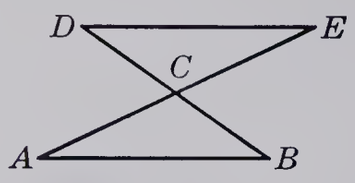
General Motors

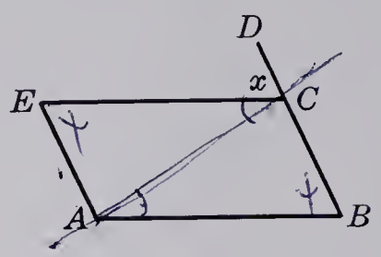
EXERCISES

A

1. Name five ways of proving lines parallel.

2. In polygon $ABCD$, $\angle A = 110°$, $\angle B = 95°$, and $\angle C = 85°$. Which two sides of the polygon are parallel? Give the reason for your answer.



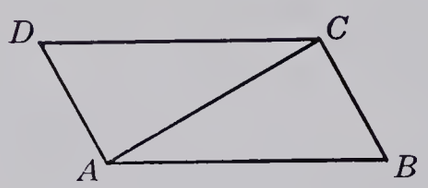


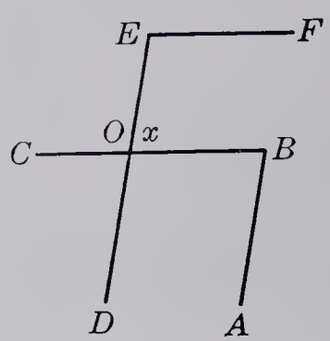
3. *Given* AE and BD bisecting each other in C.

Prove that $DE \parallel AB$.

4. *Given* figure $ABCE$ with BCD a st. line, $\angle x = \angle B$, and $\angle B = \angle E$.

Prove that $EC \parallel AB$; $AE \parallel BD$.





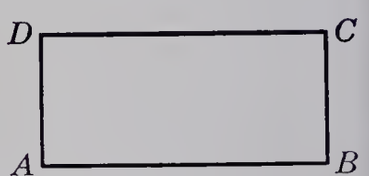
5. *Given* line AC dividing polygon $ABCD$ into two congruent triangles having BC corresponding to AD.

Prove that $AD \parallel BC$ and $AB \parallel DC$.

6. *Given* $\measuredangle E$ and B supp. and $\angle B = \angle x$.

Prove that $EF \parallel CB$.

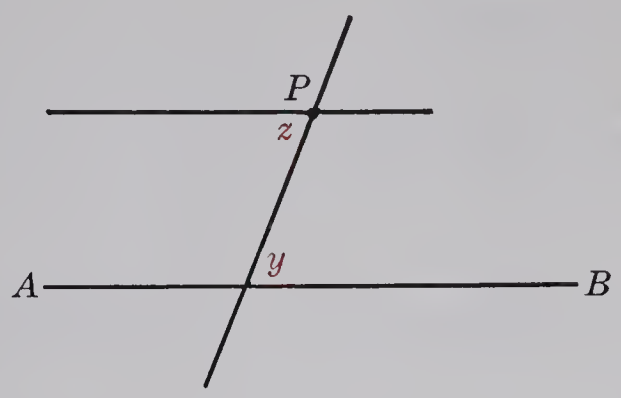
7. *Given* polygon $ABCD$ with each angle a rt. $\angle$. Without drawing auxiliary lines, prove as many facts as you can.



108. Assumption 37. Through a given point there can be only one parallel to a given line.

Notice that this assumption does not only mean (1) that through a given point one line can be drawn parallel to another, but (2) states that through the point *not more than one line* can be drawn parallel to another.

By making $\angle z = \angle y$, we can draw *one* line through P parallel to a given line AB. (We shall learn how to construct $\angle z = \angle y$ in the next chapter.)



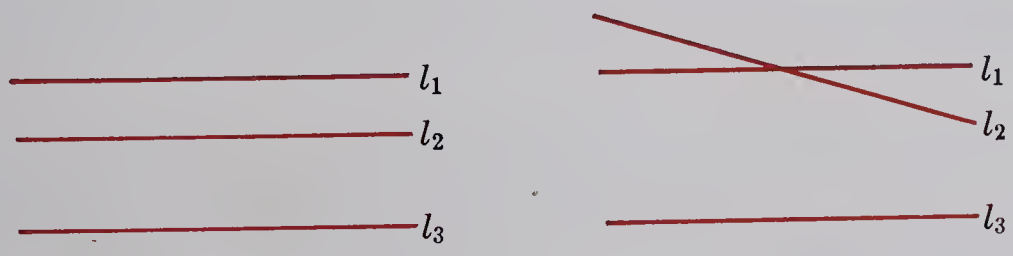
There is evidence that Euclid, the great Greek mathematician, tried to prove the second part of the statement at the foot of page 134, and not being able to do so, postulated an assumption from which it can be derived. Do you see why Asmt. 37 is known as the parallel postulate?

The belief that this assumption was not as fundamental as the other assumptions of geometry led many mathematicians over a period of 2000 years to try to prove or disprove this assumption by means of other assumptions. Their failure to do so led to the idea that a different geometry could be constructed.

In the nineteenth century Nicholas Lobachevsky (1793–1856), a Russian professor of mathematics, and John Bolyai (1802–1860), a Hungarian army officer, independently developed a new geometry based upon the assumption *Through a given point there can be any number of parallels to a given line.* A geometry of this kind is known as *non-Euclidean* geometry.

In 1854 a different non-Euclidean geometry was developed by Bernhard Riemann, a German mathematician, who assumed that any two lines in a plane intersect. A non-Euclidean geometry somewhat different from these two has been found to be of the highest importance in Einstein's theory of relativity.

★ **109. Corollary I.** Two lines parallel to a third line are parallel to each other.

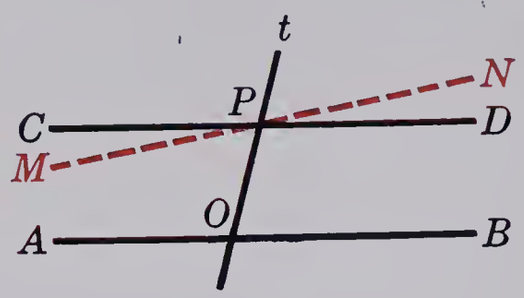


Write the complete demonstration. Use the indirect method and Assumption 37.

110. Corollary II. If a line intersects one of two parallel lines, it intersects the other.

★★ Theorem 7

111. If two parallels are cut by a transversal, the alternate interior angles are equal.



Given the two ||s AB and CD cut by the transversal t at O and P.
To prove that $\angle CPO = \angle BOP$.

Planning the Proof: 1. We can prove angles = by §§ 50, 98.
2. We shall use Asmt. 10.

Proof

STATEMENTS	REASONS
1. $\angle CPO$ and $\angle BOP$ are either = or unequal.	1. Asmt. 10.
2. If $\angle CPO \neq$ (is unequal to) $\angle BOP$, let MN be another line through P, making $\angle MPO = \angle BOP$.	2. Asmt. 10.
3. Then $MPN \parallel AB$.	3. § 104.
4. But $CPD \parallel AB$.	4. Why?
5. CD and MN are two ||s to AB through P, which is impossible.	5. Asmt. 37.
6. $\therefore MPN$ is not $\parallel AB$.	6. Asmt. 36.
7. $\therefore \angle CPO = \angle BOP$.	7. Asmt. 10.

★ **112. Corollary I.** If two parallels are cut by a transversal, the corresponding angles are equal.

113. Corollary II. If two parallels are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.

★ **114. Corollary III.** If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

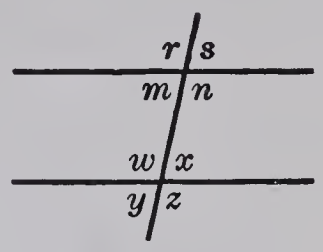
115. Corollary IV. Lines perpendicular to intersecting lines will intersect.

EXERCISES — Parallel Lines

A

1. *Prove:* If two parallels are cut by a transversal, the alternate exterior angles are equal.

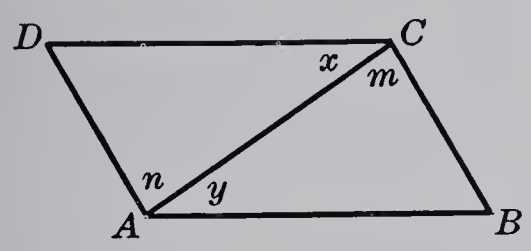
2. In this figure there are two parallels cut by a transversal. If $\angle s$ contains 51°, find the number of degrees in each of the remaining angles.

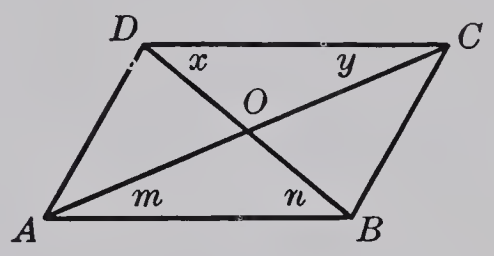


3. In the figure of Ex. 2, if $\angle r$ contains 118°, find the size of each remaining angle.

4. How many degrees are there in each of the eight angles in the figure of Ex. 2 if $\angle w$ is 15° more than twice $\angle m$?

5. *Prove:* If two lines form equal alternate exterior angles with a transversal, the lines are parallel.





6. *Given* $DC \parallel AB$ and $DC = AB$.
Prove that $AD = BC$.

7. *Given* $DC \parallel AB$ and $DC = AB$.
Prove that $AO = OC$ and that $DO = OB$.

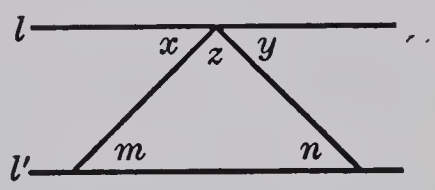
8. *Given* $l \parallel l'$.

a. *Prove* that $\angle x = \angle m$.

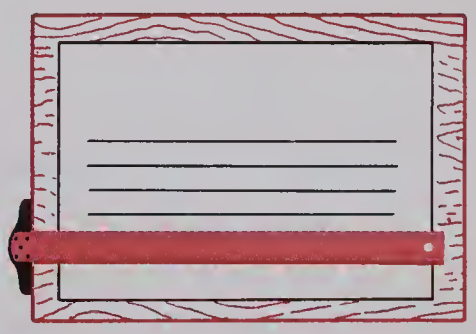
b. *Prove* that $\angle y = \angle n$.

c. *Prove* that $\angle x + \angle y + \angle z = 1$ st. $\angle$.

d. *Prove* that $\angle m + \angle z + \angle n = 1$ st. $\angle$.

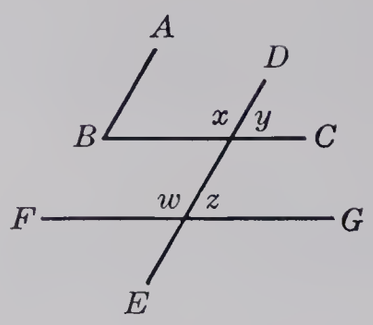


9. Why are the lines drawn with a T-square, as shown in the figure, parallel? Using the T-square, how can you draw parallel vertical lines? What other drawing instrument would be useful in drawing parallel slanting lines?

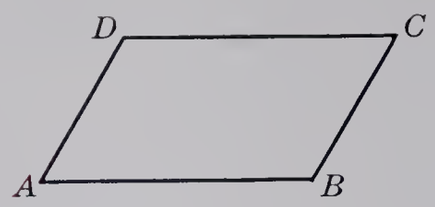


10. *Given* AC and BD the diagonals of polygon $ABCD$ bisecting each other in the point O.
Prove that $AD \parallel BC$ and $AB \parallel DC$.

B

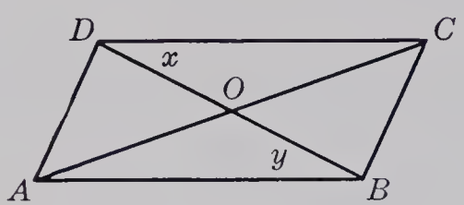


11. *Given* $AB \parallel DE$ and $BC \parallel FG$.
a. *Prove* that $\angle B = \angle z$.
b. *Prove* that $\measuredangle B$ and w are supp.

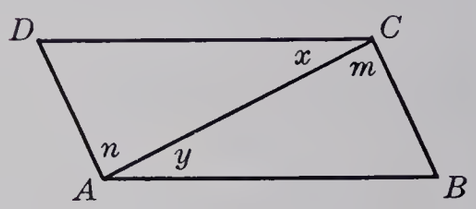


12. *Given* $AB \parallel DC$ and $AD \parallel BC$.
a. Why is $\angle A$ supp. $\angle B$?
b. Why is $\angle C$ supp. $\angle B$?
c. Why does $\angle A = \angle C$?

13. Using the figure and "given" of Ex. 12, find $\angle A$ and $\angle B$ if $\angle B$ exceeds $\angle A$ by 12°.



14. *Given* $DO = OB$; $DC \parallel AB$.
Prove that $AO = OC$ and that $DC = AB$.



15. *Given* $DC = AB$; $AD = BC$.
Prove that $DC \parallel AB$ and that $AD \parallel BC$.

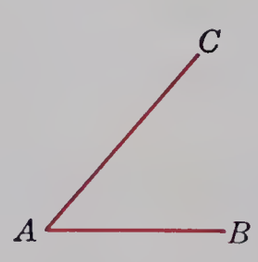
C

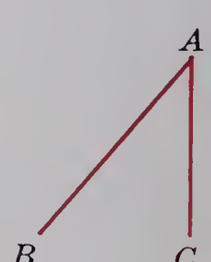
16. *Prove:* If a line is drawn through the vertex of an isosceles triangle parallel to the base, it bisects the exterior angle at the vertex.

17. If two parallels are cut by a transversal so that the two interior angles on the same side of the transversal contain $4x$ degrees and $(3x - 9)$ degrees respectively, find the number of degrees in each angle.

116. Right Side and Left Side of an Angle.

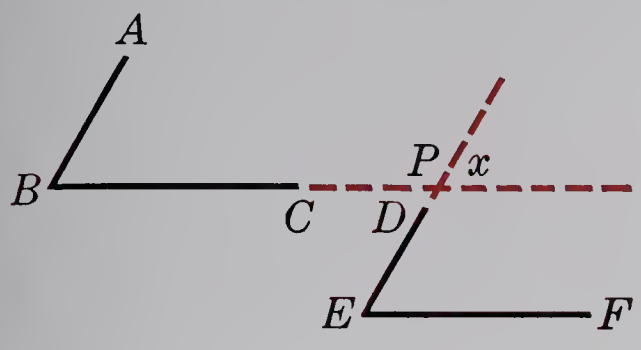
Viewed from the vertex, the side of an angle at the right is called the *right side* of the angle and the side at the left is called the *left side* of the angle. Thus AB is the right side of $\angle A$ and AC is the left side.

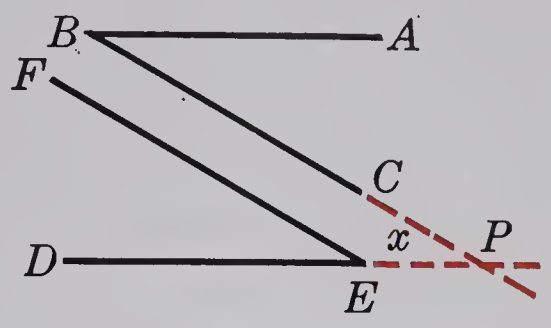




★ Theorem 8

117. If two angles have their sides parallel, right side to right side and left side to left side, the angles are equal.





Given $\measuredangle B$ and E with $BC \parallel EF$ and $BA \parallel ED$.

To prove that $\angle B = \angle E$.

Planning the Proof: 1. We can prove $\measuredangle =$ by §§ 50, 98, 111, 112.
2. We shall use §§ 111 and 112.

Proof

STATEMENTS	REASONS
1. Extend BC and ED.	1. Asmt. 13.
2. BC will intersect ED at some point P.	2. §§ 100, 108.
3. $BA \parallel EP$.	3. Why?
4. $\therefore \angle B = \angle x$.	4. Give reason for each figure.
5. $BC \parallel EF$.	5. Why?
6. $\angle E = \angle x$.	6. Why?
7. $\therefore \angle B = \angle E$.	7. Why?

EXERCISES

B

1. Prove: *If two angles have their sides parallel, right side to left side and left side to right side, the angles are supplementary.*

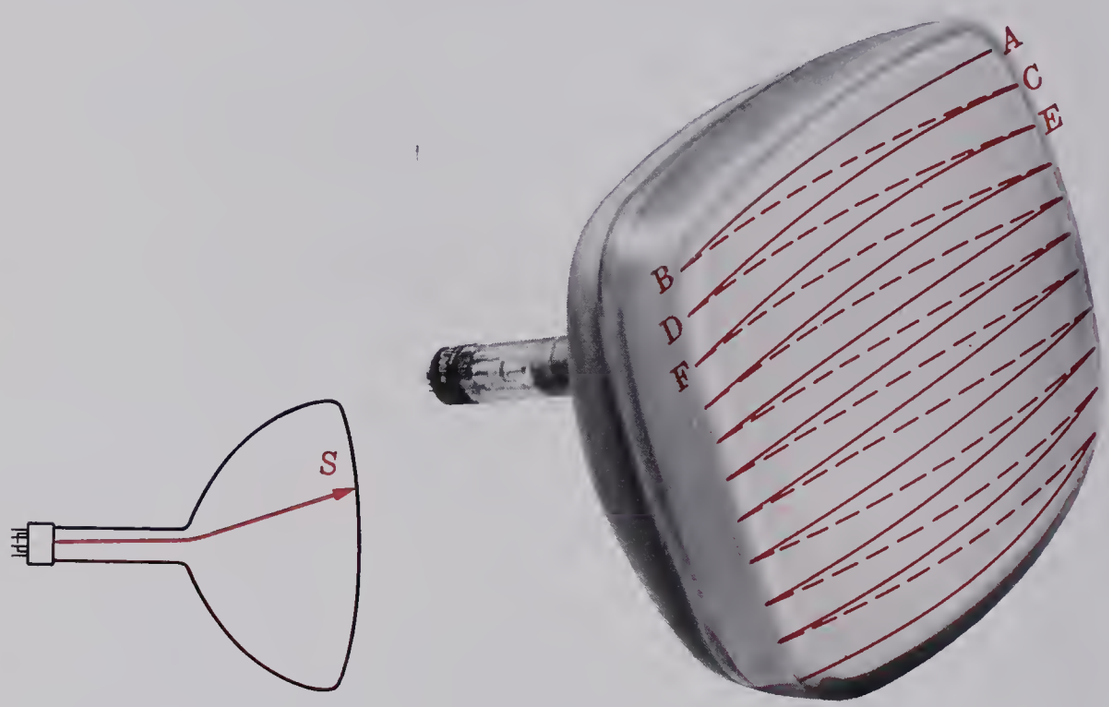
2. A line intersecting one of the equal sides of an isosceles triangle and parallel to the other forms a triangle having two equal angles.

118. Parallel Lines in Television

Your television screen is actually only one end of a tube, called a kinescope, which is shaped something like an ordinary electric light bulb

flattened at one end. A stream of electrically charged particles, called electrons, is shot from the small end of the tube. These particles form a beam which is made to trace a series of parallel lines on the inside of the television screen (S). The beam moves so rapidly that it can trace 525 parallel lines on the screen 30 times a second! Each complete group, or scanning, of 525 parallel lines gives one picture. There are, therefore, 30 pictures completed by the moving beam of electrons every second.

The electrons, or electrically charged particles, which reach the inside of the screen cause that part of the screen to glow. By a special device, many of the electrons are cut off and do not reach the screen, and this



part of the screen remains dark. It is a pattern of light and darkness which forms the images that you recognize as a picture. One picture of 525 lines replaces another so rapidly that you do not notice the end of one picture and the beginning of the next, and the effect of a motion picture is produced.

The diagram above shows the order in which the scanning beam of the television receiver forms parallel lines (only a few of the 525 lines which make up one complete picture are shown). The solid lines are those which act to form the picture on the screen, and the dotted lines show how the beam jumps back to begin scanning a new line. A complete diagram would show 525 solid lines and 525 dotted lines.

119. Indirect Reasoning in Life Situations (Optional)

Frequently we use indirect reasoning in everyday affairs when it is impossible to reach a conclusion by direct reasoning. To reach a correct conclusion by indirect reasoning we must know all the possible conclusions and eliminate all but one as false. Suppose Mr. Clay has three friends having Brown for their surname. One day Mr. Haines passes Mr. Clay on the street and says in passing, "I saw and talked to your friend Brown at the Rotary luncheon today." Mr. Clay had no direct way of knowing which Mr. Brown might have been at the Rotary luncheon, but he knew that one Mr. Brown was out of town on vacation and another Mr. Brown was confined to bed by illness. Could Mr. Clay tell which friend Brown had been at the luncheon?

EXERCISES

A

Try to solve the following:

1. One evening the picture on Mr. Frank's TV set was fluttering. When he changed from channel 8 to channel 12, the picture was good. What do you think was the cause of the poor picture, his TV set or the station?

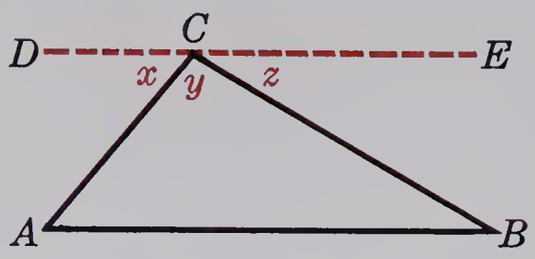
2. One evening Dick discovered that a light bulb in the kitchen would not glow. He knew that either the bulb was bad or that there was a bad fuse. He placed the bulb in a floor lamp and found the trouble. Explain how he knew the cause of the light failure.

3. At a church dinner three men, whose names were Fuller, Smith, and Baker, sat at the same table. One of the men was a teacher, one was a dentist, and the third was a machinist. Mr. Baker had never met Mr. Fuller before the dinner. The teacher had married the dentist's daughter and had bowled with the machinist. What was the teacher's name?

4. After Friday's baseball game coach Howard found a bag in one of the boys' locker rooms. He knew that it belonged to Ted, Bill, or Lawrence because no one else ever used the room. Looking into the bag, he found a bath towel, a pair of extra large baseball shoes, a right-hand fielder's glove, and other small articles. He knew that Ted and Lawrence threw with their left arms and Bill threw with his right arm. He also knew that only Ted had small feet. Which boy had left the bag?

★★ Theorem 9

120. The sum of the angles of a triangle is a straight angle.



Given the $\triangle ABC$.

To prove that $\angle A + \angle B + \angle y = 1$ st. $\angle$.

Planning the Proof: 1. We can prove that the sum of $\measuredangle s$ = another angle by Asmt. 9.

2. We shall use Asmt. 9.

Proof

STATEMENTS	REASONS
1. Through C draw $DE \parallel AB$.	1. Asmt. 37.
2. $\angle DCE$ is a st. $\angle$.	2. § 16.
3. $\angle x + \angle y + \angle z = 1$ st. $\angle$.	3. Asmt. 9.
4. $\angle x = \angle A$ and $\angle z = \angle B$.	4. Why?
5. $\therefore A + \angle y + \angle B = 1$ st. $\angle$.	5. Why?

NOTE. The followers of Pythagoras were the first to prove this theorem.

★ **121. Corollary I.** If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.

122. Corollary II. In a triangle there can be but one right angle or one obtuse angle.

★ **123. Corollary III.** An exterior angle of a triangle equals the sum of the two nonadjacent interior angles.

NOTE. It is thought that Philippus was the first to prove this corollary (about 380 B.C.).

124. Corollary IV. The acute angles of a right triangle are complementary.

★ **125. Corollary V.** If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, the triangles are congruent.

EXERCISES — Angles

A

1. Name all the ways we have had to prove (1) angles equal; (2) line segments equal; (3) triangles congruent; (4) angles supplementary; (5) angles complementary; (6) lines parallel; (7) lines perpendicular; (8) right triangles congruent (five ways). Have you located these references on pages 563–566?

2. Is it possible to have two right angles in a triangle? Why?

3. Is it possible for an isosceles triangle to be a right triangle? Why?

4. One acute angle of a right triangle equals 34°. How many degrees are there in the other acute angle?

5. How many degrees are in each angle of an equilateral triangle?

6. The vertex angle of an isosceles triangle contains 70°. Find the number of degrees in each of the other angles.

7. If one of the equal angles of an isosceles triangle contains 62°, find the number of degrees in the vertex angle.

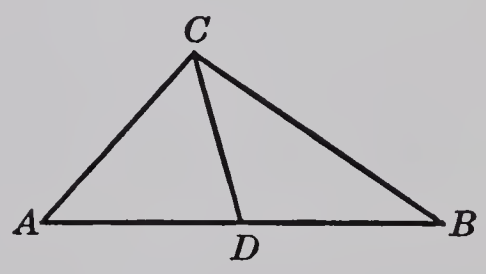
8. One of the angles of a triangle is 46°. Find each of the other two if their difference is 12°.

9. The angles of a triangle are in the ratio of 2, 3, and 4. Find the number of degrees in each angle. (Let the angles be $2x$, $3x$, and $4x$.)

10. The angles of a triangle are in the ratio of 3, 4, and 5. Find the number of degrees in each angle.

11. In this figure CD bisects $\angle ACB$, $\angle A = 48°$, and $\angle B = 34°$.

Find the number of degrees in $\angle ACB$; in $\angle DCB$.



12. An exterior angle of a triangle contains 110° and one nonadjacent interior angle contains 31°. Find the other two angles of the triangle.

13. Find the three angles of a triangle if the second is three times the first and the third is twice the second.

14. If two angles of a triangle are complementary, prove that the triangle is a right triangle.

B

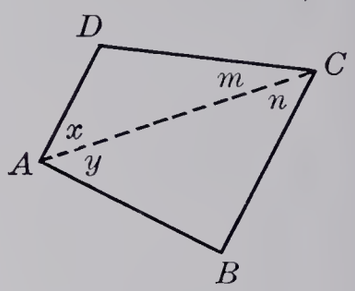
15. Prove that in an isosceles triangle the altitudes on the equal sides are equal.

16. One angle of a triangle exceeds a second by 26° and is 36° less than the third. Find the three angles.

17. If two right triangles have a leg and the opposite acute angle of one equal respectively to a leg and the opposite acute angle of the other, the triangles are congruent.

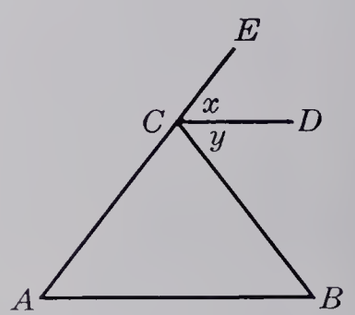
18. Are two triangles congruent if they have three angles of one equal respectively to three angles of the other? Explain.

19. $ABCD$ is a quadrilateral. Prove that $\angle A + \angle B + \angle C + \angle D = 2$ st. $\measuredangle$s.



20. Find the number of degrees in the angles formed by the bisectors of two of the angles of an equilateral triangle at their point of intersection.

21. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.



22. The perpendiculars drawn from any point in the bisector of an angle to the sides of the angle are equal.

23. If two parallels are cut by a transversal, the bisectors of the interior angles on the same side of the transversal are perpendicular to each other.

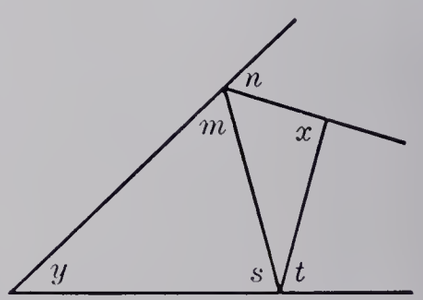
C

24. Perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides make equal angles with the base.

25. If from any point outside an acute angle perpendiculars are drawn to the sides of the angle, the angle between these perpendiculars is equal to the given angle.

26. The perpendicular from the vertex of the right angle of a right triangle to the hypotenuse divides the triangle into two triangles whose corresponding angles are respectively equal.

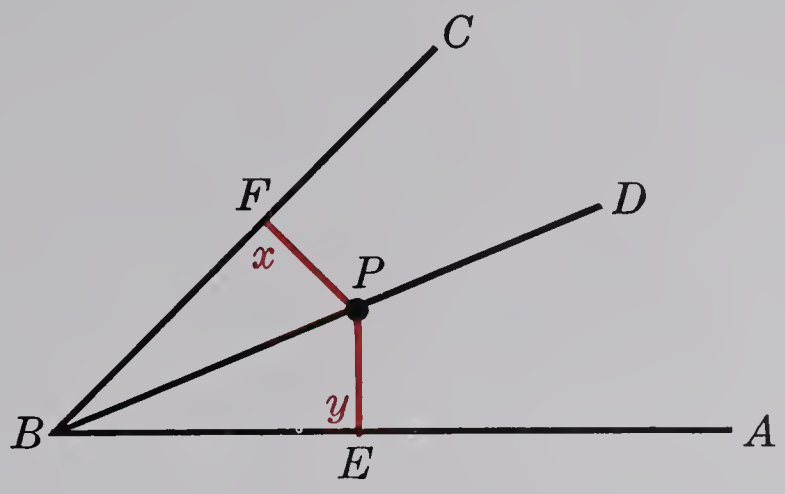
27. If the median of a triangle is equal to half of the side to which it is drawn, the triangle is a right triangle.



28. *Given* $\angle s = \angle t$ and $\angle m = \angle n$.
Prove that $\angle x = \angle m + \angle s - \angle y$.

★ Theorem 10

126. If a point is on the bisector of an angle, it is equidistant from the sides of the angle.



Given P any point in BD, the bisector of $\angle ABC$.

To prove that P is equidistant from BA and BC.

Planning the Proof: 1. We can prove line segments = by § 97.
2. We shall use § 97 a.

Proof

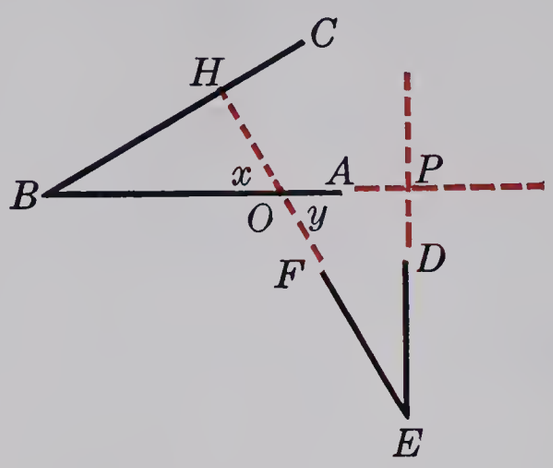
STATEMENTS	REASONS
1. Draw $PE \perp BA$ and $PF \perp BC$.	1. At any point in a line one $\perp$ and only one can be drawn to the line.
2. $\angle x$ and $\angle y$ are rt. $\measuredangle$s.	2. § 14.
3. $\triangle PEB$ and $\triangle PBF$ are rt. $\triangle$s.	3. Why?
4. $BP = BP$.	4. Why?
5. BP bisects $\angle ABC$.	5. Why?
6. $\angle EBP = \angle PBF$.	6. Why?
7. $\triangle BPE \cong \triangle PFB$.	7. § 125.
8. $\therefore PE = PF$.	8. Why?
9. $\therefore P$ is equidistant from BA and BC.	9. § 15.

EXERCISES

In $\triangle ABC$, the point O is the intersection of the bisector of $\angle A$ and the bisector of $\angle B$. Prove that O is equidistant from AC and BC.

★ Theorem 11

127. If two angles have their sides perpendicular, right side to right side and left side to left side, the angles are equal.



Given the ∡s B and E with $BA \perp ED$ and $BC \perp EF$.

To prove that $\angle B = \angle E$.

Planning the Proof: 1. We can prove ∡s = by §§ 50, 98, 111, 112, 117, 121.
2. We shall use § 50 b

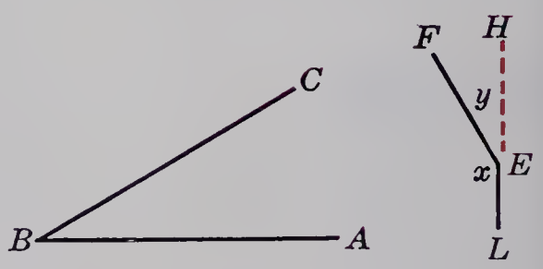
Proof

STATEMENTS	REASONS
1. If necessary, prolong the sides of the ∡s until the ⊥ sides intersect.	1. Asmt. 13.
2. ∡s B and x are comp.	2. Why?
3. ∡s E and y are comp.	3. Why?
4. But $\angle x = \angle y$.	4. Why?
5. $\therefore \angle B = \angle E$.	5. § 50 b.

EXERCISE

B

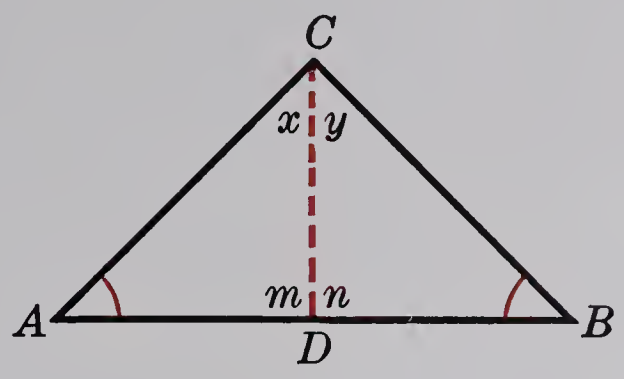
If two angles have their sides perpendicular, right side to left side and left side to right side, the angles are supplementary.



SUGGESTIONS. In the figure, $EF \perp BC$ and $EL \perp BA$. Use Theorem 11 to show the relation between $\angle B$ and $\angle y$.

★★ Theorem 12

128. If two angles of a triangle are equal, the sides opposite these angles are equal.



Given the $\triangle ABC$ with $\angle A = \angle B$.

To prove that $AC = CB$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 126.
2. We shall use § 97 a.

Proof

STATEMENTS	REASONS
1. Let CD bisect $\angle C$.	1. Asmt. 18.
2. $\angle x = \angle y$.	2. Const.
3. $\angle A = \angle B$.	3. Why?
4. $\therefore \angle m = \angle n$.	4. § 121.
5. $CD = CD$.	5. Why?
6. $\therefore \triangle ADC \cong \triangle BDC$.	6. Why?
7. $\therefore AC = BC$.	7. Why?

★ **129. Corollary.** An equiangular triangle is equilateral.

EXERCISES

A

1. State three ways of proving line segments equal. Find these methods on page 563, § 567.

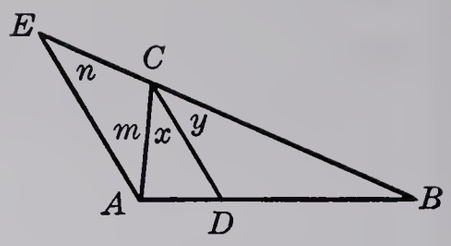
2. The bisectors of the equal angles of an isosceles triangle form with the base another isosceles triangle.

3. A line perpendicular to the bisector of an angle forms with the sides of the angle an isosceles triangle (use §§ 121 and 128).

4. If the midpoints of the sides of an equilateral triangle are joined in order, four other equilateral triangles are formed.

5. *Given* the $\triangle ABC$, $\angle x = \angle y$, and $AE \parallel CD$ meeting BC produced at E.

Prove that $AC = EC$.

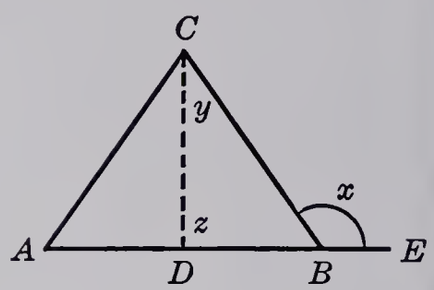


6. A line intersecting the equal sides of an isosceles triangle and parallel to the base forms another isosceles triangle.

B

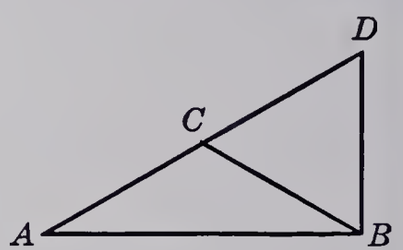
7. Find the angles of an isosceles triangle if one of the base angles is twice the vertex angle.

8. An exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one half the vertex angle.



9. If an exterior angle of a triangle is supplementary to one of the nonadjacent interior angles, the triangle is isosceles.

10. If the angle at the vertex of isosceles triangle ABC is equal to four times one of the equal angles, and if perpendicular BD to base AB meets AC produced at D, then $\triangle BCD$ is equilateral.



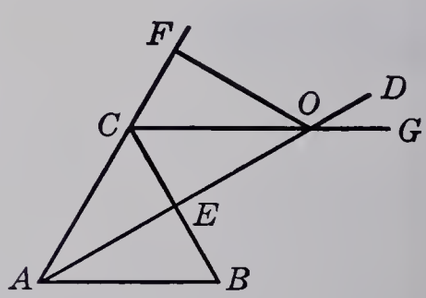
11. A line drawn through a point in the bisector of an angle parallel to one side of the angle forms with the other side and the bisector an isosceles triangle.

C

12. Prove Theorem 11 by drawing rays through E parallel to BA and BC.

13. *Given* the equilateral triangle ABC with AD bisecting $\angle A$, CG bisecting exterior $\angle BCF$ and $OF \perp AC$.

Prove that $AE = OF$.



SUGGESTION. Prove the four $\triangle \cong$.

14. If two lines are cut by a transversal so that the bisectors of the interior angles on the same side of the transversal are perpendicular to each other, the lines are parallel:

130. Converse Theorems

In § 3 we learned that all definitions are reversible. Thus we may say that "Two lines are perpendicular to each other if one of them forms two equal adjacent angles with the other," or "If two lines are perpendicular to each other, one of them forms equal adjacent angles with the other." Every definition is a *converse* of another and both are true. Every theorem has a converse, but the converse may not be true.

Let us compare Theorem 3, page 106, with Theorem 12, page 147.

Theorem 3	Theorem 12
Hypothesis: If two sides of a triangle are equal,	***Hypothesis:*** If two angles of a triangle are equal,
Conclusion: the angles opposite these sides are equal.	***Conclusion:*** the sides opposite these angles are equal.

Do you see that the conclusion of Theorem 3 is the hypothesis of Theorem 12, and that the hypothesis of Theorem 3 is the conclusion of Theorem 12? Each of these two theorems is the converse of the other.

Some theorems have more than one converse. For example, the theorem "The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base" has five converses as follows:

Converse 1. If a line is perpendicular to and bisects the base of a triangle, it bisects the vertex angle and the triangle is isosceles.

Converse 2. If the bisector of the vertex angle of a triangle bisects the base, it is perpendicular to the base and the triangle is isosceles.

Converse 3. If the bisector of the vertex angle of a triangle is perpendicular to the base, it bisects the base and the triangle is isosceles.

Converse 4. If a line through the vertex of an isosceles triangle bisects the base, it is perpendicular to the base and bisects the vertex angle.

Converse 5. If a line through the vertex of an isosceles triangle is perpendicular to the base, it bisects the base and also the vertex angle.

Do you see how these converses are formed? Converse 1 is formed by interchanging both conclusions with both hypotheses; and each of the converses 2, 3, 4, and 5 is formed by interchanging one conclusion with one hypothesis.

We define the converse of a theorem as follows:

A *converse* of a theorem is the theorem obtained by interchanging any number of conclusions with an equal number of hypotheses of the original theorem.

EXERCISES Converses

A

State the converses of the statements in Exs. 1–9 and determine, if you can, whether they are true or false:

1. If a man lives in Indianapolis, he lives in Indiana.

2. If two parallels are cut by a transversal, the alternate interior angles are equal.

3. If two triangles are congruent, the corresponding angles are equal.

4. If a point is on the bisector of an angle, it is equidistant from the sides of the angle.

5. All horses are animals.

6. Right angles are equal angles.

7. If a point lies on the perpendicular bisector of a line segment, it is equidistant from the end points of the line segment.

8. The acute angles of a right triangle are complementary.

9. A boy lives in Los Angeles only if he lives in California.

10. Write all the converses of the following theorem:

The median to the base of an isosceles triangle is perpendicular to the base and bisects the vertex angle.

131. Inverse Theorems

The *inverse* (or *opposite*) *of a theorem* having one conclusion is formed by contradicting the conclusion and one hypothesis. For example:

Theorem. If two lines form equal alternate interior angles with a transversal, the lines are parallel.

Inverse Theorem. If two lines form unequal alternate interior angles with a transversal, the lines are not parallel.

Theorem. If two angles of a triangle are equal, the sides opposite these angles are equal.

Inverse Theorem. If two angles of a triangle are unequal, the sides opposite these angles are unequal.

Theorem. The bisector of the vertex angle of an isosceles triangle bisects the base.

Inverse Theorem 1. The bisector of the vertex angle of a scalene triangle does not bisect the base.

Inverse Theorem 2. If a line through the vertex of an isosceles triangle does not bisect the vertex angle, it does not bisect the base.

EXERCISES Inverses

A

State the inverses of the following statements and determine, if you can, whether they are true or false:

1. If two angles are right angles, they are equal.
2. If two sides of a triangle are equal, the angles opposite these sides are equal.
3. When there is ice on the road, the road is slippery.
4. If a man lives in Indianapolis, he lives in Indiana.
5. If two triangles are congruent, the corresponding angles are equal.
6. If a point lies on the perpendicular bisector of a line segment, it is equidistant from the end points of the line segment.

132. Contrapositive of a Theorem

A contrapositive of a theorem having one conclusion is formed by contradicting the conclusion and one hypothesis and interchanging their respective places. For example:

Theorem. If two angles are vertical angles, they are equal.

Contrapositive. If two angles are not equal, they are not vertical angles.

Theorem. If two lines form equal alternate interior angles with a transversal, the lines are parallel.

Contrapositive. If two nonparallel lines are cut by a transversal, the alternate interior angles are unequal.

EXERCISES Contrapositives

A

State the contrapositives of the following statements and tell whether they are true:

1. If two angles are equal, their complements are equal.

2. If a man is president of the United States, he is at least 35 years old.

3. If two angles are right angles, they are equal.

133. Equivalent Statements

Let us use circles to discover a relation between a theorem and its converse and between its converse and its inverse.

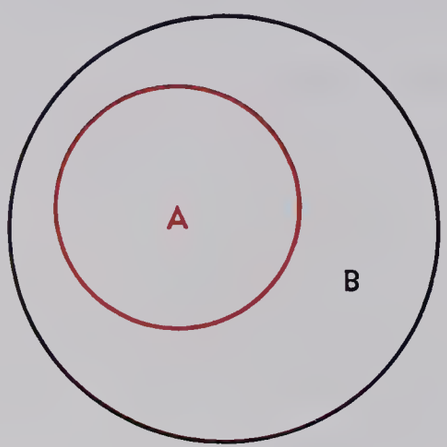


Fig. 1

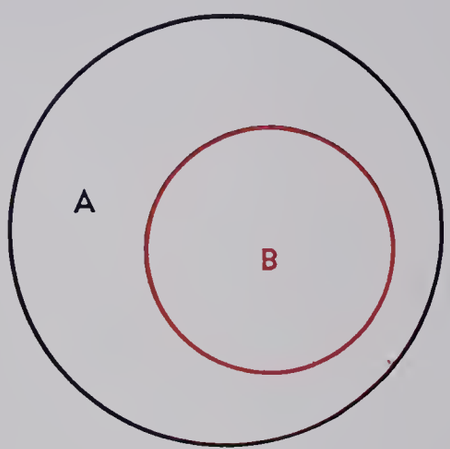


Fig. 2

In Figure 1, A is the portion of plane enclosed by the smaller circle and B is the portion of plane enclosed by the larger circle. In Figure 2, B is the portion of plane enclosed by the smaller circle and A is the portion of plane enclosed by the larger circle.

The Theorem

If a point is in A, it is in B.

or

If A, then B.

Its Converse

If a point is in B, it is in A.

or

If B, then A.

Its Contrapositive

If a point is not in B, it is not in A.

or

If not B, then not A.

Its Inverse

If a point is not in A, it is not in B.

or

If not A, then not B.

Which of the four statements above can be represented by Fig. 1? by Fig. 2? Which of the three other statements expresses the same fact as the theorem? Which of the three other statements expresses the same fact as the converse? If you have answered these questions correctly, you can accept the following assumptions.

134. Assumption 38. If a theorem having one conclusion is true, its contrapositives are true; and conversely, if a contrapositive of a theorem is true, the theorem is true.

Assumption 39. If a converse of a theorem having one conclusion is true, the corresponding inverse is true; and conversely, if an inverse of a theorem is true, the corresponding converse is true.

The relationships between a theorem, its converse, inverse, and contrapositive are illustrated by the following diagram:

If one of the statements joined by the arrows is true, the other is true. Thus, if a theorem having one conclusion is true, its contrapositives are true, and if a contrapositive of a theorem is true, the theorem is true. Also, if a converse of a theorem having one conclusion is true, the corresponding inverse is true, and vice versa.

EXERCISES

1. State the contrapositive of Theorem 6. Why is the contrapositive true?

2. State the inverse of the converse of Theorem 6.

3. Complete: The converse of Theorem 3 is Theorem __?__.

4. What is the converse of Theorem 10?

5. State the two converses of Theorem 8.
   Is either of the two converses true?

6. State the two inverses of Theorem 8.
   Is either of the inverses true?

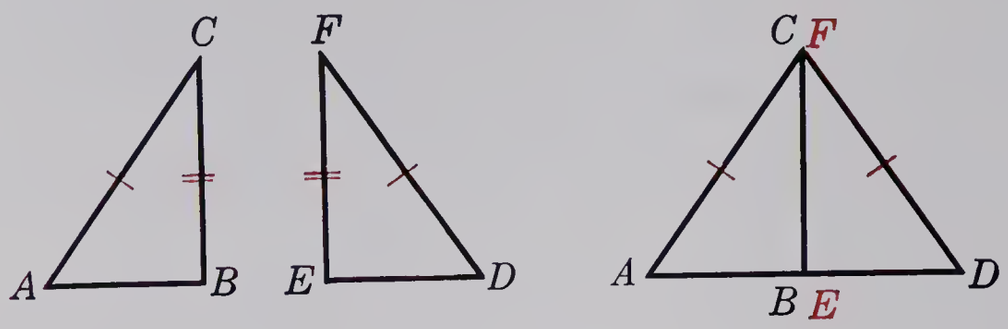
7. State the two contrapositives of Theorem 8.
   Is either of the contrapositives true?

8. State the inverse of § 124, Corollary 4.
   Is the inverse true?

9. State the contrapositive of § 124, Corollary 4.
   Is the contrapositive true?

★★ Theorem 13

135. If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.



Given rt. △ ABC and DEF with ∠ B and E rt. ∠, $AC = DF$, and $BC = EF$.

To prove that $\triangle ABC \cong \triangle DEF$.

Planning the Proof: 1. We can prove △ ≅ by §§ 95, 96, 125.
2. We shall use § 125.

Proof

STATEMENTS	REASONS
1. $BC = EF$.	1. Why?
2. Place the two △ so that BC coincides with EF, and so that A and D lie on opposite sides of BC.	2. Why possible?
3. ∠ ABC and DEF are rt. ∠.	3. Why?
4. ∴ ∠ ABD is a st. ∠.	4. Why?
5. ∴ ABD is a st. line.	5. Why?
6. ∴ ADC is a △.	6. Why?
7. $AC = DC$.	7. Why?
8. ∴ ∠ $A = \angle D$.	8. Why?
9. ∴ rt. △ $ABC \cong$ rt. △ DBC ($\triangle DEF$).	9. Why?

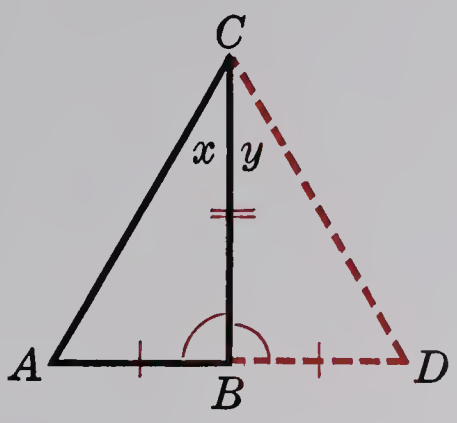
If ∠ B and E were not rt. ∠, would ADC be a △?

EXERCISES

1. *Prove:* If two altitudes of a triangle are equal, the triangle is isosceles.
2. *Given* △ ABC having $BD \perp AC$, $AE \perp BC$, and $BD = AE$. *Prove* that $AD = BE$.

★★ Theorem 14

136.* If one acute angle of a right triangle is 30°, the side opposite this angle is one half the hypotenuse.



Given rt. $\triangle ABC$ with $\angle ACB = 30°$ and $\angle ABC$ the rt. $\angle$.

To prove that $AB = \frac{1}{2} AC$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 126, 128, 129.

2. We shall use § 129.

Proof

STATEMENTS	REASONS
1. Extend AB, making $BD = AB$.	1. Asmt. 13.
2. Draw CD.	2. Asmt. 11.
3. $\triangle ABC \cong \triangle DBC$.	3. Give proof in full.
4. $\therefore \angle D = \angle A = 60°$.	4. Give proof. See § 124.
5. $\angle y = \angle x = 30°$.	5. Give proof.
6. $\therefore \angle ACD = 60°$.	6. Why?
7. $\therefore \triangle ADC$ is equiangular.	7. Why?
8. $\therefore \triangle ADC$ is equilateral.	8. Why?
9. $AB + BD = AC$.	9. Why?
10. $2\,AB = AC$.	10. Why?
11. $\therefore AB = \frac{1}{2} AC$.	11. Why?

NOTE. Plato called the 30°-60° right triangle "the most beautiful right-angled scalene triangle."

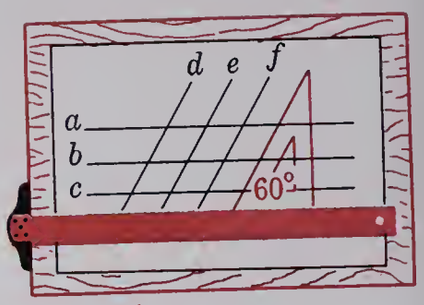
EXERCISE

In $\triangle DEF$, $\angle F = 2\ \angle E$, $\angle D = \angle F + \angle E$, and $DF = 6$ in. Find the length of FE.

*If desired, "Inequalities," pp. 505–511 inclusive, may be studied before Theorem 14.

EXERCISES

A



1. The 30°-60° right triangle and the isosceles right triangle are used in mechanical drawing. Why is $a \parallel b \parallel c$ and $d \parallel e \parallel f$?

2. *Prove:* If one acute angle of a right triangle is twice the other acute angle, the side opposite the smaller angle is half the hypotenuse.

3. If two angles of a triangle are 80° and 50°, prove that the triangle is isosceles.

4. If the exterior angle at the base of an isosceles triangle contains 135°, prove that the triangle is a right triangle.

5. If one angle of a right triangle is 30°, find the size of the angle formed by the bisectors of it and the other acute angle.

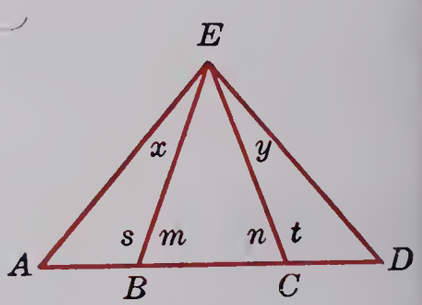
137. Analytic and Synthetic Methods of Proof

The method that you have been using for proofs is known as the *synthetic method*. Another method of proof, known as the *analytic method*, is often helpful in discovering the synthetic method.

Example. *Given* $\triangle ADE$ with $AE = DE$ and $\angle x = \angle y$.
To prove that $\angle m = \angle n$.

Analysis. Let us suppose that we do not know how to prove that $\angle m = \angle n$. Before attempting to write the proof in formal order, we shall try to discover the analytic proof.

We are to prove two angles equal. We recall the various methods of proving angles equal. These methods are found to be §§ 50, 98, 111, 112, and 117. After studying these methods, we believe we cannot use any but §§ 50 c and 98 a. Why can we not use § 98 b? § 50 d?



We can use § 98 a if we know $BE = CE$. Why? Or we can use § 50 c if we know $\angle s = \angle t$. Why?

Is $BE = CE$ if $\angle s = \angle t$? Is the converse true?

We can prove $BE = CE$ if we can prove $\triangle ABE \cong \triangle DCE$. Why? $AE = DE$ and $\angle x = \angle y$. Why? To prove these triangles congruent, we need three parts of one equal respectively to three parts of the other.

Now we can prove them congruent if we can show that $\angle A = \angle D$ (A.S.A.). We know that $\angle A = \angle D$. Why? Then $\angle m = \angle n$.

The analytic proof can be written as follows:

Analytic Proof

STATEMENTS	REASONS
1. $\angle m = \angle n$ if $BE = CE$.	1. § 98 a.
2. $BE = CE$ if $\triangle ABE \cong \triangle CED$.	2. § 97 a.
3. $\triangle ABE \cong \triangle CED$ if $\angle A = \angle D$, $AE = DE$, and $\angle x = \angle y$.	3. A.S.A.
4. $\angle A = \angle D$ if $AE = DE$.	4. § 98 a.
5. $AE = DE$.	5. Given.
6. $\angle x = \angle y$.	6. Given.
7. $\therefore \angle m = \angle n$.	7. Statements 1–6.

The synthetic proof can now be written by reversing the order of the above statements.

Synthetic Proof

STATEMENTS	REASONS
1. $\angle x = \angle y$.	1. Given.
2. $AE = DE$.	2. Given.
3. $\angle A = \angle D$.	3. § 98 a.
4. $\triangle ABE \cong \triangle CED$.	4. A.S.A.
5. $BE = CE$.	5. § 97 a.
6. $\angle m = \angle n$.	6. § 98 a.

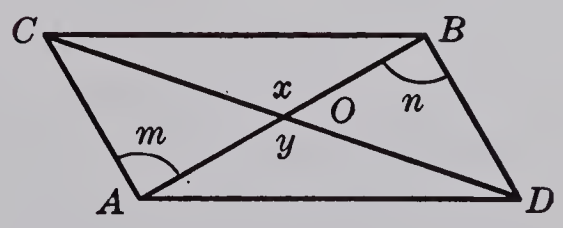
If in the future you cannot prove a theorem by the synthetic method, try the analytic method.

EXERCISES

A

1. *Given* figure $ADBC$ with $AO = OB$, $\measuredangle m$ and n rt. $\measuredangle$, and AOB and COD intersecting st. lines.

Prove that $CB = AD$.



Analysis.

$CB = AD$ if $\triangle COB \cong \triangle AOD$.

$\triangle COB \cong \triangle AOD$ if $AO = OB$, $CO = OD$, and $\angle x = \angle y$.

But $AO = OB$ and $\angle x = \angle y$. Why?

$CO = OD$ if $\triangle CAO \cong \triangle BOD$. Why?

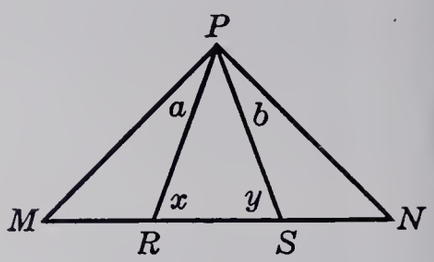
$\triangle CAO \cong \triangle BOD$ if $\angle m = \angle n$, $\angle AOC = \angle BOD$, and $AO = OB$. Why?

But $\angle m = \angle n$, $\angle AOC = \angle BOD$, and $AO = OB$. Why?

Write the analytic proof in full, following the model above.

2. *Given* $\triangle MNP$ with $PM = PN$ and $MR = SN$.

Prove $PR = PS$.



Ex. 2-4

3. *Given* $\angle x = \angle y$ and $\angle a = \angle b$.

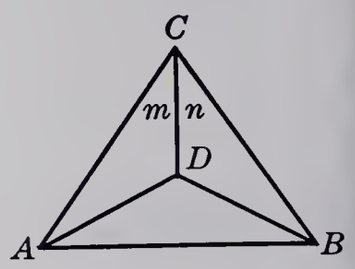
Prove $\angle M = \angle N$.

4. *Given* $\angle x = \angle y$ and $MR = SN$.

Prove that $MP = NP$.

5. *Given* $\triangle ABC$ with $\angle DAB = \angle DBA$ and $\angle CDA = \angle CDB$.

Prove $\angle m = \angle n$.



Ex. 5-6

6. *Given* $CA = CB$ and $\angle m = \angle n$.

Prove $\angle DAB = \angle DBA$.

7. *Given* $\triangle ABC$ with $\angle A = 30°$, $\angle C = 60°$, M a point in AC, and $\angle MBC = 60°$.

Prove $AM = BM = CM$.

8. Prove that two isosceles triangles are congruent if the base and vertex angle of one are equal respectively to the base and vertex angle of the other.

9. If the altitudes drawn to two sides of a triangle are equal, the triangle is isosceles.

10. Equilateral triangles BCD and ACE are constructed outward on $\triangle ABC$.

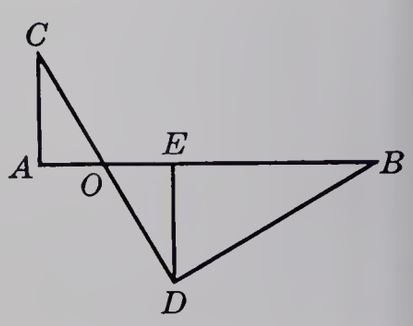
Prove that distance AD = distance BE.

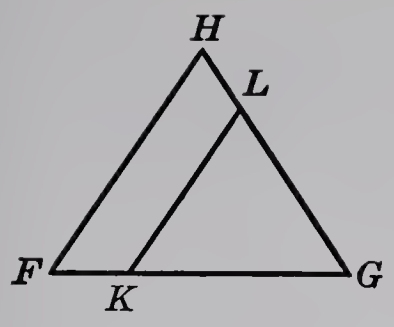
11. If two lines are cut by a transversal so that the bisectors of the alternate interior angles are parallel, the lines are parallel.

B

12. *Given* CD intersecting AB at O, $AC \perp AB$, $DE \perp AB$, and $CD \perp DB$.

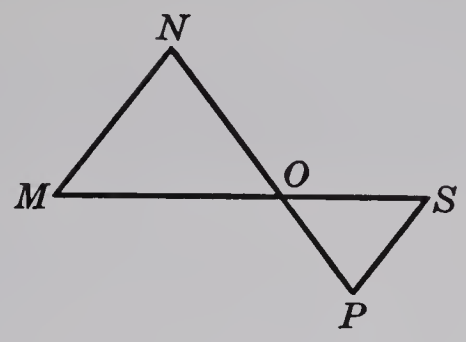
Prove that the corresponding angles of $\triangle AOC$ and DBE are equal.





13. *Given* isosceles $\triangle FGH$ with $FH = GH$ and $KL \parallel FH$.

Prove $KL = LG$.



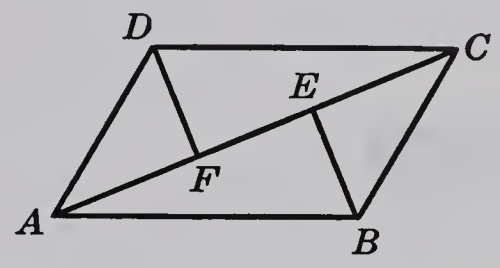
14. *Given* NP intersecting MS at O with $MN = NO$ and $OP = PS$.

Prove as many facts as you can.

C

15. If the bisector of an exterior angle at one vertex of a triangle is parallel to the side opposite the vertex, the triangle is isosceles.

16. If the opposite sides of quadrilateral $ABCD$ are equal, then DF and BE, the perpendiculars to the diagonal AC, are equal.



17. If, from a point within a right angle, perpendiculars are drawn to the sides of the right angle and each perpendicular is produced its own length, the line joining the extremities of the produced lines and the vertex of the right angle is a straight line.

18. *Given* the quadrilateral $ABCD$ with $AB \parallel DC$, $\angle ADC = \angle BCD$, and E and F points in CD such that $CE = DF$.

Prove $AE = BF$.

SUGGESTIONS. Draw $\perp$s from D and C to AB. Prove $AD = BC$.

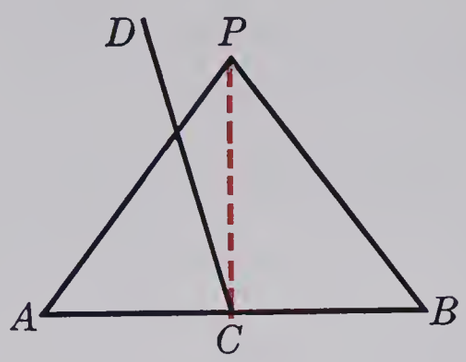
138. Proof by the Coincidence Method

The theorem on the next page is proved by a form of indirect proof known as the *coincidence method*. It consists in drawing a geometric figure and proving that it coincides with a given geometric figure. We know that if two figures coincide, each has all the properties of the other.

In Theorem 15 on the next page, we are to prove that a certain point lies on a certain line. At present we cannot prove it directly. We prove it indirectly by showing that an auxiliary line on which the given point lies coincides with the given line.

★ Theorem 15

139. A point equidistant from the end points of a line segment lies on the perpendicular bisector of the line segment.



Given CD the $\perp$ bisector of AB, and point P such that $PA = PB$.
To prove that P lies on CD.

Planning the Proof: 1. We have no direct method.
2. We shall use the indirect method.

Proof

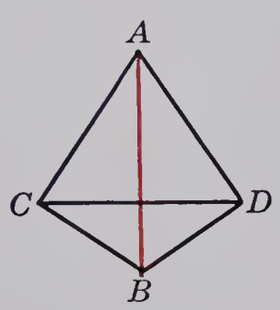
STATEMENT	REASONS
1. Through C, the midpoint of AB, draw PC.	1. Asmt. 17, 11.
2. $AC = CB$.	2. Why?
3. $PA = PB$.	3. Why?
4. $PC = PC$.	4. Why?
5. $\therefore \triangle ACP \cong \triangle CBP$.	5. Why?
6. $\angle PCA = \angle PCB$.	6. Why?
7. $PC \perp AB$.	7. Why?
8. $\therefore PC$ coincides with CD and P lies on CD.	8. Asmt. 20.

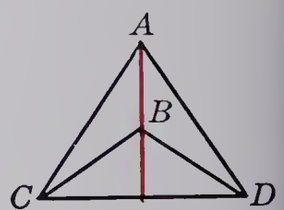
★ **140. Corollary.** A line passing through two points each equidistant from the end points of a line segment is the perpendicular bisector of the segment.

SUGGESTIONS. Both A and B lie on the $\perp$ bisector of CD. Why? How many lines can be drawn through A and B?

EXERCISE. Why is the inverse of Theorem 2 true?

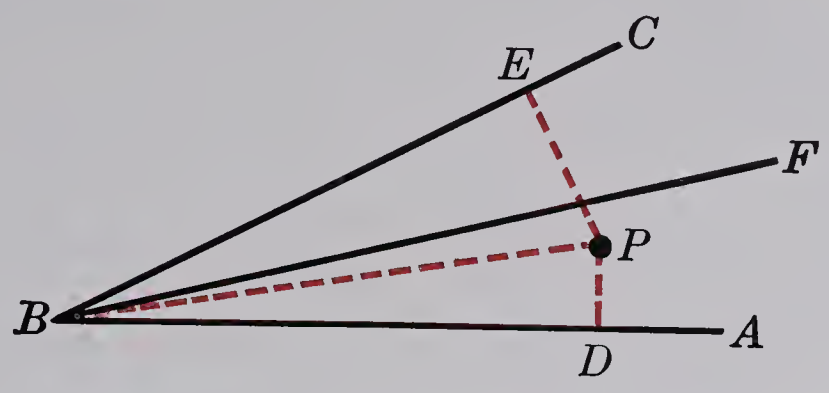
NOTE. If desired, pages 190–198 may be studied at this time.





★ Theorem 16

141. If a point is equidistant from the sides of the angle, it is on the bisector of the angle.



Given BF the bisector of $\angle ABC$, and point P equidistant from BA and BC.

To prove that P lies on BF.

Planning the Proof: 1. We can prove that a point lies on a line by § 139.
2. We shall use the indirect method.

Proof

STATEMENTS	REASONS
1. Draw BP.	1. Why possible?
2. Draw $PD \perp BA$ and $PE \perp BC$.	2. Asmt. 20.
3. $\angle PDB$ and $\angle PEB$ are rt. $\measuredangle$s.	3. Why?
4. $\triangle PDB$ and $\triangle PEB$ are rt. $\triangle$s.	4. Why?
5. $PE = PD$.	5. § 15.
6. $BP = BP$.	6. Why?
7. Rt. $\triangle PDB \cong$ rt. $\triangle PEB$.	7. § 135.
8. $\angle PBD = \angle PBE$.	8. Why?
9. BP bisects $\angle ABC$.	9. Why?
10. But BF bisects $\angle ABC$.	10. Why?
11. $\therefore BP$ coincides with BF and P lies on BF.	11. Asmt. 18.

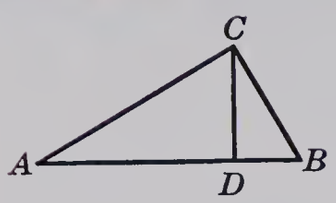
EXERCISES

1. *Given* $\triangle ABC$ with point P on the bisectors of $\angle A$ and $\angle B$.
Prove that P is on the bisector of $\angle C$.

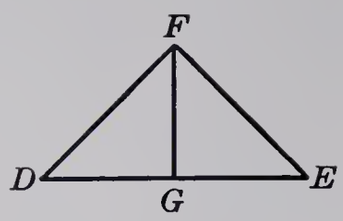
2. *Given* $\triangle ABC$ with point P the intersection of the bisectors of $\angle A$ and $\angle B$ and $AC = BC$.
Prove that P is on the perpendicular bisector of AB.

EXERCISES

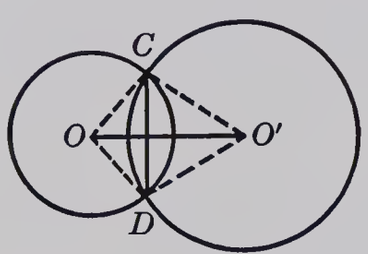
A



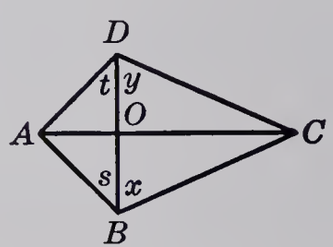
1. *Given* rt. $\angle ACB$, $\angle A = 30°$, $CD \perp AB$, $AC = 866$, $AB = 1000$.
Find CD, BC, DB, and AD.



2. *Given* $\triangle DEF$ with $DF = EF$ and $DG = GE$.
Prove $GF \perp DE$.



3. *Given* ⓢ O and O' intersecting at C and D.
Prove that OO' is the $\perp$ bisector of CD.



4. *Given* figure $ABCD$ with $\angle s = \angle t$ and $\angle x = \angle y$.
Prove $OB = OD$.

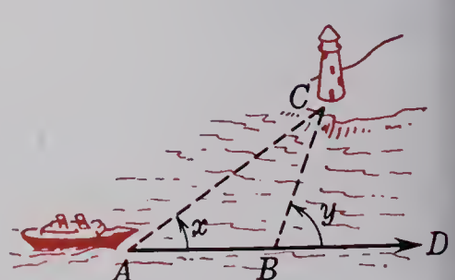
B

5. If from a point in a perpendicular to a line, equal oblique line segments are drawn to the line, they cut off equal segments from the foot of the perpendicular.

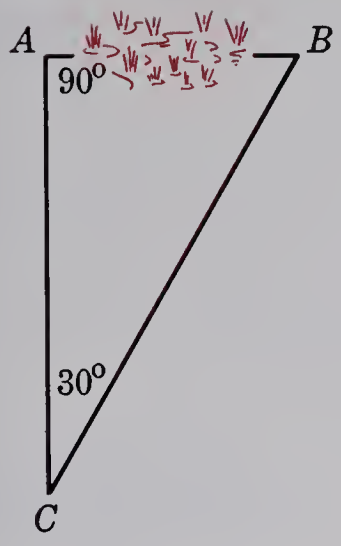
6. Prove informally by § 139 and Asmt. 38, § 134:
If a point is not on the perpendicular bisector of a line segment, it is not equidistant from the end points of the segment.

C

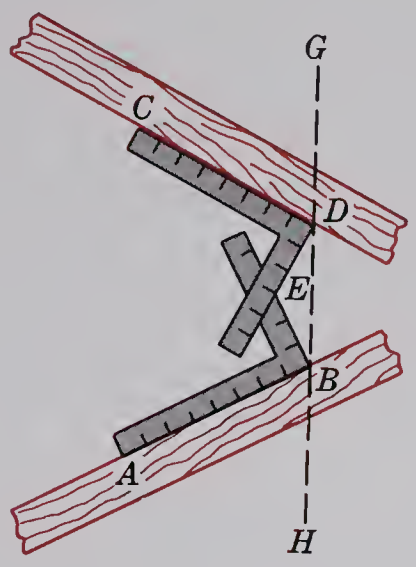
7. The distance of a ship, sailing in the direction AD, from a lighthouse can be found by the following method:
At A, $\angle x$ is measured; the ship is steered in a straight course until a position B is found where $\angle y = 2\ \angle x$; the distance AB, having been measured by the log of the ship, is said to equal the distance from B to the lighthouse. Can you prove that $AB = BC$?



EXERCISES Applications

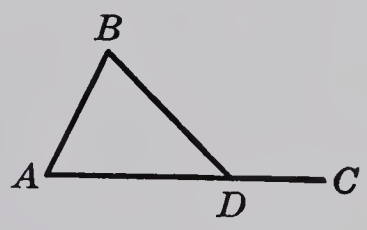


1. If A is visible from B but cannot be measured on account of a swamp, show what measurements are needed in the figure above to determine AB. Point C can be determined by trial when $\angle ACB = 30°$.

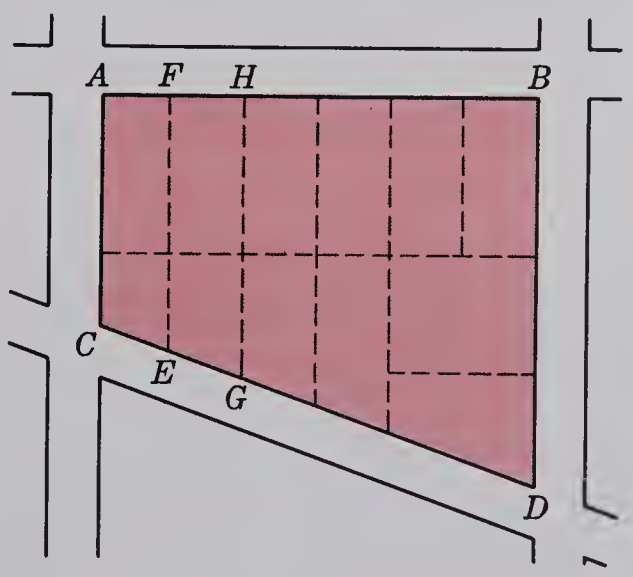


2. The figure above shows how to cut two converging timbers on a line, GH, which makes equal angles with the timbers. Two carpenter's squares are placed so that $BE = DE$. Prove that $\angle ABD = \angle CDB$.

3. Distances of ships from objects on land are often found in the following manner: A sailor, whose ship is at A, moving in direction AC, measures the $\angle A$ between his direction and an object B on shore. When he reaches a point D where $\angle CDB$ is twice $\angle A$, he knows that $DB = DA$. Prove that this is true. The distance AD is obtained from the ship's log.

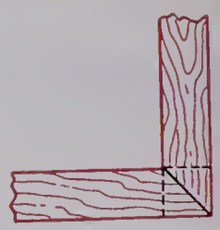


4. A real-estate man wishes to divide the plot of ground $ABDC$ into lots by constructing lines parallel to the street, AC. If $\angle ACE$ is 110°, how many degrees must he make $\angle CEF$, EGH, etc.? If AC and BD are $\perp AB$, how many degrees are in $\angle CDB$?

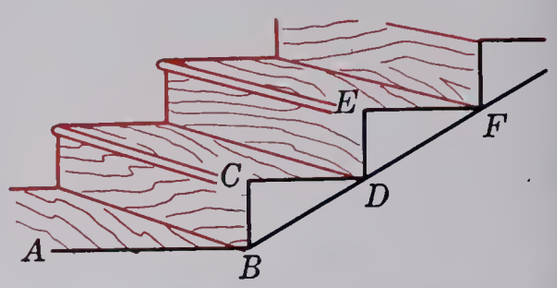


5. Explain by an illustration what measurements a carpenter could make to saw a board at a 45° angle.

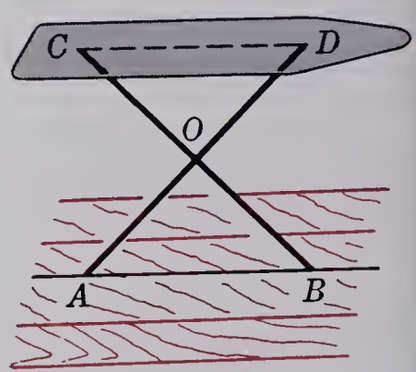
6. At what angle must a carpenter cut each of two 2×4's to form a right angle as shown in the figure?



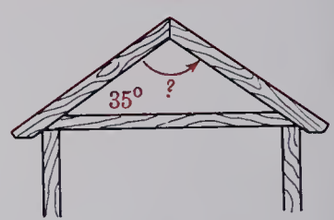
7. The timber BF supports a stairway and makes an angle of $150°$ with the floor AB. What size must the carpenter make the angles CBD and CDB of $\triangle CBD$ so that $CB \perp AB$ and $CD \parallel AB$? Compare the lengths of BC and BD.



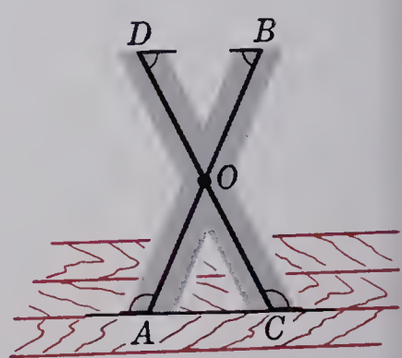
8. An ironing board is supported as shown in the figure. If $AO = OD$ and $BO = OC$, prove that the ironing board CD is parallel to the floor AB. How do the lengths of AB and CD compare? Is the ironing board raised or lowered when C remains fixed and D is moved away from C?



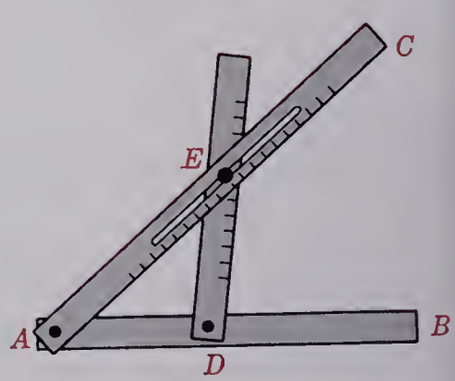
9. The rafters of a roof make an angle of $35°$ with the level line. What angle do the rafters form at the ridge? Why are the rafters of equal length?



10. In making a sawbuck a boy nails two pieces of wood together so that $AO = OC$, $OD = OB$, and $\angle AOC = 50°$. At what angle must he cut the pieces of wood at A and C so that they will stand firmly on a floor? How many degrees must $\measuredangle D$ and B contain in order that DB be parallel to AC? Where should he brace the sawbuck to make it more rigid?

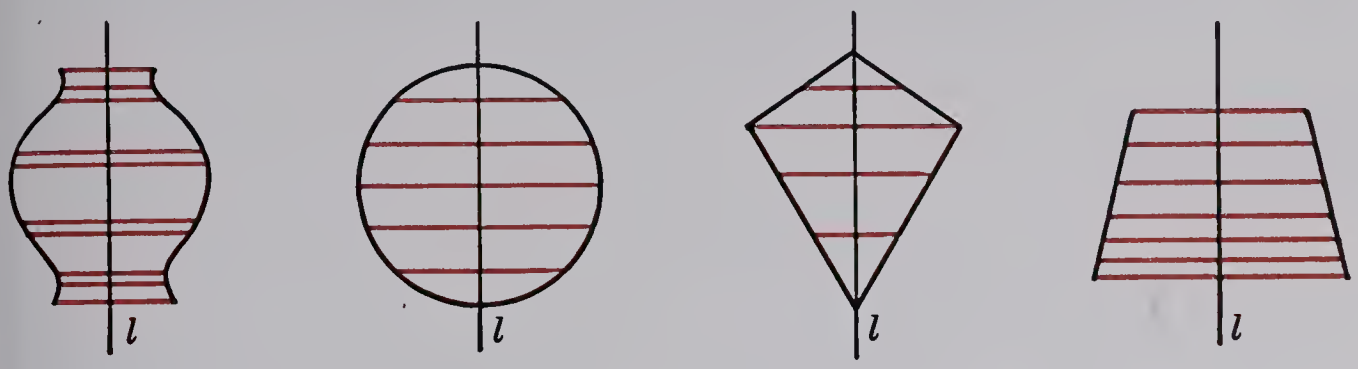


11. The figure at the right illustrates an instrument that can be used to find half an angle. The pin at E moves in a slot cut in bar AC. $AD = DE$. Prove that if BDE is adjusted to fit an angle, the angle at A is half of it.



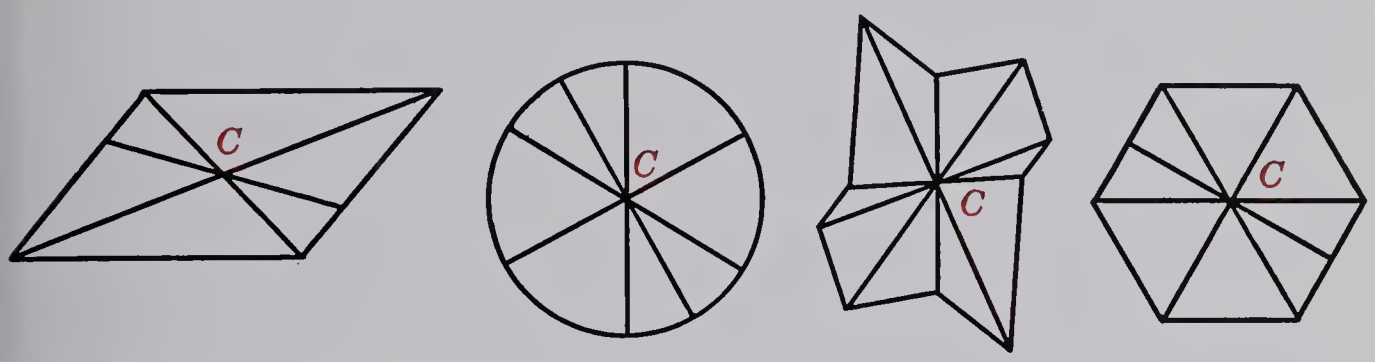
142. Symmetry

A geometric figure is said to be *symmetric* with respect to a line as an axis if this line bisects all line segments perpendicular to the line and



terminated by the figure. The figures above are symmetric with respect to the line l. Each figure has *axial* symmetry. Notice that in each case l is the perpendicular bisector of the colored line segments.

A geometric figure is symmetric with respect to a point if the point bisects all line segments which pass through it and are terminated by the figure.



The figures above are symmetric with respect to the point C. They have *central* symmetry.

Nature has made abundant use of symmetry in both plant and animal life, as the pictures below suggest. The flower at the left is of the Silversword plant, found in Hawaii National Park. The butterfly is a tiger swallowtail. What types of symmetry are illustrated? The beauty of a

Hawaii National Park | American Museum of Natural History | Hugh Spencer

plant or animal is due, to a large extent, to the symmetry which gives it balance. Flowers, fruits, and the lower forms of animal life often possess central symmetry. Leaves, trees, landscapes with their reflections in still water, and higher forms of animal life often possess axial symmetry.

EXERCISES

A

1. Which of the figures in the first two rows on page 165 have both axial and central symmetry?

2. Where is the axis of symmetry of the letter A? of the letter D? of the letter M?

3. Name two letters of the alphabet which have two axes of symmetry.

4. Name two letters of the alphabet which have central symmetry.

5. Complete: If a figure is symmetric with respect to a line as an axis, the axis divides it into two _ _?_ _ parts.

Solids as well as plane figures can have symmetry. A solid can be symmetric with respect to a point, a line, or a plane. For example, a sphere is symmetric with respect to its center, it is symmetric with respect to any of its diameters as an axis, and it is symmetric with respect to any plane passing through its center.

143. Location of Points

The location of all the points which satisfy certain given conditions has always been a very interesting topic of mathematics. At this time we shall consider some simple problems on the location of points. These problems will lead us to discover some important theorems which we shall prove later.

Example. Locate ten points which are equidistant from two given parallel lines.

Solution. Locate point E which seems to be equidistant from AB and CD. Likewise locate nine other points. We see that these ten points appear to lie in a straight line parallel to AB and CD and midway between them.

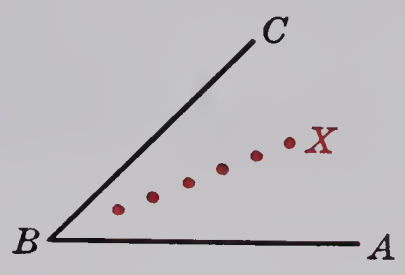
EXERCISES

A

1. How many points can you locate which are equidistant from AB and CD in the Example, § 143? Will these points all lie in a straight line?

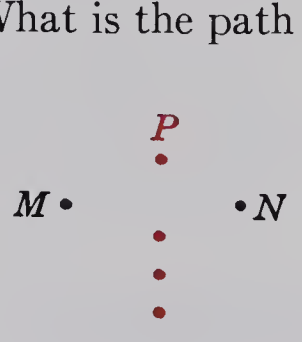
2. A point P moves so that it is always halfway between two parallel lines. What is its path?

3. Draw $\angle ABC$. Locate within the angle a point X that seems to be equidistant from the two sides. In the same manner locate five other points. How many points can be found that are equidistant from AB and CB? Where will these points lie?



4. A boy on a hike in the country comes to a fork in the road and decides to go through the fields, keeping the same distance from each of the two roads. What is his path?

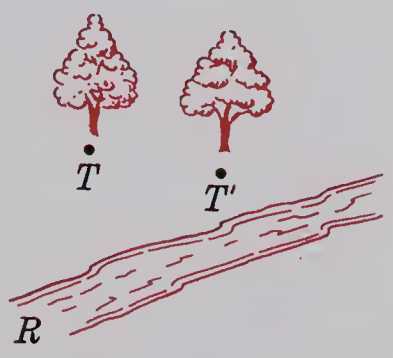
5. Two parallel lines, AB and CD, are 3 inches apart. Locate several points that are twice as far from AB as from CD. What is the path of a point that moves so that it is twice as far from AB as from CD?



6. Locate a point P equidistant from M and N. Locate several other points equidistant from M and N. Do these points all lie in a straight line? What relation does the line containing all points equidistant from M and N have to the line segment joining M and N?

7. What is the path of a point on this page if the point moves so that it is always 1 inch from an edge?

8. A treasure is located on the bank of a river R and at equal distances from two trees, T and T'. How would you locate the treasure?



9. Locate a point A that is 2 inches from another point O. Locate another point B that is 2 inches from O. In like manner locate ten other points that are 2 inches from O. Do these points lie in a straight line or in a curve line? If point A moves so that it is always 2 inches from O, what is the name of the path it will form?

10. What is the path of the center of an automobile wheel as the car runs on a straight, level road?

All the points which satisfy certain conditions form a figure which is called a *locus of points*. *Locus of points* and *locus of a point* have the same general meaning, as you will learn in Chapter 11.

11. What is the locus of points, or locus of a point, equidistant from two parallel lines?

12. What is the locus of points within an angle equidistant from the sides of the angle?

13. What is the locus of points 1 inch from a given point?

14. What is the locus of points equidistant from two given points?

15. What is the locus of the center of a circle that rolls on the outside of a larger circle?

16. What is the locus of the center of a circle that rolls on the inside of a larger circle?

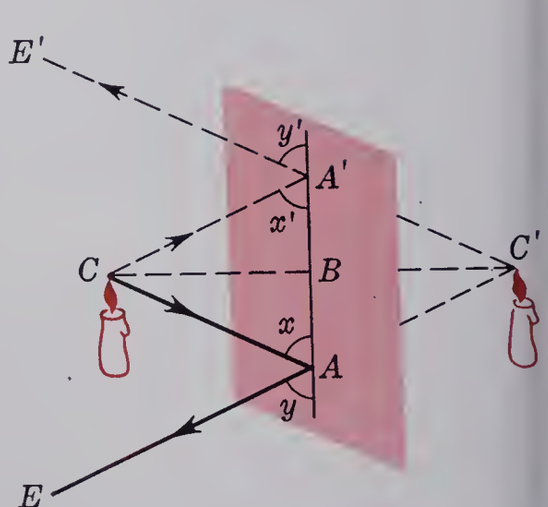
17. What is the locus of the tip of the hour hand of a watch?

NOTE TO TEACHER. Co-ordinate geometry, pp. 518–523 inclusive, may be studied now.

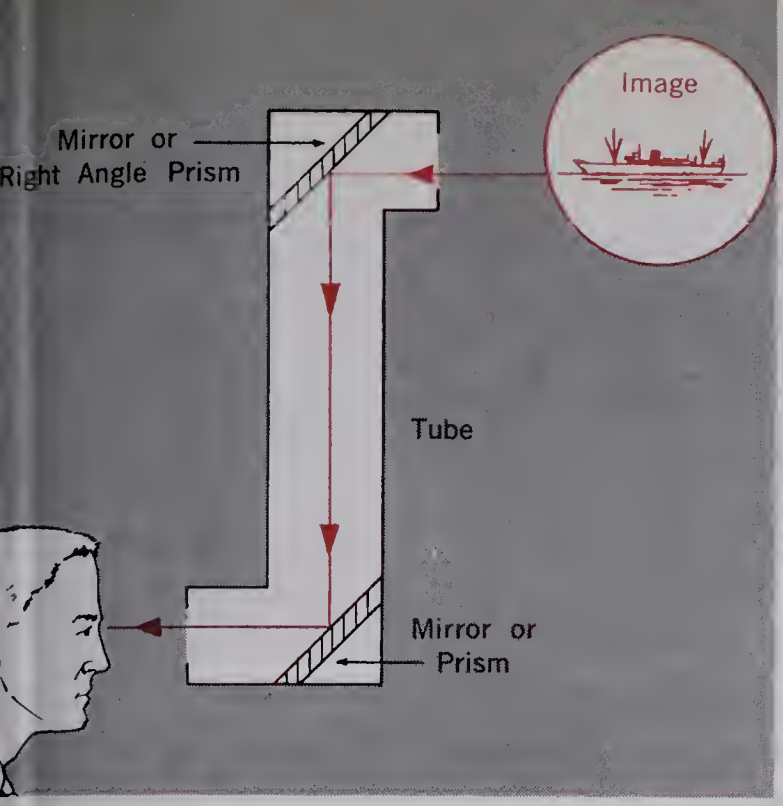
EXERCISES Miscellaneous

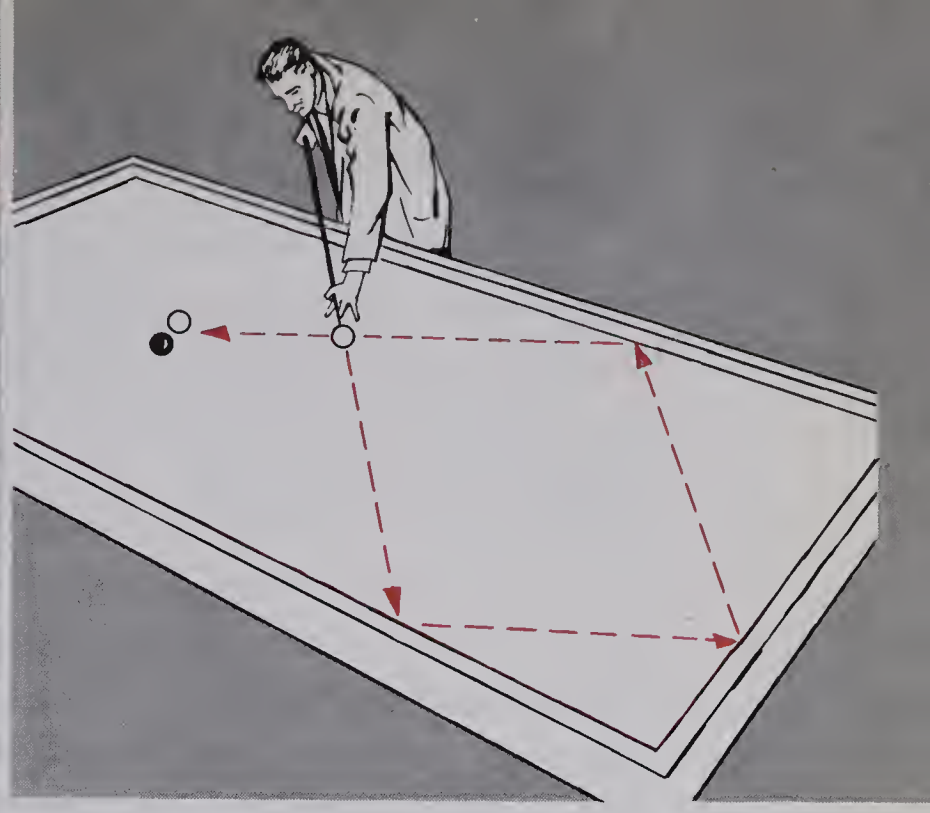
B

1. It is shown in physics that when a ray of light from a candle C strikes a plane mirror at A, it is reflected along AE (E being the position of the observer's eye) in such a way that the *incident ray* CA and the *reflected ray* AE make equal angles, x and y, with the mirror. $A'E'$ is another reflected ray, which meets AE in C'. Prove that CC' is $\perp AA'$ and that $CB = BC'$.



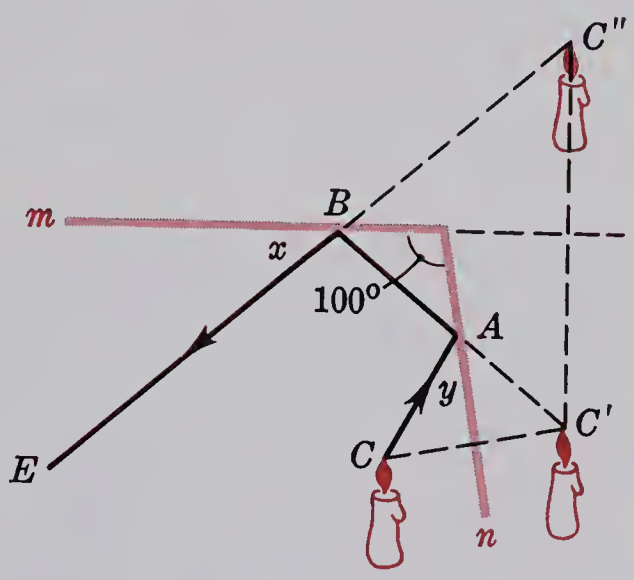
2. Prove that any other reflected ray seems to come from C', called the image of C.





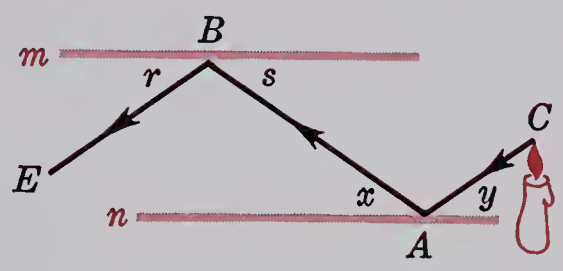
How do the periscope and the game of billiards illustrate the principle we are studying?

3. In this figure, m and n represent two plane mirrors set at an angle of 100°. From E a candle appears to be at C'', C'' being the image of C' in m, and C' being the image of C in n. What is the path of the ray of light from C reflected by both mirrors to E? How many degrees are in $\angle x$ if $\angle y = 50°$? if $\angle y = 40°$?



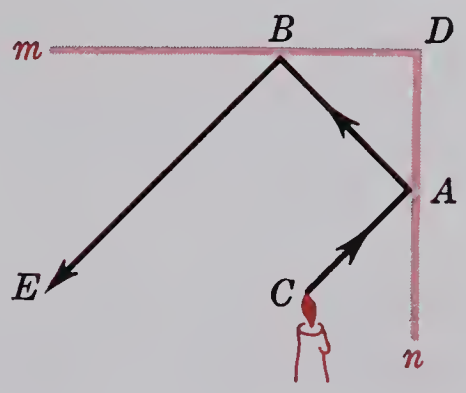
4. In the figure of Ex. 3, find $\angle y$ if $\angle x = 24°$.

5. In this figure, m and n are two parallel plane mirrors. Show that the incident ray CA is parallel to the reflected ray BE.



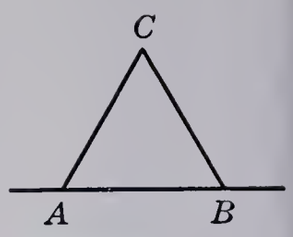
6. If the lower ends of two ladders having equal lengths are so placed that they are equidistant from a high wall, prove that the upper ends of the ladders will reach the same height on the wall.

7. If two plane mirrors, m and n, are perpendicular, show that the incident ray CA is parallel to the reflected ray BE.

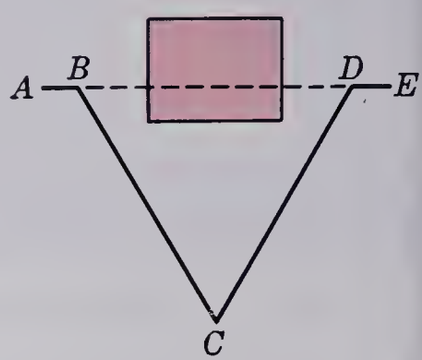


SUGGESTION. Extend lines BD and CA so that they meet.

8. Sometimes in surveying, a triangle is formed as follows: A line segment AB, 50 feet long, is measured on a given line; the point C is located by the midpoint of a 100-foot tape whose ends are held at A and B. How many degrees are there in each of $\measuredangle A$, B, and C?



9. Forming a triangle by the method explained in Ex. 8, a surveyor is able to extend a line AB beyond an obstacle and to determine the distance from B to D. Explain how the line is extended and prove that DE is in line with AB. How is BD found?

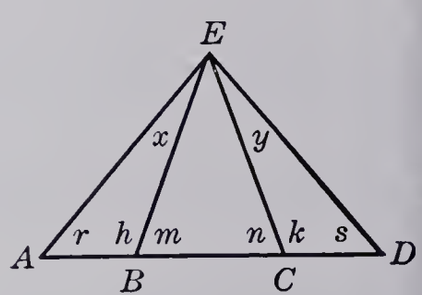


10. In $\triangle ADE$, if $AE = DE$ and $\angle x = \angle y$, prove that

a. $\angle h = \angle k$.

b. $\angle m = \angle n$.

c. $BE = CE$.

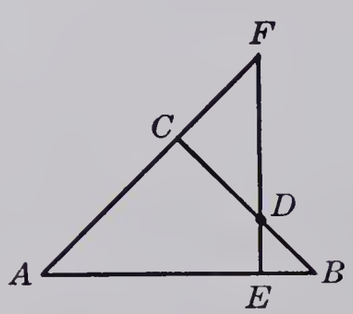


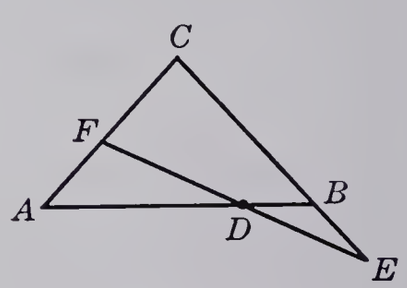
11. In the figure of Ex. 10, if $\angle r = \angle s$ and $\angle x = \angle y$, prove that $BE = CE$.

12. The exterior angle at the base of an isosceles triangle is equal to the angle formed by the bisectors of the base angles.

13. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular is drawn to the opposite side, it makes with the base an angle equal to one half the vertex angle of the triangle.

C



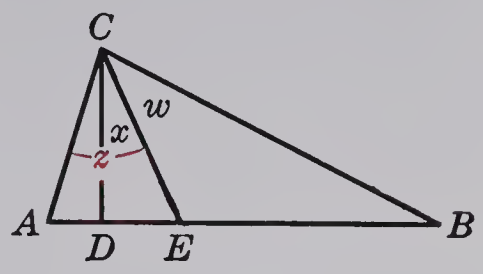


14. If through any point D in one of the equal sides BC of an isosceles $\triangle ABC$, DE is drawn $\perp$ to the base AB, meeting AC extended at F, then $\triangle CFD$ is isosceles.

15. D is any point in the base AB of isosceles $\triangle ABC$. The side CB is extended from B to E so that $BE = BD$. Then ED is drawn and extended to meet AC at F.

Prove that $\angle CFE = 3\ \angle CEF$.

16. The angle formed by the altitude and the bisector of the vertex angle of a triangle is equal to one half the difference between the base angles.



SUGGESTIONS.

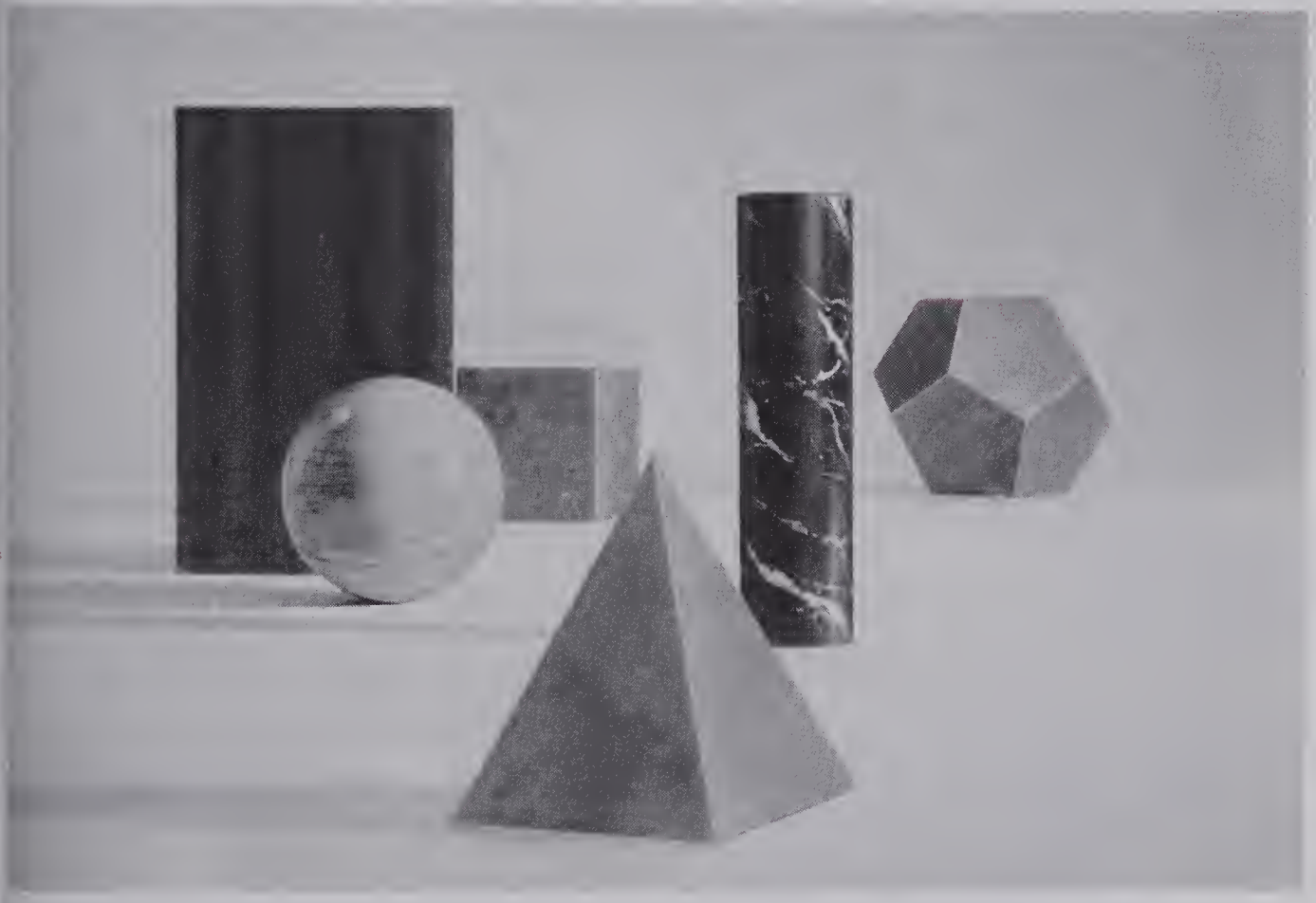
$\angle A + \angle z - \angle x = \angle B + \angle w + \angle x.$

Solve the equation for $\angle x$.

17. An altitude of a triangle divides the vertex angle into two parts whose difference equals the difference of the base angles.

SPACE GEOMETRY

NOTE TO TEACHERS. The work on space geometry on pages 171–178 and elsewhere in the book is optional.

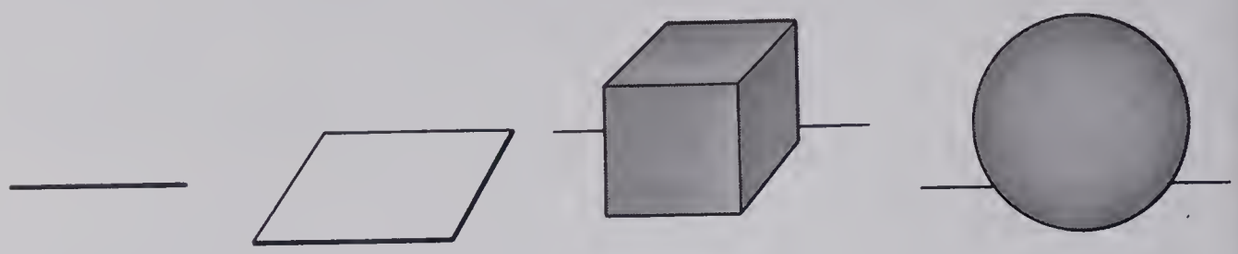


General American Transportation Corporation

Physical Solids

The figures we have studied thus far are called *plane* figures, since all the points and lines of each figure lie in the same plane. In space geometry

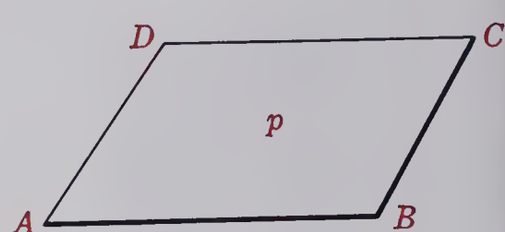
the figures also consist of points and lines, but not all the points and lines of each figure necessarily lie in the same plane.



Figures in a plane have one or two dimensions. A line has one dimension (length) and a rectangle has two dimensions (length and width). Figures in space have one, two, or three dimensions. A cube and a sphere have three dimensions (length, width, and thickness).

144. Plane Surfaces

Since we have dealt with lines in plane geometry, we may say that a plane is a surface such that a straight line joining any two points of the surface lies wholly in the surface. Although a plane is indefinite in extent, in space we usually picture it as a rectangle seen obliquely. The quadrilateral $ABCD$ represents a plane. This plane may be referred to as plane AC or as plane p. The lines AB and BC are drawn heavier than AD and DC to make the plane appear horizontal.



EXERCISES

1. If you should illustrate the intersection of a plane and a line by means of a card and a needle, what would you call the intersection?

2. What kind of line is formed when a piece of paper is folded in a sharp crease?

3. In your classroom find several cases of the intersection of two planes.

4. Can two different planes contain the same two lines?

5. How many straight lines can be drawn through one point? through two points? through any three points?

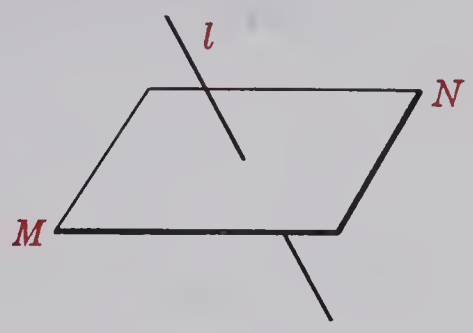
6. How many planes can contain a given point? two given points? three given points not in a straight line?

7. How many planes can contain a given straight line? two given intersecting straight lines? two parallel lines?

8. How many dimensions has a line? a plane?

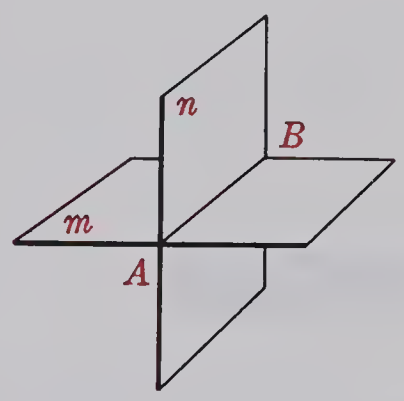
145. Fundamental Properties of Planes

From the preceding exercises the assumptions relating to planes will be apparent:



a. The intersection of a straight line and a plane is a point.

If there were two points in common, the line would lie in the plane.



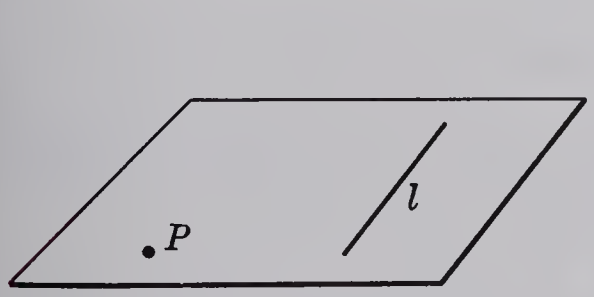
b. The intersection of two planes is a straight line.

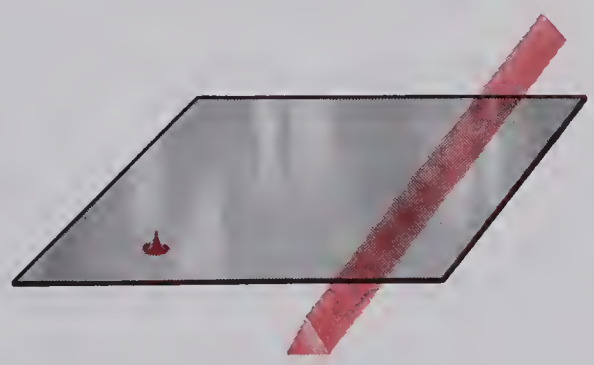
The intersection of planes m and n is the straight line AB.

c. A plane is determined by three points not in the same straight line.

The figure at the right shows the plane surface of a sheet of plate glass resting on three pegs.

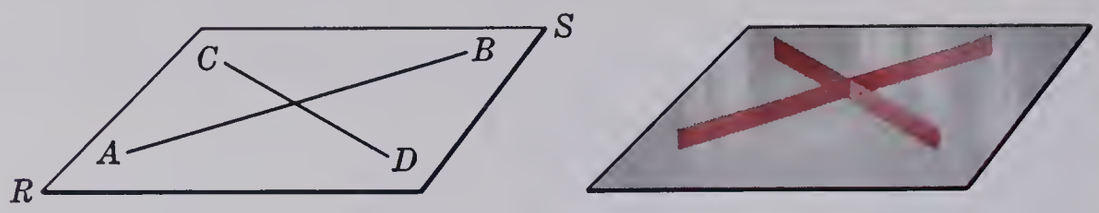
d. A plane is determined by a line and a point not on the line.



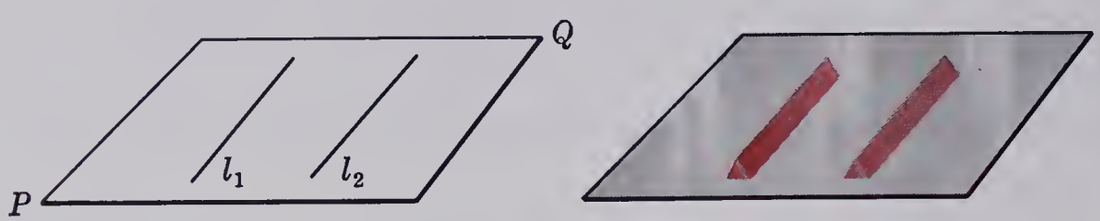


The figure at the right above shows the plane surface of a sheet of glass resting on the point of a tack and one straight edge of a bar.

e. A plane is determined by two intersecting lines.



f. A plane is determined by two parallel lines.



SUGGESTIONS. What are parallel lines? Can a plane contain both l_1 and l_2? Show that no other plane can contain l_1 and a point in l_2.

EXERCISES

1. Can you hold two pencils in such a manner that they do not lie in a plane?

2. Why is it that a table with three legs will always stand firmly on a level floor and a table with four legs will sometimes rock?

3. Why are tripods used for the support of a surveyor's transit and a camera?

4. Are all triangles plane figures? Why?

5. Is any figure having four sides necessarily a plane figure?

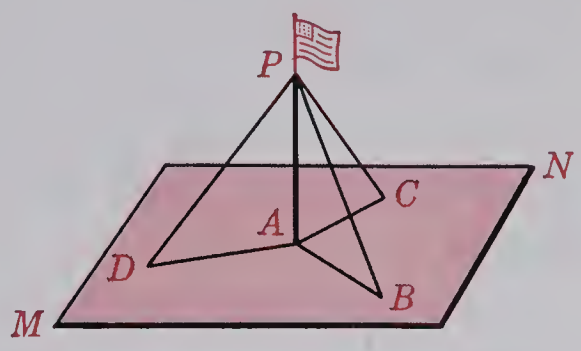
6. How many applications of a straightedge are necessary to determine whether or not a surface is a plane surface?

7. Draw two intersecting planes.

8. AB, AC, and AD are three straight lines through point A but not in the same plane. How many planes are determined by these lines if they are taken two at a time?

9. AB, CD, and EF are three parallel lines, but not in the same plane. How many planes are determined by these lines if they are taken two at a time?

10. In the figure a flagpole PA is shown as perpendicular to plane MN representing the ground. Three wires PD, PB, and PC brace the flagpole. (a) Is $PA \perp AD$, AB, and AC? (b) If $PA = 150'$ and $AB = 80'$, find PB. (c) If $AB = AC$, prove that $PB = PC$.



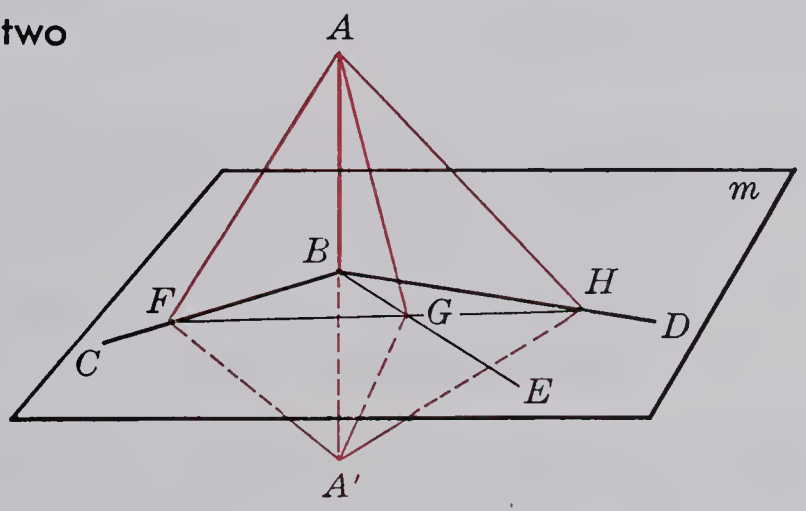
146. Foot of a Line

The point of intersection of a straight line and a plane is called the *foot* of the line.

147. Line Perpendicular to a Plane

A *line* is *perpendicular to a plane* if it is perpendicular to every line in the plane passing through its foot. If a line meets a plane and is not perpendicular to it, it is said to be *oblique* to the plane.

148. Theorem. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.



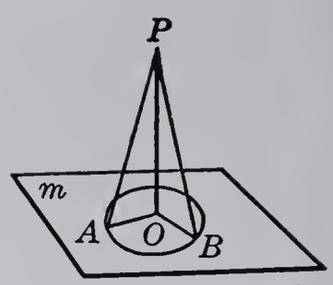
Given $AB \perp BC$ and BD, and m the plane of the intersecting lines BC and BD.

To prove $AB \perp$ plane m.

SUGGESTIONS.

1. Through B draw any other line BE in plane m. Why possible?
2. Draw any line intersecting BC at F, BE at G, and BD at H. Why possible?
3. Extend AB to A' so that $BA' = BA$ and draw AF, AG, AH, $A'F$, $A'G$, and $A'H$. Why possible?
4. $AF = A'F$, $AH = A'H$, and $FH = FH$. $\triangle AFH \cong \triangle A'FH$.
5. Then $\angle AFG = \angle A'FG$ and $\triangle AFG \cong \triangle A'FG$.
6. $AG = A'G$, $BG \perp AB$, and $AB \perp m$.

EXERCISES



1. *Given* $PO \perp$ plane m at O, the center of a circle in the plane.

Prove that $PA = PB$ and $\angle PAO = \angle PBO$.

2. In a plane, how many lines may be drawn perpendicular to a given line at a given point in the line?

3. In space, how many lines may be drawn perpendicular to a given line at a given point in the line?

4. Are all lines perpendicular to a vertical line horizontal?

5. Are all lines perpendicular to a horizontal line vertical?

6. Can two flagpoles at opposite ends of a large building be both parallel and vertical?

In the following exercises, illustrate by means of cardboard and pencil:

7. If a line is perpendicular to a line in a plane, can it be perpendicular to the plane? Is it necessarily perpendicular to the plane?

8. How many lines may be drawn perpendicular to a plane at a given point in the plane? through a given external point?

9. How many planes may be drawn perpendicular to a line at a point in the line? through an external point?

149. Assumptions on Perpendicular Lines and Planes

a. Through a given point there can be one line, and only one, perpendicular to a plane.

b. Through a given point there can be one plane, and only one, perpendicular to a line.

150. Parallel Planes and Lines

A *line* and a *plane* are *parallel* if they do not meet even if extended. Two *planes* are *parallel* if they do not meet even if extended.

151. Skew Lines

Two *skew lines* are lines that do not lie in the same plane. Are parallel lines ever skew lines? Why?

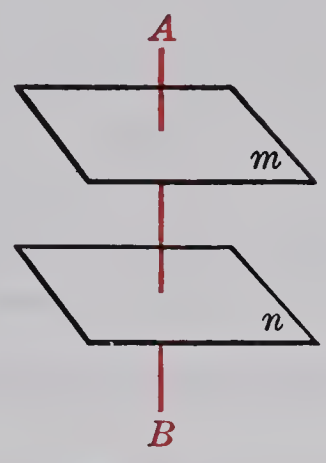
EXERCISES

1. Are two plumb lines parallel?

2. Can two skew lines intersect? be perpendicular to each other?

3. Can two skew lines both be vertical? horizontal?

4. If two planes are parallel, is every line in one of the planes parallel to the other plane?

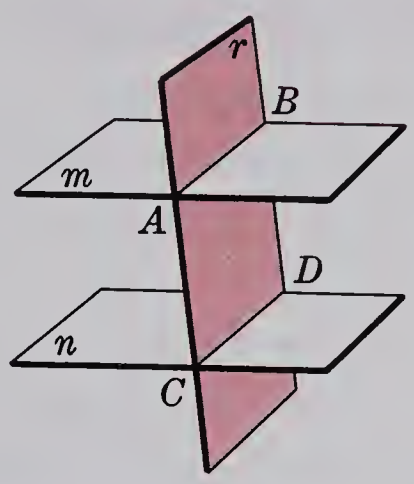


152. Theorem. **Two planes perpendicular to the same line are parallel.**

Given planes m and n each $\perp AB$.

To prove $m \parallel n$.

SUGGESTION. Prove indirectly, using §§ 149 *b* and 150.



153. Theorem. **If two parallel planes are cut by a third plane, the lines of intersection are parallel.**

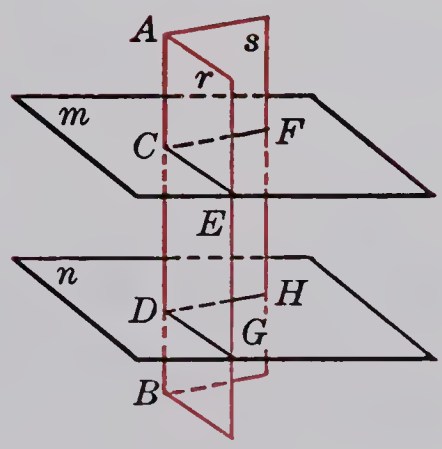
Given the $\parallel$ planes m and n intersected by plane r in AB and CD respectively.

To prove $AB \parallel CD$.

Proof

STATEMENTS	REASONS
1. AB and CD lie in the same plane r.	1. Given.
2. AB in m cannot meet CD in n.	2. § 150.
3. $AB \parallel CD$.	3. § 99.

154. Theorem. **A line perpendicular to one of two parallel planes is perpendicular to the other.**



Given the $\parallel$ planes m and n and $AB \perp m$.

To prove $AB \perp n$.

(See next page for proof.)

Proof

STATEMENTS	REASONS
1. Through AB pass plane r intersecting m and n in CE and DG respectively. Also through AB pass plane s intersecting m and n in CF and DH respectively.	1. §§ 145 b, 145 d.
2. $CE \parallel DG$ and $CF \parallel DH$.	2. § 153.
3. $AB \perp CE$ and CF.	3. § 147.
4. $\therefore AB \perp DG$ and DH.	4. § 114.
5. $\therefore AB \perp n$.	5. § 148.

155. Corollary. Through a point outside a plane there can be one plane, and only one, parallel to a given plane.

REVIEW QUESTIONS — Chapter 5

1. What do you know about two perpendiculars to the same line at different points in the line? at the same point in the line?

2. Through a point outside a line how many lines can be drawn parallel to the line?

3. How many acute angles has every triangle?

4. What is the name of a triangle having two acute angles and one right angle? two acute angles and one obtuse angle? three acute angles?

5. What is the sum of the angles of a triangle?

6. What kind of theorem is formed by interchanging any number of conclusions with an equal number of hypotheses of a given theorem?

7. What kind of theorem is formed by negating the conclusion and one hypothesis of a given theorem having one conclusion?

8. Under what conditions is an exterior angle of a triangle an acute angle? a right angle?

9. What is the location of points that are equidistant from two given points?

10. If one angle of a triangle is a right angle, what is the relation between the other two angles?

11. In an isosceles right triangle how many degrees are there in each angle?

12. When two parallel lines are cut by a transversal, what pairs of angles are equal?

13. In what right triangle is one leg equal to one half the hypotenuse?

14. How many degrees are there in one angle of an equilateral triangle?

15. How many degrees are there in the angle formed by the bisectors of the two acute angles of a 30°-60° right triangle?

16. The vertex angle of an isosceles triangle is one fourth of a base angle. How many degrees are there in each angle?

17. What is the difference between axial and central symmetry?

18. State six ways of proving right triangles congruent.

SUMMARY of Principal Bases for Proof

156. Two right triangles are congruent

a. If the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

b. If the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.

157. Two line segments are equal

a. If they are sides opposite equal angles in a triangle.

b. If they are perpendiculars drawn from a point in the bisector of an angle to the sides of the angle.

158. Two angles are equal

a. If they are alternate interior or corresponding angles of parallel lines.

b. If their sides are parallel (or perpendicular), right side to right side and left side to left side.

c. If they are the third angles of two triangles having two angles of one equal respectively to two angles of the other.

159. Two lines are perpendicular

a. If one of them is parallel to a third line and the other is perpendicular to the third line.

b. If two points of one line are each equidistant from two points of the other.

160. Two lines are parallel

a. If they do not intersect even if extended.
b. If a pair of (1) alternate interior angles or (2) corresponding angles are equal.
c. If two interior angles on the same side of the transversal are supplementary.
d. If they are perpendicular to a third line.
e. If they are parallel to a third line.

161. Two angles are complementary

If they are the acute angles of a right triangle.

162. Two angles are supplementary

If they are the interior angles on the same side of a transversal of two parallel lines.

WORD LIST

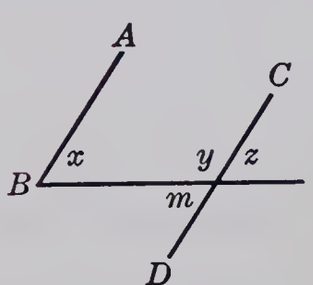
Can you spell and use all the following words correctly?

alternate	consecutive	interior	parallel
analytic	indirect	locus	synthetic

TEST 9 — Supplying Reasons

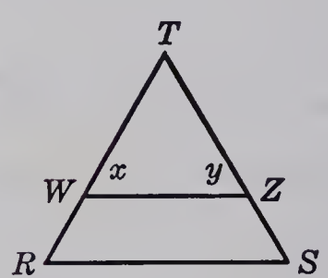
[10 Minutes]

Supply assumptions, definitions, or theorems for the following:



If $AB \parallel CD$,

1. $\angle x = \angle z$.
2. $\angle m = \angle x$.
3. $\angle x$ and $\angle y$ are supp.
4. $\angle m = \angle z$.



If $RT = TS$ and $WZ \parallel RS$,

5. $\angle x = \angle R$.
6. $\angle R = \angle S$.
7. $\angle y = \angle S$.
8. $\angle x = \angle y$.
9. $WT = TZ$.

TEST 10 — Beginning a Proof

[20 Minutes]

Draw the figure for each of the following theorems and state accurately, in terms of the letters on the figure, the hypothesis and conclusion:

1. If a line intersects the equal sides of an isosceles triangle and is parallel to the base, the new triangle formed is also isosceles.

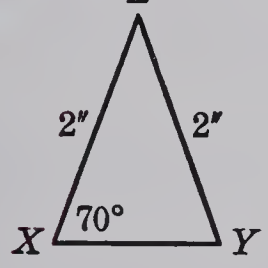
2. If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles are equal.

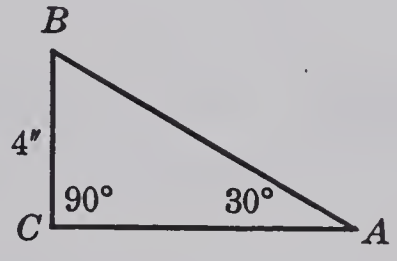
3. If the altitude of a triangle bisects the side to which it is drawn, the triangle is isosceles.

4. If two lines are cut by a transversal so that the bisectors of two corresponding angles are parallel, the lines are parallel.

TEST 11 — Applications

[20 Minutes]



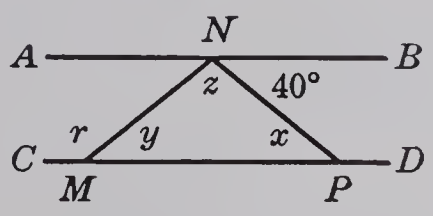


1. How many degrees are there in $\angle Y$? in $\angle Z$?

2. How many inches are there in AB?

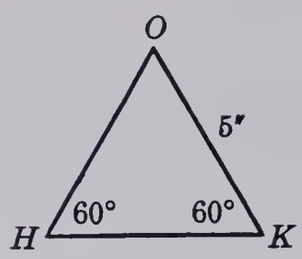
3. A wheel makes 30 revolutions per minute. Through how many degrees does it turn in one second?

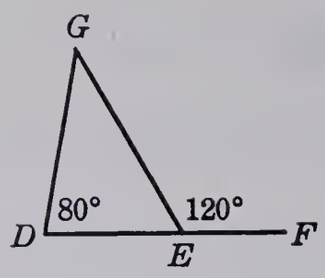
4. How many degrees are there in each base angle of an isosceles right triangle?



Ex. 5

5. If $AB \parallel CD$ and $MN = PN$, how many degrees are there in $\angle x$? in $\angle z$? in $\angle r$?





6. How many inches are in HO?

7. How many degrees are in $\angle G$?

8. If x represents the number of degrees in an angle, what represents the number of degrees in its complement?

9. If two parallels are cut by a transversal, how many degrees are in the angle formed by the bisectors of two interior angles on the same side of the transversal?

10. If the vertex angle of an isosceles triangle is 40°, how many degrees are in the angle included by the bisectors of the base angles?

11. One acute angle of a right triangle is half the other. How many degrees are there in each acute angle?

TEST 12 — Multiple-choice Questions

Write the answer (a), (b), (c), or (d) which you think is correct in each exercise. Give only one answer for each exercise.

1. If line t is a transversal of lines l and m, the corresponding angles

a. are always equal.
b. are never equal.
c. may be equal.
d. are never supplementary.

2. If two lines are perpendicular to a third line,

a. they are parallel to each other.
b. they are oblique to each other.
c. they are perpendicular to each other.
d. they intersect.

3. Through a given external point the number of lines that can be drawn parallel to a given line is:

a. two.
b. an infinite number.
c. none.
d. one.

4. In a right triangle, the angle opposite the hypotenuse is

a. an acute angle.
b. an obtuse angle.
c. a right angle.
d. not (a), (b), or (c).

5. If a point is equidistant from the vertices A and B of $\triangle ABC$, it is

a. on the bisector of $\angle C$.
b. equidistant from AC and BC.
c. on the median to AB.
d. on the perpendicular bisector of AB.

6. The converse of "If $a = b$, then $c = d$" is:

a. If $a \neq b$, then $c \neq d$.
b. If $c \neq d$, then $a \neq b$.
c. If $c = d$, then $a = b$.
d. If $a = b$, then $c \neq d$.

7. If $\triangle ABC$ has $\angle C = 90°$ and $\angle B = 60°$, then

a. $BC = \frac{1}{2} BA$.
b. $AC = 2\ BC$.
c. $AC = \frac{1}{2} AB$.
d. $BC = 2\ AC$.

8. The total number of ways you have had of proving right triangles congruent is

a. 4. **b.** 3. **c.** 7. **d.** 6.

9. A line intersecting the equal sides of an isosceles triangle and parallel to the base forms

a. a right triangle.
b. an obtuse triangle.
c. an isosceles triangle.
d. an oblique triangle.

10. The locus of points one inch from a given line consists of

a. one line.
b. one point.
c. two lines.
d. Not (**a**), (**b**), or (**c**).

11. The capital letter H has

a. central and axial symmetry.
b. only central symmetry.
c. neither axial nor central symmetry.
d. only axial symmetry.

CHAPTER 6

Constructions

Can we construct the geometric figures
we have been studying?
For example, can we construct
the bisector of an angle
or a perpendicular to a line
at a point on the line?
In this chapter
we shall attempt to answer such questions.

163. Procedure in Making Constructions

We said in § 9 that the only instruments used in making geometric constructions are the compasses and straightedge. We use the straightedge for drawing straight lines and the compasses for drawing arcs and measuring the lengths of segments. Before drawing a line it is necessary to locate two of its points. A point is located by finding the intersection of two lines (either straight or curved).

To keep our work organized and clear while making constructions, we shall follow this plan:

1. State the problem.
2. Represent the given parts with a drawing.
3. State specifically what is given.
4. State what is to be constructed.
5. Make the construction, describing each step and giving an authority for it.
6. State that the required construction has been made.
7. Prove the statement of Step 6.
8. Usually, discuss the limitations of the construction, its special cases, and the number of solutions possible.

How are the steps outlined above like those used in proving a theorem? How are they different? (See page 68.)

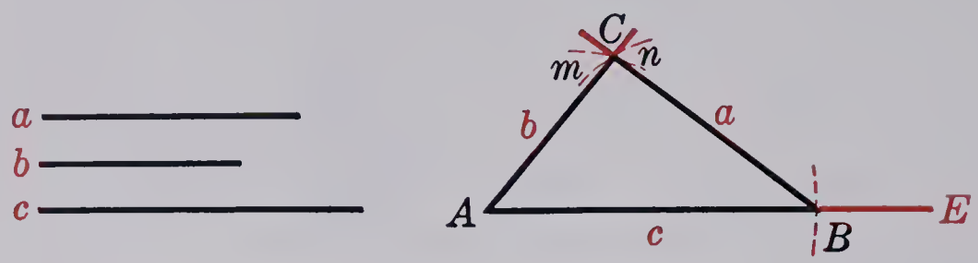
Since we know that our hands and eyes make small errors in using the compasses and straightedge, we do not claim that our completed constructions are perfect. We only mean that if we could carry out the steps perfectly, the constructions would be the required constructions.

164. Proposition, Theorem, Problem

You know that a *proposition* is a statement in which we propose something for consideration. If a proposition sets forth a statement to be proved, we call it a *theorem*, but if it sets forth a construction to be made or a computation to be performed, we call it a *problem*. The propositions of this chapter are called problems because they set forth constructions to be made. Why have the propositions of the preceding chapters been called theorems?

★★ Problem 1

165. To construct a triangle, given the three sides.



Given a, b, and c, the sides of a $\triangle$.

To construct the $\triangle$.

Construction

STATEMENTS	REASONS
1. On any line AE construct $AB = c$.	1. § 9.
2. With A as a center and a radius $= b$ draw an arc m.	2. Asmt. 15.
3. With B as a center and a radius $= a$ draw an arc n intersecting arc m at C.	3. Asmt. 15.
4. Draw AC and BC.	4. Asmt. 11.
Then ABC *is the required* $\triangle$.	

Proof

STATEMENTS	REASONS
$\triangle ABC$ has sides $= a$, b, and c.	Const.

Discussion. If $a + b < c$ or if $a + b = c$, the construction is impossible because the arcs would not intersect outside of c. Since the shortest distance between two points is the length of the straight line segment between them, two circles will intersect only when the distance between their centers is less than the sum of their radii.

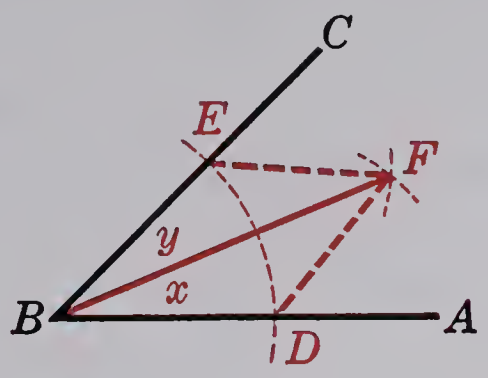
EXERCISES Constructions

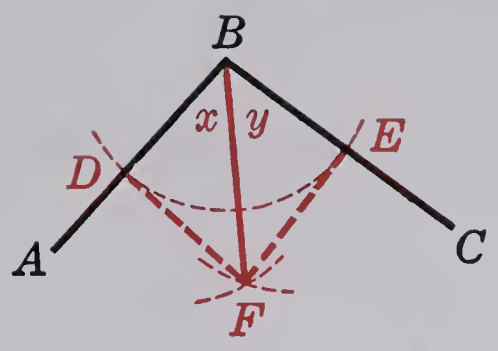
A

1. Construct an equilateral triangle given one side.
2. Construct a triangle having sides of $1\frac{1}{2}$, 2, and 3 inches respectively.
3. Why is a triangle rigid?
4. Can you construct a triangle whose sides are 3″, 4″, and 7″?

★★ Problem 2

166. To bisect a given angle.





Given $\angle ABC$.

To construct its bisector.

Construction

STATEMENTS	REASONS
1. With B as a center and any radius draw an arc intersecting BA at D and BC at E.	1. Asmt. 15.
2. With D and E as centers and with $=$ radii draw arcs which will intersect at F.	2. Asmt. 15.
3. Draw BF.	3. Asmt. 11.
Then BF is the required bisector.	

Proof

1. Draw DF and EF.	1. Asmt. 11.
In $\triangle BDF$ and BEF,	
2. $BD = BE$ and $DF = EF$.	2. Const.
3. $BF = BF$.	3. Why?
4. $\therefore \triangle BDF \cong \triangle BEF$.	4. Why?
5. $\therefore \angle x = \angle y$.	5. Why?
6. $\therefore BF$ bisects $\angle ABC$.	6. Why?

Discussion. The construction is always possible.

EXERCISES Constructions

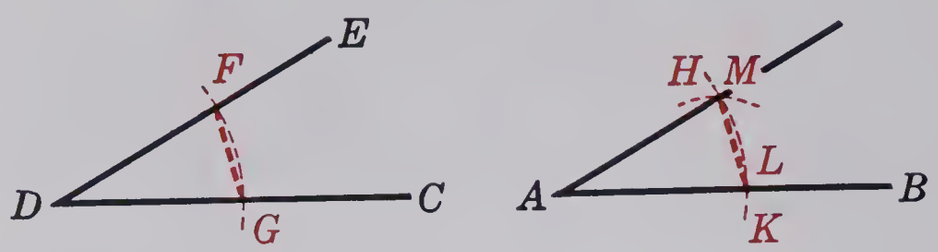
A

1. Draw four angles with different sizes and construct their bisectors.

2. Draw an acute triangle and construct the bisectors of its angles. Do the same for a right triangle and an obtuse triangle.

★★ Problem 3

167. With a given vertex and a given side to construct on angle equal to a given angle.



Given vertex A, side AB, and $\angle CDE$.

To construct an $\angle = \angle CDE$ and having A as the vertex and AB as one side.

Construction

STATEMENTS	REASONS
1. With D as a center and any radius draw an arc intersecting DE at F and DC at G.	1. Asmt. 15.
2. With A as a center and a radius $= DF$, draw arc HK intersecting AB at L.	2. Why possible?
3. With L as a center and a radius $= GF$, draw an arc intersecting $\overset{\frown}{HK}$ at M.	3. Why possible?
4. Draw AM.	4. Why possible?
Then $\angle BAM = \angle CDE$.	

Proof: (The proof is left to the student.)

SUGGESTION. Draw GF and LM.

Discussion: Is the construction always possible?

EXERCISES Construction

A

With a given vertex and a given side construct an angle equal to a given obtuse angle.

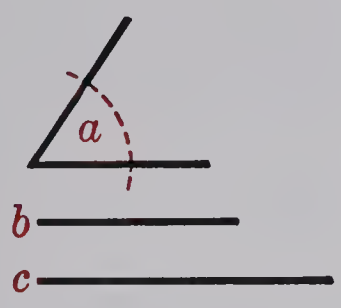
168. Construction Problems in Proofs

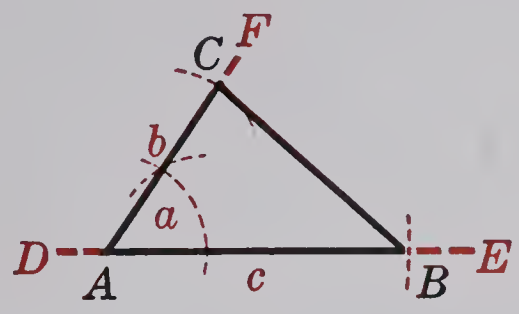
The statements of construction problems may be used as reasons in proofs in the same way as theorems are used. The wording is usually

changed. For example, Prob. 3 may be stated, "An angle can be constructed equal to a given angle."

★★ Problem 4

169. To construct a triangle, given two sides and the included angle.





Given sides b and c and the included $\angle a$.

To construct the $\triangle$.

Construction

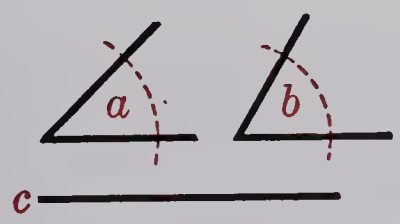
STATEMENTS	REASONS
1. On any line DE construct $AB = c$.	1. § 9.
2. At A construct $\angle BAF = \angle a$.	2. § 167.
3. On AF construct $AC = b$.	3. Why possible?
4. Draw BC.	4. Why possible?
Then ABC is the required $\triangle$.	

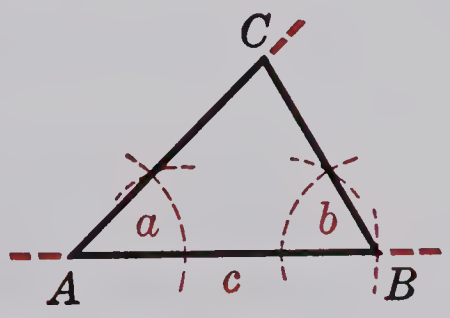
Proof: (The proof and discussion are left to the student.)

Discussion

★★ Problem 5

170. To construct a triangle, given two angles and the included side.





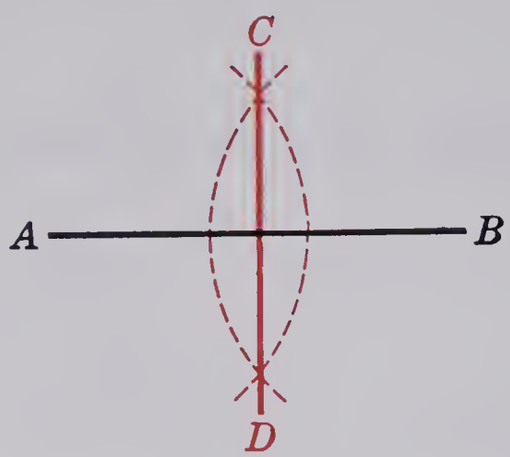
Given $\measuredangle a$ and b and the included side c.

To construct the $\triangle$.

(The construction and proof are left to the student.)

★★ Problem 6

171. To construct the perpendicular bisector of a given line segment.



Given the line segment AB.
To construct the $\perp$ bisector of AB.

Construction

STATEMENTS	REASONS
1. With A and B as centers and with $=$ radii draw arcs intersecting at C and D.	1. Asmt. 15.
2. Draw CD.	2. Why possible?
Then CD is the $\perp$ bisector of AB.	

Proof

1. C is equidistant from A and B.	1. Const.
2. D is equidistant from A and B.	2. Why?
3. $\therefore$ CD is the $\perp$ bisector of AB.	3. § 140.

EXERCISES — Constructions

A

1. Can the method of § 171 be used to bisect a line segment? to construct a perpendicular?

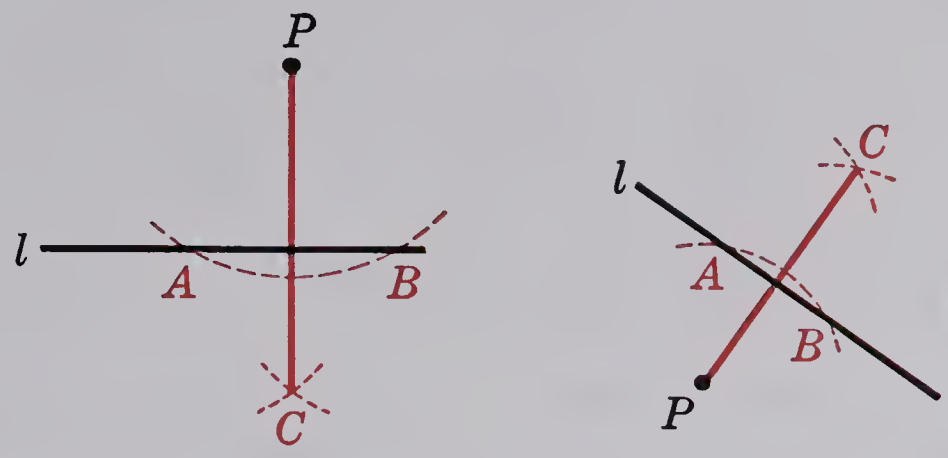
2. See if you can construct the perpendicular bisector of a line segment by locating two points on the same side of the line segment and each equidistant from the end points of the line segment.

3. Construct a right triangle, having given the two legs.

4. Divide a line segment into four equal parts.

★ Problem 7

172. To construct a perpendicular to a given line from a given point outside the line.



Given line l and point P without l.

To construct a $\perp$ to l from P.

Construction

STATEMENTS	REASONS
1. With P as a center draw an arc that will intersect l in two points, A and B.	1. Asmt. 15.
2. With A and B as centers and with = radii, draw arcs intersecting at C.	2. Why possible?
3. Draw PC.	3. Why possible?
Then $PC \perp AB$.	

Proof: (The proof is left to the student.)

SUGGESTION. Show that P and C are each equidistant from A and B.

EXERCISES Constructions

B

1. Construct perpendiculars to the sides of an angle from a point within the angle.

2. Construct the three altitudes of an acute triangle.

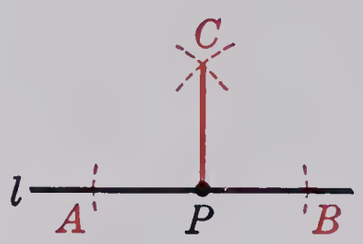
3. Construct the three altitudes of an obtuse triangle.

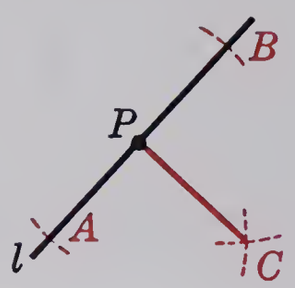
4. Construct a right triangle and then construct its three altitudes.

5. From Exs. 2, 3, and 4, what common relation do the altitudes of a triangle seem to have?

★★ Problem 8

173. To construct a perpendicular to a given line at a given point in the line.





Given line l and point P in line l.

To construct a $\perp$ to l at P.

(The construction and proof are left to the student.)

EXERCISES Constructions

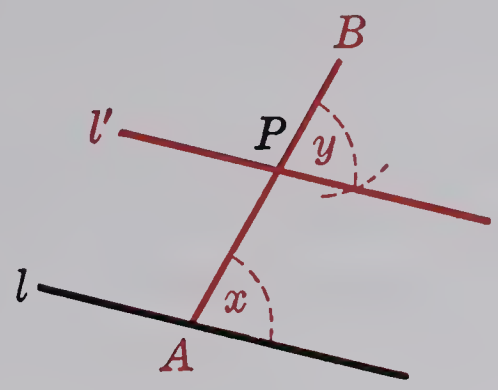
1. Construct a 45° angle. (What part of a right angle is this?)
2. Construct a 60° angle. (Construct an equilateral $\triangle$.)
3. Construct a 30° angle; a 15° angle.
4. Construct a 75° angle. ($75° = 60° + 15°$.)
5. Construct the bisectors of the angles of an acute triangle; of a right triangle; of an obtuse triangle. What common relation do the bisectors of the angles of a triangle seem to have?

B

6. Construct perpendiculars to the sides of a triangle from a point outside the triangle.
7. Construct a right triangle and then construct the perpendicular bisectors of its legs.
8. Construct the perpendicular bisectors of the sides of an acute triangle; of an obtuse triangle. What common relation do the perpendicular bisectors of the sides of a triangle seem to have?
9. Construct a square (a quadrilateral having four equal sides and a right angle).

★ Problem 9

174. Through a given point to construct a line parallel to a given line.



Given the point P and the line l.

To construct the line through $P \parallel l$.

Construction

STATEMENTS	REASONS
1. Draw BA through P and any point A in l.	1. Why possible?
2. With P as the vertex and PB as a side construct $\angle y = \angle x$.	2. § 167.
Then $l' \parallel l$.	

Proof

1. $\angle y = \angle x$.	1. Const.
2. $\therefore l' \parallel l$.	2. Why?

EXERCISES Constructions

A

1. Make the construction for Prob. 9 by constructing a pair of equal alternate interior angles.

2. Make the construction for Prob. 9 by constructing a perpendicular from P to l and then constructing a perpendicular to this perpendicular at P.

3. Construct a line through the vertex of a triangle parallel to the base.

4. Through the vertices of a triangle construct parallels to the opposite sides.

5. Construct a polygon of four sides having the opposite sides parallel.

175. Directions for the Solution of Construction Problems

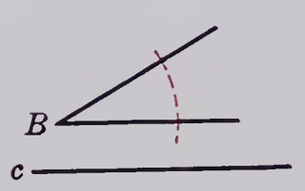
If the solution of a construction problem is not apparent, the following directions will be found useful:

1. Sketch (do not construct) the figure as it will appear when completed.
2. In the sketch indicate the parts of the figure that are given.
3. Study the sketch to discover (a) which given part should be constructed first; (b) how the additional parts can be constructed by means of known constructions and theorems.
4. After determining the method, construct the figure.

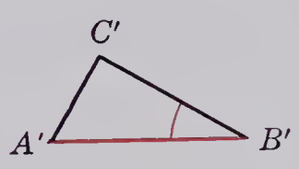
Example. Construct a right triangle, given the hypotenuse and an acute angle.

Given the hypotenuse c and an acute $\angle B$.

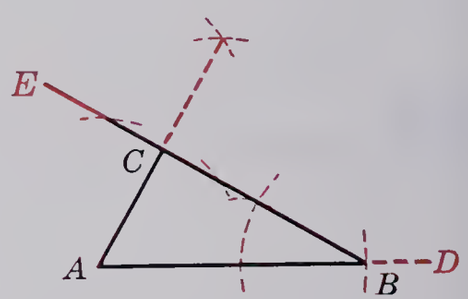
To construct the rt. $\triangle$.



Given



Sketch



Construction

Analysis. At this point in the solution of the problem we determine the method of construction and perform the construction, proceeding according to the directions above as follows:

1. We sketch the $\triangle$ as it will appear when completed and label its vertices A', B', and C'.

2. In the sketch we indicate the given hypotenuse and the given $\angle B'$ by heavy lines, colored lines, or check marks.

3 and 4. From a study of the sketch we see that we can now construct the hypotenuse c and its adjacent $\angle B$. On AD we construct $AB = c$ (§ 9) and $\angle ABE = \angle B$ (§ 167). It remains to locate point C on BE. From the sketch we know that $\angle C'$ is a rt. $\angle$. Then $A'C' \perp B'C'$. This suggests the construction of a $\perp$ from A to BE. We next construct this $\perp$ from A to BE (§ 172), completing the required rt. $\triangle ABC$.

We are now ready to write the *construction* and *proof* of the problem.

EXERCISES — Constructions

A

1. Construct a right triangle, given one leg and the adjacent acute angle.

2. Bisect one side of a given triangle. Through this point of bisection construct a line parallel to another side of the triangle.

3. Construct the three medians of a triangle. (Do the medians appear to meet in a point?)

4. Construct an isosceles triangle, given the base and altitude.

5. Construct an isosceles triangle, given one of the equal sides and the vertex angle.

6. Construct an isosceles triangle, given the base and the sum of its equal sides.

7. Construct an isosceles right triangle, given a leg.

B

8. Construct an angle equal to the sum of two given angles. (Construct two adjacent angles equal respectively to the given angles.)

9. Construct an angle equal to the sum of the angles of a given triangle.

10. Given an acute scalene triangle ABC. Construct the altitude on AB, the median on AB, and the bisector of angle C.

11. Construct one of the base angles of an isosceles triangle, given the vertex angle. (What is the sum of the three angles of a triangle?)

12. Construct a 30°-60° right triangle, given the hypotenuse.

13. Construct a right triangle, given the hypotenuse and one leg.

14. Construct a 30°-60° right triangle, given the shorter leg.

C

15. Construct $\triangle ABC$, given AB, AC, and the median on AB.

16. Divide a line segment into eight equal parts.

17. Construct an equilateral triangle, given the altitude.

18. Construct an isosceles triangle, given one of the equal sides and the altitude to the base.

19. Construct a triangle, given two angles and the side opposite one of them.

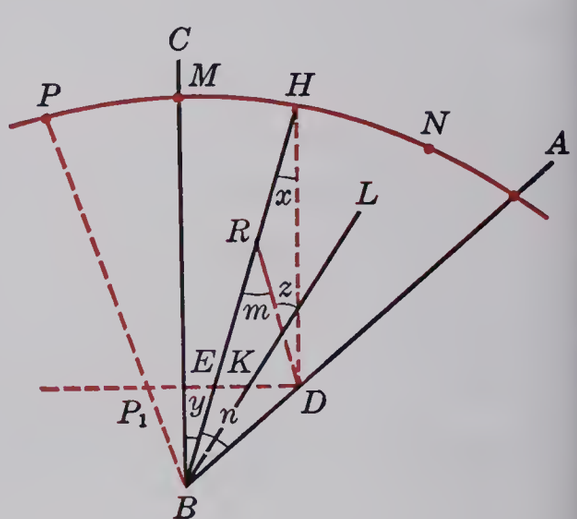
176. Trisection of an Angle

The *trisection of any given angle* (division of an angle into three equal parts), the *duplication of the cube* (the construction of $\sqrt[3]{2}$, or finding the side of a cube whose volume is double that of a given cube), and the *quadrature of the circle* (squaring a circle, or finding the square whose area is equal to that of a given circle) are the three famous geometric problems of antiquity.

It can be proved that these problems cannot be solved by using only the straightedge and compasses, but the proofs are beyond the scope of high-school mathematics. The early Greeks must have known of the impossibility of trisecting an angle by the use of the straightedge and compasses because they constructed other curves to use in the trisection of an angle.

Nicomedes (about 180 B.C.) invented a curve known as the *conchoid* by which an angle can be trisected and a cube can be duplicated. No doubt he invented this curve for one or both of these constructions.

*Trisecting an Angle Using the Conchoid.** Let $\angle ABC$ be the given angle to be trisected. From D, any point in AB, draw $DE \perp BC$. Taking P_1, any point on DE, draw BP_1. Extend BP_1 so that $P_1P = 2\,BD$. Then P is a point on the upper branch of the conchoid. Other points, such as M, N, and Q, of the curve may be located in the same manner, and a smooth curve may be drawn connecting them. The curve $PMNQ$ is part of the conchoid.



At D erect a $\perp$ to ED intersecting the conchoid at H. Draw BH intersecting ED at K. Draw BL bisecting $\angle ABH$. Then BL and BH trisect $\angle ABC$.

Proof. Take R, the midpoint of KH. Draw DR. $DH \parallel BC$. Why? $\angle x = \angle y$. Why? $RD = RH$ (see § 205). $\angle x = \angle z$. Why? $DB = DR$ (Asmt. 7). $\angle n = \angle m$. Why? But $\angle m = \angle x + \angle z$. Why? Then $\angle n = \angle m = \angle x + \angle z = 2\,\angle x = 2\,\angle y$. Then $\angle HBC = \angle LBH = \angle ABL$, and $\angle ABC$ is trisected.

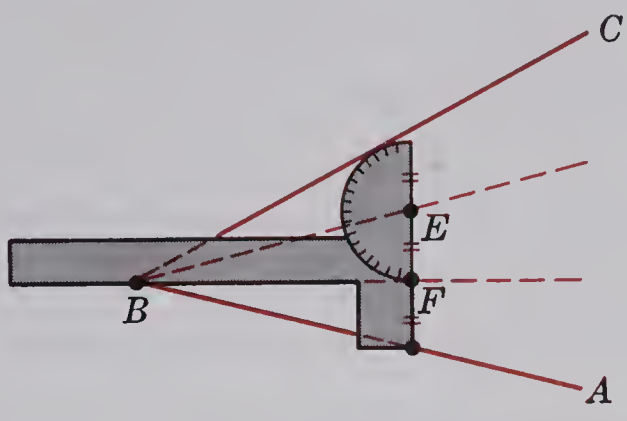
*The algebraic equation of the conchoid is $x^2y^2 = (b + y)^2(a^2 - y^2)$.

Since there are an infinite number of points of the curve and only a finite number of them can be constructed, this method of trisection is not a purely geometric one.

EXERCISES — Miscellaneous Exercises

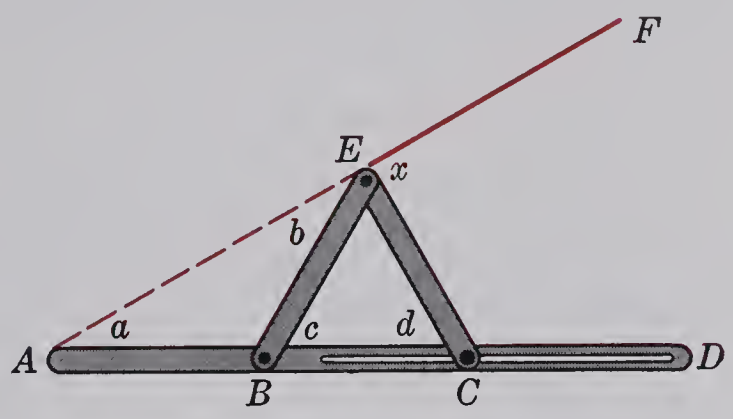
C

1. The figure at the right shows how to trisect an angle ABC by means of an instrument known as an angle trisector. BE and BF are the lines of trisection. Make a trisector and trisect an angle with it.



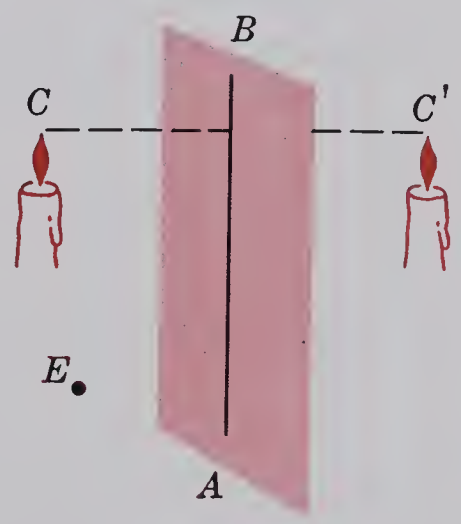
2. Construct a right angle and trisect it by using the angle trisector.

3. Another instrument, supposedly invented by the famous French mathematician Blaise Pascal, is shown at the right. It is composed of three bars hinged at B and E and having $AB = BE = EC$ with C free to move back and forth along AD. Show that if E is placed at the vertex of a given angle x with EC along one side and A on the other side extended, $\angle a = \frac{1}{3} \angle x$.



4. A ray of light starting from C is reflected by a plane mirror AB to point E. Locate the point where the ray strikes the mirror. (See Ex. 1, p. 168).

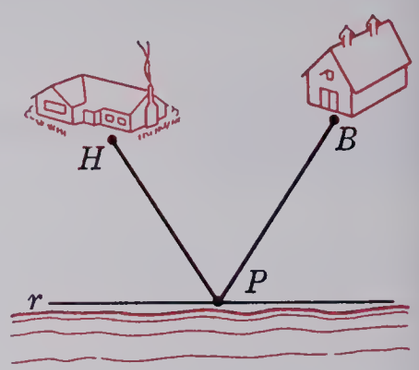
5. Charles, standing 10 feet in front of a plane mirror, sees in the mirror an object located 6 feet in front of the mirror. Construct the paths of the incident and reflected rays, using a scale of 1 inch = 4 feet.



6. Two plane mirrors are perpendicular to each other. A ray of light from an object C is reflected by the mirrors to the eye E. Construct the path of the ray of light.

7. Two plane mirrors are parallel. A ray of light from an object O is reflected by the mirrors to the eye at E. Construct the path of the ray.

8. A man wishes to pipe water from a pump P on a river r to his house H and his barn B which are on the same side of the river. By means of a construction show where he should place the pump in order to use the least amount of pipe.



9. A man sees his eye reflected by two plane mirrors placed at an angle of 60°. Construct the path of the ray of light which starts from his eye and after being reflected by the two mirrors returns to his eye.

REVIEW QUESTIONS — Chapter 6

1. How is a line located? How is a point located?

2. Explain the difference between a problem and a theorem.

3. Name in order the seven steps in the solution of a problem.

4. Make the following constructions:

a. Construct a triangle, given the three sides.

b. Bisect a given angle.

c. With a given vertex and a given side construct an angle equal to a given angle.

d. Construct a triangle, given two sides and the included angle.

e. Construct a triangle, given two angles and the included side.

f. Construct the perpendicular bisector of a given line segment.

g. Construct a perpendicular to a given line from a given point not on the line.

h. Construct a perpendicular to a given line at a given point in the line.

i. Through a given point construct a line parallel to a given line.

WORD LIST

Can you spell and use all the following words correctly?

adjacent	included	parallel	proposition
construction	isosceles	perpendicular	theorem
discussion	miscellaneous	problem	trisection

SUMMARY of Construction Problems

177. Bisectors

a. To bisect a given angle.
b. To construct the perpendicular bisector of a given line segment.

178. Perpendiculars

a. To construct a perpendicular to a given line from a given point outside the line.
b. To construct a perpendicular to a given line at a given point in the line.

179. Parallel

Through a given point to construct a line parallel to a given line.

180. Equal angles

With a given vertex and a given side to construct an angle equal to a given angle.

181. Triangles

a. To construct a triangle, given the three sides.
b. To construct a triangle, given two sides and the included angle.
c. To construct a triangle, given two angles and the included side.

TEST 13 Constructions

[25 Minutes]

Make the following geometric constructions as accurately as you can:

1. Construct a triangle congruent to a given triangle.

2. Construct an altitude of a given triangle.

3. Construct a circle having a given diameter.

4. Construct a line passing through a given point and parallel to a given line.

5. Construct an isosceles triangle, given one of the equal sides and the vertex angle.

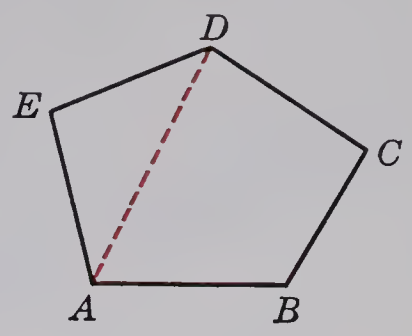
CHAPTER 7

Polygons

You have already studied polygons having three sides. Now you will extend your study to include polygons having more than three sides.

182. Polygon

A *polygon* is a closed broken line in a plane. $ABCDE$ is a polygon. Points A, B, C, D, and E are the *vertices* of the polygon, and AB, BC, CD, DE, and EA are the *sides*. The *perimeter* of a polygon is the sum of its sides. A *diagonal* of a polygon is a line segment joining any two non-adjacent vertices, as AD.



183. a. Polygons Classified as to Angles

An *equiangular* polygon is a polygon having all its angles equal.

A *convex* polygon is a polygon having each angle less than a straight angle. A *concave* polygon is a polygon having at least one angle greater than a straight angle. Unless otherwise stated, "polygon" will mean "convex polygon."

Equiangular Convex Concave Equilateral Regular

b. Polygons Classified as to Sides

An *equilateral* polygon is a polygon having all its sides equal. Polygons are named according to the number of sides, as follows:

Number of Sides	Kind of Polygon	Number of Sides	Kind of Polygon
3	Triangle	8	Octagon
4	Quadrilateral	9	Nonagon
5	Pentagon	10	Decagon
6	Hexagon	15	Pentadecagon
7	Heptagon	n	n-gon

184. Regular Polygon

A *regular* polygon is a polygon which is both equilateral and equiangular.

185. Parallelogram

A *parallelogram* (▱) is a quadrilateral having two pairs of parallel sides.

A *rhombus* is a parallelogram that has two equal adjacent sides.

A *rectangle* is a parallelogram having one right angle. The corollary in § 191 proves that all angles of a rectangle are right angles.

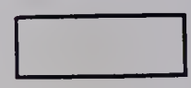
A *square* is a rectangle having two equal adjacent sides.

Study the following outline to see the relationships between these figures.

Quadrilateral . . . . .

A. Parallelogram . . . . .

(1) Rhombus . . . . .

(2) Rectangle . . . . . 

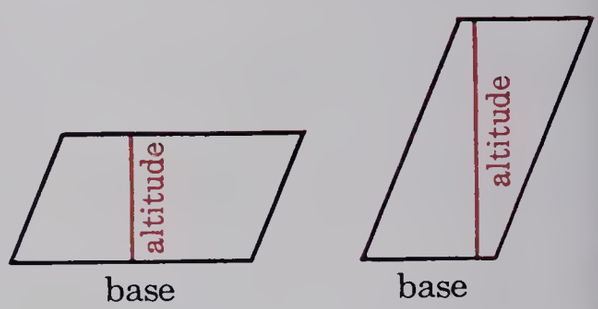
(a) Square . . . . .

B. Trapezoid . . . . .

NOTE. Since a rhombus and a rectangle are parallelograms, every theorem which applies to a parallelogram will also apply to a rhombus and a rectangle.

186. Base and Altitude of a Parallelogram

Any side of a parallelogram may be considered as its *base*. The base is usually the side on which the figure stands. The *altitude* of a parallelogram is the perpendicular to the base from any point in the opposite side.



EXERCISES

1. Can you prove that a square is a rhombus?

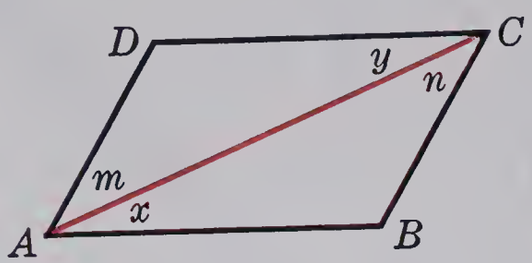
2. Cut a parallelogram from cardboard and stand it on each of its sides in turn. In each case sketch an altitude.

A. Devaney, Inc., N. Y.

The structure shown is the framework for a dome of a large engineering building in an automobile plant. How many types of polygons can you find in the picture? Have you noticed geometric forms in buildings under construction in your own community?

★ Theorem 17

187. A diagonal of a parallelogram divides it into two congruent triangles.



(The demonstration is left to the student.)

SUGGESTIONS. Why is $AB \parallel DC$? $AD \parallel BC$?

★★ **188. Corollary I.** The opposite sides of a parallelogram are equal.

★★ **189. Corollary II.** The opposite angles of a parallelogram are equal.

190. Corollary III. Any two consecutive angles of a parallelogram are supplementary.

(The proof of this corollary is based on § 113.)

191. Corollary IV. All angles of a rectangle are right angles.

192. Corollary V. Segments of parallels included between parallels are equal.

193. Corollary VI. Two parallels are everywhere equidistant.

NOTE. From Cor. VI we know that the altitudes of a parallelogram are equal.

194. Corollary VII. All sides of a rhombus are equal.

EXERCISES

A

1. Which corollary states that all sides of a square are equal?

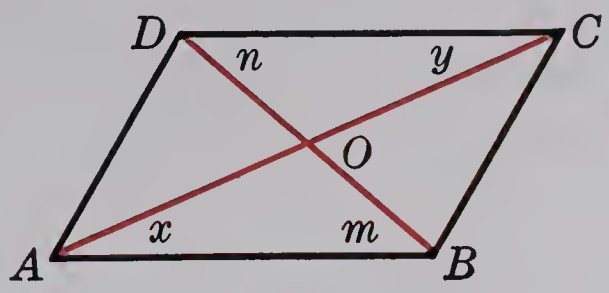
2. If points E and F trisect the diagonal AC of $\square ABCD$, then $BE = DF$.

3. Draw a rhombus $ABCD$. Prove that the altitude from C to AB equals the altitude from C to AD.

4. Two opposite angles of a parallelogram contain $(8x - 3)°$ and $(5x + 9)°$. How many degrees are there in each angle?

★ Theorem 18

195. The diagonals of a parallelogram bisect each other.



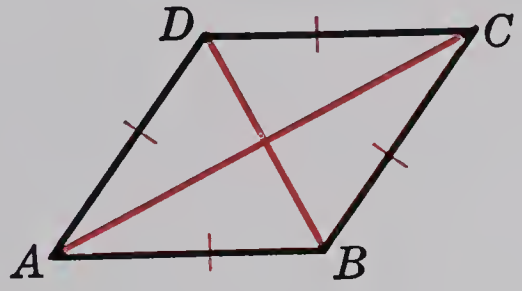
Given the $\square\, ABCD$ with the diagonals AC and BD intersecting at O.
To prove that $AO = OC$ and $BO = OD$.

Planning the Proof: 1. We can prove line segments equal by §§ 97, 157, 188, 192, 193, 194.
2. You can use § 97 a.

Proof: (The proof is left to the student.)

Theorem 19

196. The diagonals of a rhombus are perpendicular to each other.



Given the rhombus $ABCD$ with the diagonals AC and BD.
To prove that $AC \perp BD$.

Planning the Proof: 1. We can prove lines $\perp$ by §§ 51, 159.
2. You can use § 159 b.

Proof: (The proof is left to the student.)

EXERCISES — Parallelograms

A

1. Name the ways you have learned in this chapter for proving

a. Two line segments equal.
b. Two angles equal.
c. Two triangles congruent.
d. Two lines perpendicular.
e. Two angles supplementary.

Have you found these ways on pages 563–565?

2. If one angle of a parallelogram contains 65°, how many degrees are there in each of the other angles?

3. One angle of a parallelogram is three times as large as another. Find the number of degrees in each of the angles of the parallelogram.

4. Is a square a rectangle? a parallelogram? a rhombus?

5. Is a rectangle a square? a parallelogram? a rhombus?

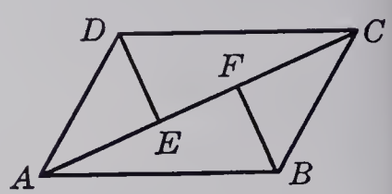
6. Name some of the properties of a square.

7. The diagonals of a rhombus bisect the angles through which they pass.

8. The diagonals of a rectangle are equal.

B

9. Perpendiculars drawn to a diagonal of a parallelogram from opposite vertices are parallel and equal.

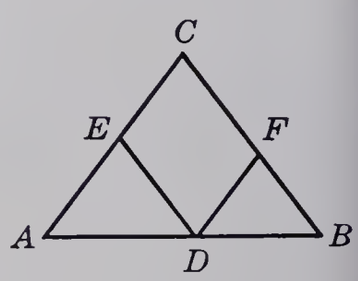


10. The diagonals of rhombus $ABCD$ intersect in O. BO is 6 inches and $\angle ADC$ is 120°. Find DO, AD, and DC.

11. If the bisectors of two opposite angles of a parallelogram are terminated by the sides, the bisectors are parallel and equal.

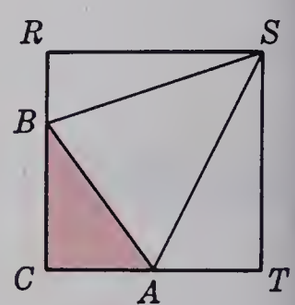
C

12. If from any point in the base of an isosceles triangle parallels to the equal sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the equal sides.



13. The bisectors of two consecutive angles of a parallelogram are perpendicular to each other.

14. In the figure at the right $\angle C$ is a right angle, SB bisects $\angle ABR$, SA bisects $\angle TAB$, $SR \perp CB$, and $ST \perp CA$. Prove that $CTSR$ is a square.

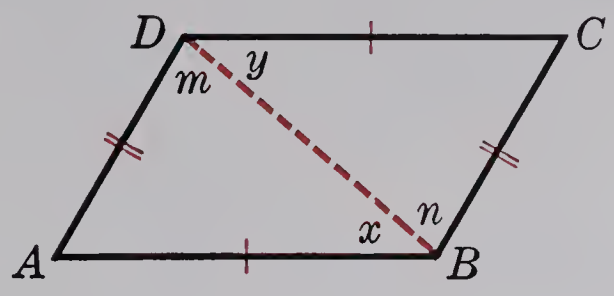


SUGGESTION. Use Theorem 10.

15. Prove that the median to one side of a triangle and the line segment joining the midpoints of the other two sides bisect each other.

★★ Theorem 20

197. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.



Given quadrilateral $ABCD$ with $AB = DC$ and $AD = BC$.

To prove that $ABCD$ is a $\square$.

Planning the Proof: 1. We can prove that a quadrilateral is a $\square$ by § 185.
2. We shall use § 185.

Proof

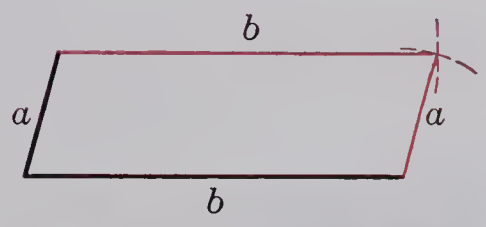
STATEMENTS	REASONS
1. Draw BD.	1. Why possible?
In $\triangle ABD$ and BCD,	
2. $AD = BC$ and $AB = DC$.	2. Why?
3. $BD = BD$.	3. Why?
4. $\therefore \triangle ABD \cong \triangle BCD$.	4. Why?
5. $\therefore \angle x = \angle y$.	5. Why?
6. $\therefore AB \parallel DC$.	6. Why?
7. $\angle m = \angle n$.	7. Why?
8. $\therefore AD \parallel BC$.	8. Why?
9. $\therefore ABCD$ is a $\square$.	9. § 185.

EXERCISES Parallelograms

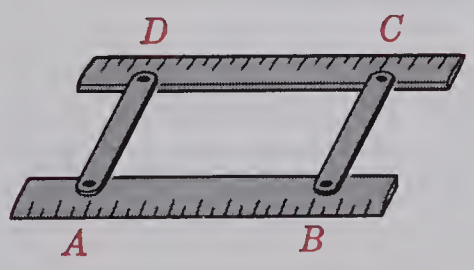
A

1. Name two ways of proving that a quadrilateral is a parallelogram.

2. Explain why the construction shown gives a parallelogram.

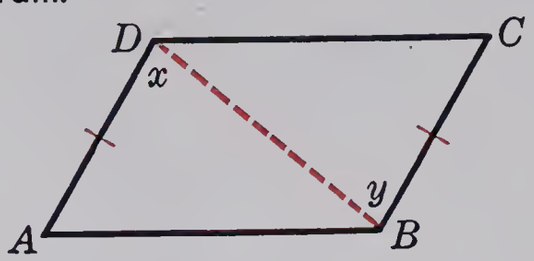


3. Why is the edge DC of the parallel ruler shown in the figure always parallel to the edge AB?



★★ Theorem 21

198. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.



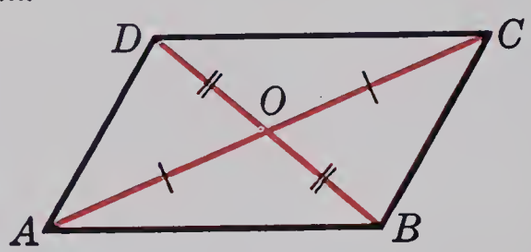
Given quadrilateral $ABCD$ with $AD = BC$ and $AD \parallel BC$.
To prove that $ABCD$ is a $\square$.

Planning the Proof: 1. We can prove that a quadrilateral is a $\square$ by §§ 185, 197. 2. You can use § 197 or § 185.

Proof: (The proof is left to the student.)

Theorem 22

199. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



Given quadrilateral $ABCD$ with $AO = OC$ and $DO = OB$.
To prove that $ABCD$ is a $\square$.

Planning the Proof: 1. We can prove that a quadrilateral is a $\square$ by §§ 185, 197, 198. 2. You can use § 197 or § 198.

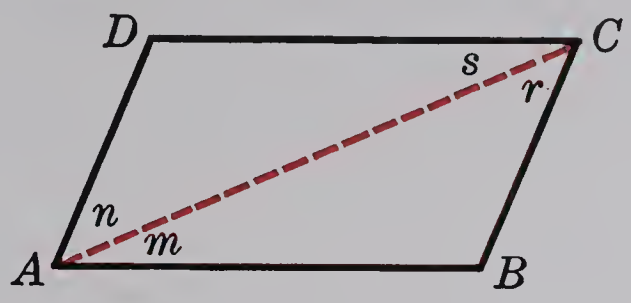
Proof: (The proof is left to the student.)

EXERCISE

Join two unequal sticks with a nail through their mid-points and join their consecutive ends with rubber bands to form a quadrilateral. What special quadrilateral do you have regardless of how you rotate the sticks about their point of joining?

Theorem 23

200. If the opposite angles of a quadrilateral are equal, the quadilateral is a parallelogram.



Given quadrilateral $ABCD$ having $\angle A = \angle C$ and $\angle B = \angle D$.

To prove that $ABCD$ is a $\square$.

Planning the Proof: 1. We can prove a quadrilateral is a $\square$ by §§ 185, 197, 198, 199.

2. You can use § 185.

Proof: (The proof is left to the student. HINT: Draw AC. $\angle m + \angle B + \angle r = ?$ $\angle s + \angle D + \angle n = ?$ $\angle A + \angle B + \angle C + \angle D = ?$ Now use § 106.)

201. Discovering a Proof

Pupils sometimes say, "I can understand a proof after it has been demonstrated, but I do not know how to discover a proof." If you have this difficulty, make a careful study of what follows.

The general statements on making a proof which are given on pages 103–105 are restated below with comments:

I. Read the theorem carefully, being certain that you know the meaning of each word used. Determine which is the hypothesis and which is the conclusion.

II. Draw a figure to illustrate each point, angle, and line described in the hypothesis. Make the figure as general as possible. For example, when a triangle is given, do not draw an isosceles triangle.

Letter the figure, using capital letters for points and small letters (lower case) for angles. If you wish to name a line by one letter, use a small letter.

III. From the hypothesis state in terms of the figure what is given.

IV. From the conclusion state in terms of the figure what you are to prove.

V. Plan a proof for the theorem. See page 210.

VI. Give the proof. Remember that each statement in the argument must be a complete sentence and that it must be supported by the given information, a definition, one of the assumptions which we have accepted as true, or a previously proved theorem or corollary.

Planning the proof. Skill in planning the proof comes with patient practice just as skill comes in playing tennis, in swimming, or in playing the piano.

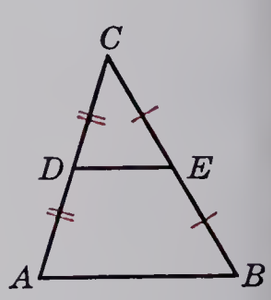
You have studied indirect proof and two kinds of direct proof—analytic and synthetic. You can always try the indirect method when the synthetic and analytic methods fail. The indirect method is often useful in proving a theorem when its converse has been proved. In such a case you show by its converse that one of the possibilities is false.

In using the synthetic method of proof you start with the *given* facts and work toward the conclusion. In using the analytic method you start with the conclusion, saying that it is true if other facts are true, that these facts are true if others are true, and so on until you arrive at the *given* facts.

The best method of discovering a proof is the analytic-synthetic method, which is a combination of the analytic and synthetic methods. Discovering a proof in geometry may be compared to finding an automobile route from one city to another. If you wish to select an automobile route from the town "Given" to another town "Conclusion," you study your map to see which roads lead out of Given towards Conclusion and see which roads lead into Conclusion from the direction of Given. Then you select the best of these roads going into Conclusion and connect it with a road coming from Given. The following example illustrates this selection of roads.

Given $\triangle ABC$ with $AD = DC$, $BE = EC$, and line segment DE.

To prove that $DE \parallel AB$.

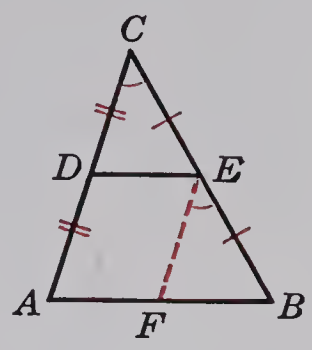


First write the *given* facts as the first steps of the proof and the conclusion as the last step of the proof, namely:

Proof

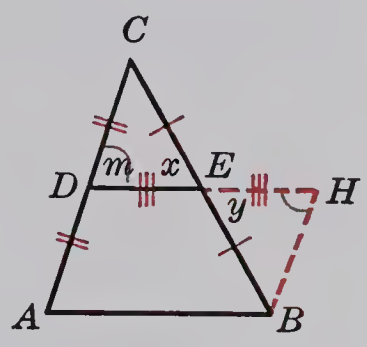
STATEMENTS	REASONS
1. ABC is a $\triangle$.	1. Given.
2. $AD = DC$ and $BE = EC$.	2. Given.
. . . . .	
$DE \parallel AB$.	

There is no statement (road) that leads directly from the given to the conclusion. By referring to § 160 we find that there are five ways of proving lines parallel, none of which leads directly from what is given. This fact suggests the use of construction lines. Let us try making a parallelogram, and perhaps we can use § 185 to reach the conclusion.



Draw $EF \parallel DA$. The figure $AFED$ looks like a $\square$. We can prove that it is a $\square$ if we can show that $EF = DA$, or its equal CD. We can show that $EF = CD$ if we can prove $\triangle FBE \cong \triangle DEC$. But we must stop here, for we are unable to prove the $\measuredangle\!\!\!\measuredangle \cong$.

Let us try another way of making a parallelogram. Extend DE to H so that $EH = DE$. Draw BH. From the figure we know that $\angle x = \angle y$ and that the two triangles are congruent (S.A.S.). It seems that we can prove that $ABHD$ is a $\square$ and then we shall know that $DE \parallel AB$.



There are five ways of proving that a quadrilateral is a parallelogram. What are they? We cannot use § 185. Why? We cannot use §§ 197, 199. Why? Let us try § 198. We know that $DC = BH$. Therefore $AD = BH$. If we can prove $AD \parallel BH$, $ABHD$ is a $\square$. There are six ways of proving two lines parallel. Since $\angle m = \angle H$, $AD \parallel BH$. Why? Then $ABHD$ is a $\square$ and $DE \parallel AB$.

The synthetic arrangement of the proof follows:

Proof

STATEMENTS	REASONS
1. ABC is a $\triangle$.	1. Given.
2. $AD = DC$ and $BE = EC$.	2. Given.
3. Extend DE to H, so that $EH = DE$.	3. Asmt. 13.
4. Draw BH.	4. Asmt. 11.
5. $\angle x = \angle y$.	5. § 50 d.
6. $\triangle CDE \cong \triangle BHE$.	6. S.A.S.
7. $DC = BH$.	7. § 97 a.
8. $\therefore AD = BH$.	8. Asmt. 7.
9. $\angle m = \angle H$.	9. § 98 b.
10. $\therefore AD \parallel BH$.	10. § 160 b(1).
11. $\therefore ABHD$ is a $\square$.	11. § 198.
12. $\therefore DE \parallel AB$.	12. § 185.

EXERCISES — Parallelograms

A

1. Name the four ways of proving that a quadrilateral is a parallelogram. Have you found these methods on pages 566–567?

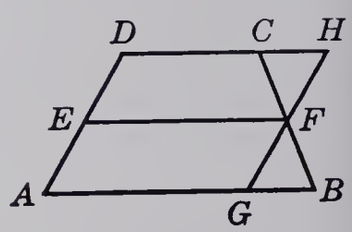
2. If on the same line segment as a base two congruent isosceles triangles are drawn with their vertices on opposite sides of the base, a rhombus is formed.

3. A rectangle is a square if it has perpendicular diagonals.

4. If the median of a triangle is produced its own length through the side to which it is drawn, and the extremity of the part produced is joined to the ends of the side, a parallelogram is formed.

B

5. *Given* $ABCD$, a quadrilateral, with $AB \parallel DC$, $AE = ED$, $EF \parallel AB$, $GH \parallel AD$, and DC produced to H.



Prove as many facts as you can about the figure.

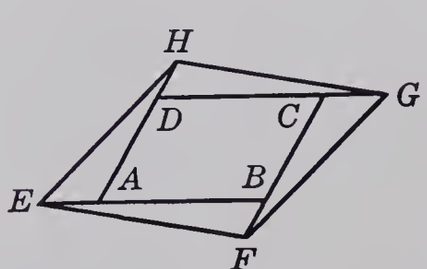
6. If the midpoints of the four sides of a parallelogram are joined in order, another parallelogram is formed.

C

7. A parallelogram is a rectangle if its diagonals are equal.

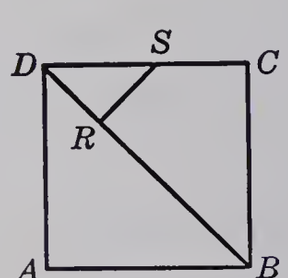
8. The bisectors of the angles of a parallelogram which is not a rhombus form a rectangle. Why is a rhombus excluded?

9. If the midpoints of the four halves of the diagonals of a parallelogram are joined in order, another parallelogram is formed.



10. *Given* $\square ABCD$ with sides extended in succession so that $EA = CG$ and $BF = DH$.

Prove that $EFGH$ is a $\square$.

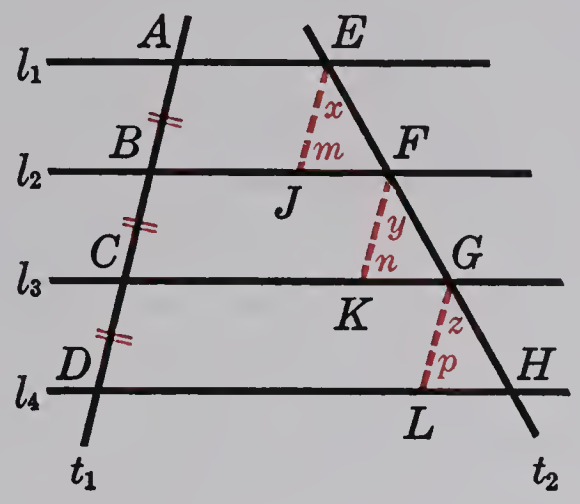


11. *Given* $ABCD$ is a square, $RB = CB$, $SR \perp DB$.

Prove that $DR = RS = SC$.

★★ Theorem 24

202. If parallels intercept* equal segments on one transversal, they intercept equal segments on any transversal.



Given the ‖s l_1, l_2, l_3, and l_4 intercepting the = segments AB, BC, and CD on transversal t_1, and the segments EF, FG, and GH on transversal t_2.

To prove that $EF = FG = GH$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 157, 188, 192, 193, 194, 195.
2. We shall use § 97 a.

Proof

STATEMENTS	REASONS
1. $l_1 \parallel l_2 \parallel l_3 \parallel l_4$.	1. Why?
2. Draw EJ, FK, and GL each $\parallel t_1$.	2. § 174.
3. $ABJE$, $BCKF$, and $CDLG$ are Ⓢ.	3. Why?
4. $\therefore EJ = AB$, $FK = BC$, and $GL = CD$.	4. Why?
5. But $AB = BC = CD$.	5. Why?
6. $\therefore EJ = FK = GL$.	6. Why?
7. $EJ \parallel FK \parallel GL$.	7. Why?
8. $\therefore \angle x = \angle y = \angle z$.	8. Why?
9. $\angle m = \angle n = \angle p$.	9. § 158 b.
10. $\therefore \triangle EJF \cong \triangle FKG \cong \triangle GLH$.	10. Why?
11. $\therefore EF = FG = GH$.	11. Why?

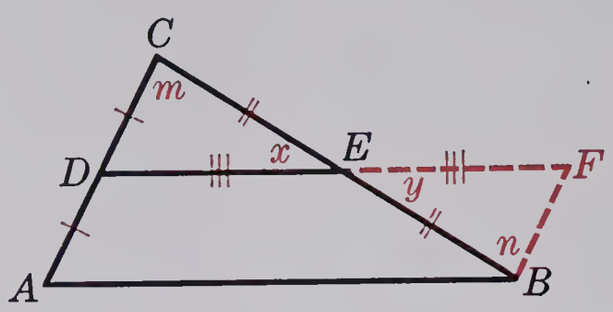
203. Corollary. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.

SUGGESTION. Through the vertex draw a line parallel to the base.

*To intercept is to cut off, or to bound, a portion of a line, plane, surface, or solid.

★ Theorem 25

204. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to one half of it.



Given $\triangle ABC$ with $CD = DA$ and $CE = EB$.

To prove that $DE \parallel AB$ and $DE = \frac{1}{2} AB$.

Planning the Proof: 1. We can prove (a) lines $\parallel$ by §§ 160, 185; (b) line segments $=$ by §§ 97, 157, 188, 192, 193, 194, 195, 202, 203.

2. We shall use (a) § 185; (b) § 188 and Asmt. 4.

Proof

STATEMENTS	REASONS
1. Extend DE to F so that $EF = DE$.	1. Why possible?
2. Join B to F.	2. Why possible?
3. $\triangle CDE \cong \triangle EBF$.	3. Give full proof.
4. $\therefore CD = BF$ and $\angle m = \angle n$.	4. Why?
5. $AD = DC$.	5. Why?
6. $\therefore AD = BF$ and $AC \parallel BF$.	6. Give reasons.
7. $\therefore ABFD$ is a $\square$.	7. Why?
8. $\therefore DF \parallel AB$, i.e., $DE \parallel AB$.	8. Why?
9. $DF = AB$.	9. Why?
10. $\therefore DE = \frac{1}{2} AB$.	10. Why?

NOTE. This theorem has many applications, not all easily recognized. Watch for figures in which the midpoints of two sides of a triangle are given.

EXERCISES — Line Segments

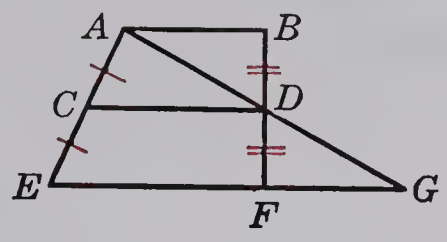
A

1. In the figure above, find DE if $AB = 26$.

2. In the figure above, find AB if $DE = 13.25$.

3. The three sides of a triangle are 8, 10, and 13. Find the lengths of the line segments joining the midpoints of the sides.

4. *Given* $AB \parallel EF$, C the midpoint of AE, D the midpoint of BF, EF and AD produced, meeting at G.



a. *Prove* that $AB = FG$.

b. *Prove* that $CD = \frac{1}{2}\, EG$.

c. *Prove* that $CD = \frac{1}{2}(AB + EF)$.

5. The perpendiculars drawn from the midpoints of two sides of a triangle to the third side are equal.

B

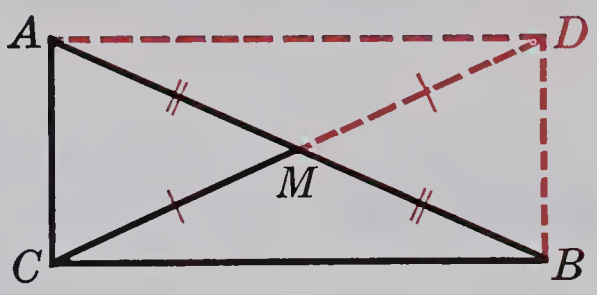
6. The line segment joining the midpoints of two adjacent sides of a quadrilateral is equal and parallel to the line segment joining the midpoints of the other two sides. (Draw a diagonal.)

7. The line segments joining the midpoints of the sides of a quadrilateral, taken in order, form a parallelogram.

8. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other. (Base the proof on Ex. 7.)

Theorem 26

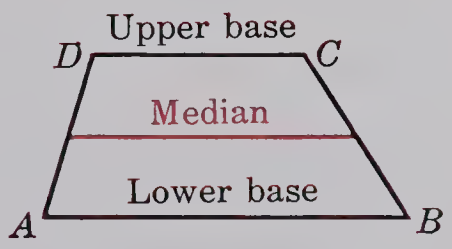
205. The midpoint of the hypotenuse of a right triangle is equidistant from its vertices.



SUGGESTIONS. Extend CM so that $MD = CM$. Draw AD and BD; then $CBDA$ is a $\square$. Why? Since $\angle ACB$ is a rt. angle, $CBDA$ is a rectangle. Then $CD = AB$, and $CM = AM = BM$.

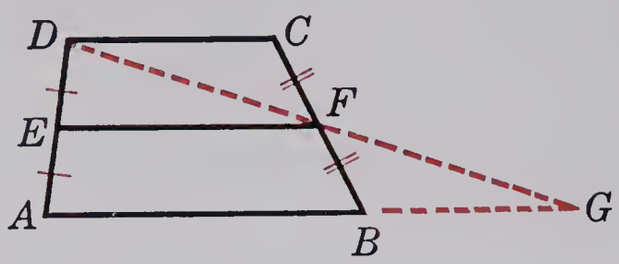
206. Trapezoids

A *trapezoid* is a quadrilateral having one and only one pair of parallel sides. The parallel sides are called the *bases*. The *altitude* of a trapezoid is a perpendicular to one base from any point in the other base. The *median* is the line segment joining the midpoints of the nonparallel sides. An *isosceles* trapezoid is one that has equal nonparallel sides.



Theorem 27

207. The median of a trapezoid is parallel to the bases and equal to one half their sum.



Given the trapezoid $ABCD$ with the median EF.

To prove that $EF \parallel AB$ and DC, and $EF = \frac{1}{2}(AB + DC)$.

Planning the Proof: 1. We can prove (a) lines $\parallel$ by §§ 160, 185, 204; (b) one line segment $= \frac{1}{2}$ another by §§ 136, 204.
2. We shall use (a) § 204; (b) § 204.

Proof

STATEMENTS	REASONS
1. Draw DF.	1. Why possible?
2. Extend DF to meet AB produced at G.	2. Why possible?
3. $\triangle FCD \cong \triangle FBG$.	3. Give full proof.
4. $\therefore DF = FG$ and $DC = BG$.	4. Why?
5. $EF \parallel AG$.	5. § 204.
6. Then $EF \parallel DC$.	6. Why?
7. $EF = \frac{1}{2} AG$, or $\frac{1}{2}(AB + BG)$.	7. Why?
8. $\therefore EF = \frac{1}{2}(AB + DC)$.	8. Why?

EXERCISES — Miscellaneous

A

1. The bases of a trapezoid are 8 and 10 inches respectively. Find the median.

2. The parallel sides of a trapezoid are 12 and 16 inches respectively and the nonparallel sides are 3 and 5 inches. Find the median.

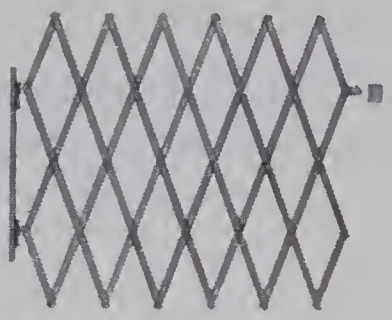
3. The median of a trapezoid is 18 feet and the upper base is 12 feet 8 inches. Find the other base.

4. The angles adjacent to one of the nonparallel sides of a trapezoid are supplementary.

5. The median of a trapezoid bisects each diagonal.

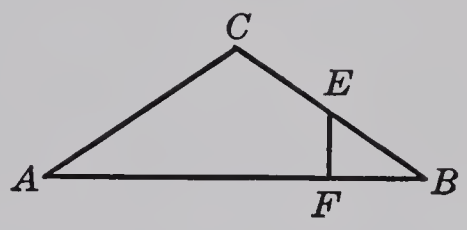
6. A line parallel to the bases of a trapezoid and bisecting one of the nonparallel sides bisects the other nonparallel side.

7. This is a picture of an extension gate. Show that each of the quadrilaterals is a parallelogram.



B

8. Draw an isosceles trapezoid and prove as many facts as you can about it. State each as a theorem. Prove only facts which are true because the trapezoid is isosceles. Additional lines may be added if you wish.

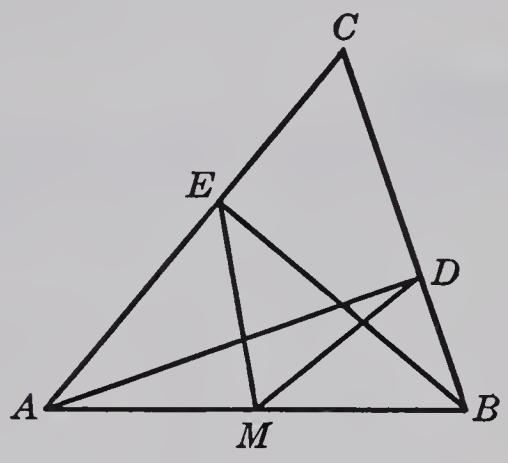


9. *Given* triangle ABC having $AC = BC$, $BE = EC$, and $EF \perp AB$.

Prove that $FB = \frac{1}{4} AB$.

10. If the midpoint of the base of an isosceles triangle is joined to the midpoints of the equal sides, a rhombus is formed.

11. In $\triangle ABC$ altitudes from the vertices A and B meet the opposite sides at D and E, respectively. If M is the midpoint of AB, prove $EM = DM$.

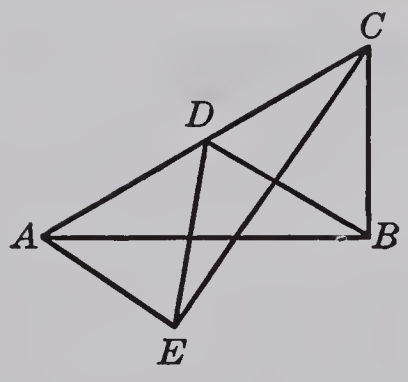


C

12. *Prove:* If the non-parallel sides of a trapezoid make equal angles with the base, the trapezoid is isosceles.

13. In this figure, $AB \perp BC$, $AE \perp EC$, BD bisects AC, and $BD = 12$ inches.

Find ED.

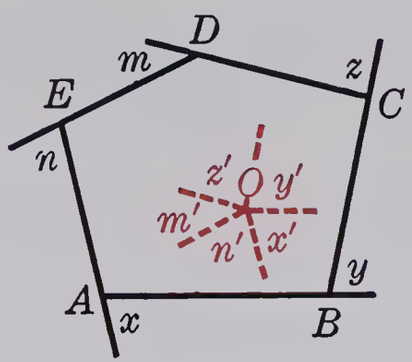


14. The bisectors of two exterior angles of a triangle meet at an angle which is equal to one half the third exterior angle.

15. The bisectors of the angles of a parallelogram which is not a rhombus form a rectangle. Why is a rhombus excluded?

★ Theorem 28

208. The sum of the exterior angles of a polygon, formed by extending each of its sides in succession, is two straight angles.



Given the polygon $ABCDE$ with ext. $\measuredangle$ x, y, z, m, and n.

To prove that $\angle x + \angle y + \angle z + \angle m + \angle n = 2$ st. $\measuredangle$.

Planning the Proof: 1. We can prove a sum of $\measuredangle$ = 2 st. $\measuredangle$ by Asmt. 22.
2. We shall use Asmt. 22.

Proof

STATEMENTS	REASONS
1. Through some point O draw lines ∥ the sides of the polygon.	1. Why possible?
2. $\angle x' + \angle y' + \angle z' + \angle m' + \angle n' = 2$ st. $\measuredangle$.	2. Asmt. 22.
3. $\angle x' = \angle x$, $\angle y' = \angle y$, $\angle z' = \angle z$, etc.	3. § 158 b.
4. $\therefore \angle x + \angle y + \angle z + \angle m + \angle n = 2$ st. $\measuredangle$.	4. Why?

209. Corollary. Each exterior angle of a regular polygon of n sides contains $\frac{360°}{n}$.

EXERCISES — Exterior Angles

A

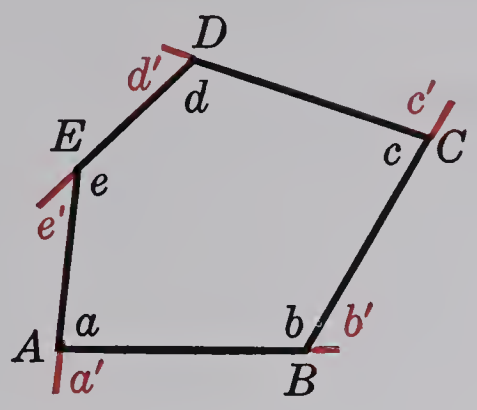
1. Imagine a regular polygon drawn on the floor. Starting at any point in a side, walk once around the polygon. Through what angle do you turn at each vertex? Through how many degrees do you turn in making the circuit?

2. Name the polygons having 3, 4, 5, . . ., 10 sides and state the sum of the exterior angles of each polygon.

3. Why are two of the exterior angles of a right triangle obtuse?

★ Theorem 29

210. The sum of the interior angles of a polygon having n sides is $(n-2)$ straight angles.



Given the polygon $ABCDE$ with angles a, b, c, d, and e.

To prove that $\angle a + \angle b + \angle c + \angle d + \angle e = (n-2)$ st. $\measuredangle$s.

Planning the Proof: 1. We can prove the values of sums of angles by Asmt. 22, §§ 52, 53, 120, 123, 161, 162, 208.
2. We shall use § 208.

Proof

STATEMENTS	REASONS
1. Extend the sides of $ABCDE$ in succession to form the exterior $\measuredangle$s a', b', c', d', and e'.	1. § 92; Asmt. 13.
2. $\angle a + \angle a' = 1$ st. $\angle$; $\angle b + \angle b' = 1$ st. $\angle$; $\cdots$.	2. Asmt. 24.
3. $\angle a + \angle a' + \angle b + \angle b' + \angle c + \angle c' + \cdots = n$ st. $\measuredangle$s.	3. Asmt. 1.
4. But $\angle a' + \angle b' + \angle c' + \angle d' + \angle e' = 2$ st. $\measuredangle$s.	4. § 208.
5. $\therefore \angle a + \angle b + \angle c + \angle d + \angle e = (n-2)$ st. $\measuredangle$s.	5. Asmt. 2.

NOTE. If we represent the sum of the interior angles by I and the sum of the exterior angles by E, statements 3, 4, and 5 of the proof above may be summarized as follows: $I + E = n$, and $E = 2$. Then $I = n - 2$.

211. Corollary. Each interior angle of a regular polygon of n sides contains $\dfrac{180(n-2)^\circ}{n}$.

EXERCISES

Prove Theorem 29 by drawing line segments from the vertices to a point within the polygon.

EXERCISES — Miscellaneous

A

Example 1. Find the number of degrees in each interior angle of a regular nonagon.

Solution. From § 209, each ext. $\angle = \frac{360°}{n}$.

Substituting $n = 9$, each ext. $\angle = \frac{360°}{9}$, or $40°$.

Then each int. $\angle = 180° - 40° = 140°$.

Example 2. Find the number of sides of a regular polygon if each interior angle is 135°.

Solution. Each ext. $\angle = 180° - 135° = 45°$.

By § 209 no. of degrees in each ext. $\angle = \frac{360}{n}$.

$$\therefore 45 = \frac{360}{n};$$

whence $45\,n = 360$,

and $n = 8$, no. of sides.

Example 3. Find the number of sides of a polygon if the sum of the interior angles is 1080°.

Solution. Let n = no. of sides of the polygon.

From § 210, $(n - 2)180$ = no. of degrees in the sum of the int. $\measuredangle$s.

Substituting, $(n - 2)180 = 1080$,

or $180\,n - 360 = 1080$;

whence $180\,n = 1440$,

and $n = 8$, no. of sides.

1. How many degrees are there in the sum of the exterior angles of a triangle? of a quadrilateral? of any polygon?

2. Using § 209, find the number of degrees in each exterior angle of an equilateral triangle; a regular pentagon; a regular hexagon.

3. Find the number of straight angles in the sum of the interior angles of a quadrilateral; a pentagon; a hexagon; a heptagon; an octagon; a nonagon; a decagon. Find the number of degrees in the sum of the interior angles of each polygon.

4. Find the number of degrees in each interior angle of a regular pentagon; a regular hexagon; a regular heptagon; a regular octagon; a regular polygon of fifteen sides.

5. How many degrees are there in the sum of the interior angles of a parallelogram? a rhombus? a trapezoid?

6. The sum of three angles of a trapezoid is 290°. How many degrees are there in the fourth angle?

7. As the number of sides of a regular polygon increases, does an exterior angle increase or decrease? an interior angle?

8. A surveyor gave the following measurements for a field in the form of a quadrilateral: $AB = 18$ rods, $\angle B = 128°$; $BC = 12.5$ rods, $\angle C = 102°$; $CD = 21$ rods, $\angle D = 56°$; $DA = 26.3$ rods, $\angle A = 72°$. Can you find an error in his work?

9. One angle of a triangle contains 84°. Find the two other angles if they have the ratio 1 : 3.

10. One leg of a right triangle is 4 inches and the hypotenuse is 8 inches. Find the length of the median to the hypotenuse.

B

11. How many sides has a regular polygon if each exterior angle contains 40°? 45°? 60°? $51\frac{3}{7}°$? 120°?

12. How many sides has a regular polygon if each interior angle contains 108°? 144°? 150°? 162°? (First find the exterior angle.)

13. Find the number of sides of a polygon if the sum of its interior angles is 540°; 900°; 8 rt. ∡s; 7 st. ∡s.

14. Two exterior angles of a triangle contain 200°. How many degrees are there in the third exterior angle?

15. If the sum of seven angles of an octagon is 1000°, find the number of degrees in the remaining angle.

16. How many sides has an equiangular polygon if one ext. $\angle = 22.5°$?

17. What is the smallest angle that any equiangular polygon may have? What is the largest exterior angle that any equiangular polygon may have?

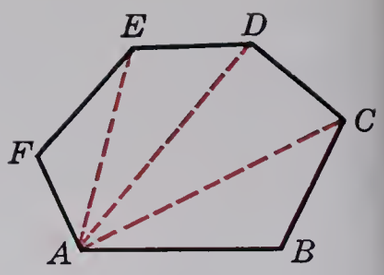
18. Find the number of sides of a polygon if the sum of its interior angles is twice the sum of its exterior angles.

19. By how many degrees is the sum of the angles of a polygon increased when the number of sides is increased by 4?

20. *Given* quadrilateral $ABCD$ with $AD = BC$, diagonal $AC =$ diagonal BD, and O the intersection of the diagonals.

Prove $\triangle ABO$ isosceles.

21. Prove Theorem 29, making use of the accompanying figure.



SUGGESTIONS. From any vertex, as A, draw diagonals as shown. If the polygon has n sides, show that $(n-2)$ triangles are formed. What is the sum of the angles of $(n-2)$ triangles?

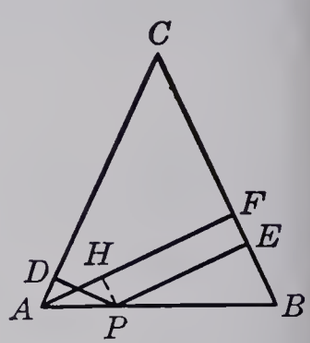
C

22. Linoleum designs and tiled floors are often made of regular polygons. How many squares must be placed about a point to fill the plane about it? How many equiangular triangles? What other regular polygon can be used to fill the plane about a point?

23. Show that a square and two regular octagons can be used to fill the plane about a point.

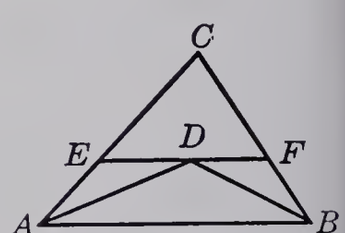
24. *Given* isos. $\triangle ABC$ with P any point in AB, $PD \perp AC$, $PE \perp BC$, and $AF \perp BC$.

Prove that $AF = DP + PE$.



25. If from a point within a given angle perpendiculars are drawn to the sides, the angle formed by the perpendiculars is a supplement of the given angle.

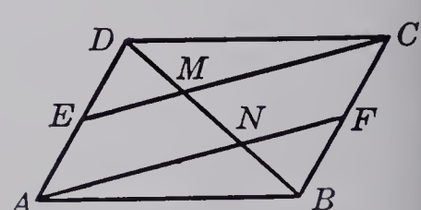
26. The bisector of the right angle of a right triangle bisects the angle included by the altitude and median on the hypotenuse.



27. *Given* $\triangle ABC$ with AD and BD the bisectors of $\measuredangle A$ and B respectively, and $EDF \parallel AB$.

Prove that $EF = AE + BF$.

28. The difference between two consecutive angles of a parallelogram is 80°. Find all the angles.



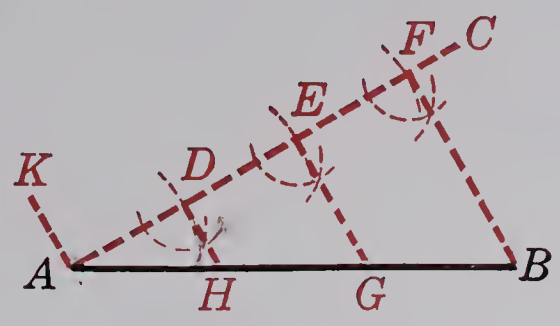
29. *Given* $\square ABCD$ with E and F the midpoints of AD and BC respectively, and the diagonal BD.

Prove that AF and EC trisect BD.

30. State and prove the converse of Theorem 26.

★ Problem 10

212. To divide a given line segment into any number of equal parts.



Given the line segment AB.

To divide AB into any number of $=$ parts (as three).

Construction

STATEMENTS	REASONS
1. Draw a line from A to some point C not in AB.	1. Why possible?
2. Construct three $=$ line segments, AD, DE, and EF on AC.	2. Why possible?
3. From the last point of division F, draw FB.	3. Why possible?
4. Through E and D draw lines $\parallel FB$, intersecting AB at G and H respectively. *Then* $AH = HG = GB$.	4. Why possible?

Proof

1. Through A draw $AK \parallel BF$.	1. Why possible?
2. $AK \parallel DH$ and EG.	2. Why?
3. $AD = DE = EF$.	3. Why?
4. $\therefore AH = HG = GB$.	4. Why?

EXERCISES

A

1. In the figure above, after point G is found, show two ways of finding H without drawing a line through D parallel to FB.

2. Divide a line segment into four equal parts by two methods.

3. Divide the base of a given triangle into five equal parts.

4. Using Problem 10, find a line segment that is three-fifths of a given line segment.

EXERCISES — Construction

A

Before attempting these exercises, review § 175.

Construct a square, given

1. One side.
2. The perimeter.
3. The diagonal.
4. The sum of its diagonals.

Construct a rectangle, given

5. Two adjacent sides.
6. One side and a diagonal.
7. The diagonals and the angle between them.

Construct a rhombus, given

8. The diagonals.
9. The perimeter and one diagonal.
10. One side and one angle.

Construct a parallelogram, given

11. Two adjacent sides and the included angle.
12. The diagonals and the angle between them.

Construct a quadrilateral, given

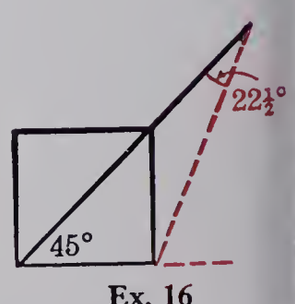
13. The four sides and the angle between two adjacent sides.
14. Two consecutive angles and the three sides which include them.
15. Three angles and the two sides included by them.

B

Construct a square, given

16. The sum of the diagonal and one side.

17. The difference between the diagonal and one side.

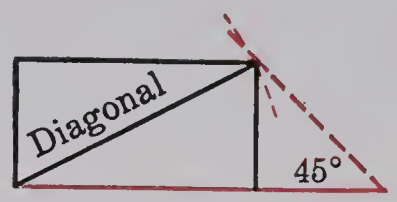


Ex. 16

Construct a rectangle, given

18. One side and the angle formed by the diagonals.

19. The perimeter and one diagonal.



Ex. 19

Construct a rhombus, given

20. One angle and a diagonal.

21. The altitude and one diagonal.

C

Construct a parallelogram, given

22. One side, one angle, and one diagonal.

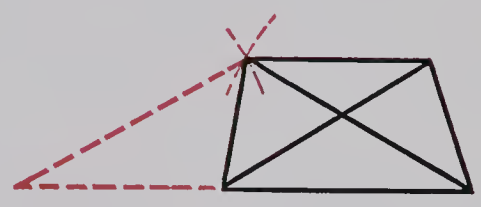
23. Two adjacent sides and an altitude.

24. One side and the diagonals.

Construct a trapezoid, given

25. Four sides.

26. The bases and the diagonals.



Ex. 26

27. The bases and lower base angles.

28. The nonparallel sides and the difference of the bases.

Construct an isosceles trapezoid, given

29. The bases and the altitude.

30. The bases and one angle.

Ex. 30

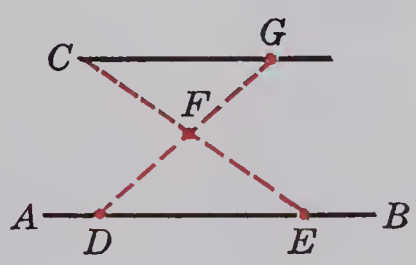
31. The median, altitude, and one of the bases.

EXERCISES Miscellaneous

A, B, and C

1. If a diagonal of a parallelogram bisects one angle, the figure is a rhombus.

2. A man wishing to build a fence through C parallel to AB proceeded in this way: He set stakes at two points, D and E, on line AB, another at F, the midpoint of CE, and another at G in line with DF so that $FG = DF$. Prove that a fence passing through C and G is parallel to AB.

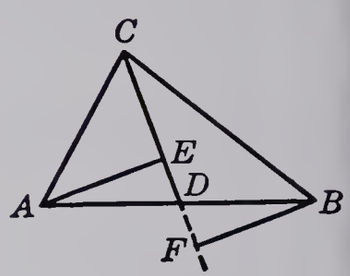


3. If a diagonal of a quadrilateral bisects two of its angles, it is the perpendicular bisector of the other diagonal.

4. The perpendicular bisector of one leg of a right triangle bisects the hypotenuse.

5. The line segments joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles. (See § 204 and § 187).

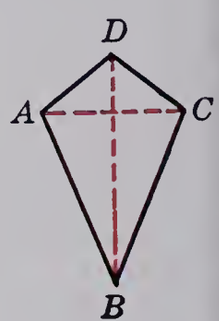
6. The line segments joining the midpoints of the sides of a rhombus, taken in order, form a rectangle.



7. Perpendiculars drawn from the end points of the base of a triangle to the median on the base are equal.

8. In the figure of Ex. 7, draw AF and BE. Prove that $AFBE$ is a parallelogram.

9. Show that the greatest number of diagonals possible in a polygon of n sides is $\frac{n(n-3)}{2}$.



10. A quadrilateral in which two pairs of adjacent sides are equal is called a *kite*. Prove that the longer diagonal divides the kite into two congruent triangles. Prove that the shorter diagonal divides the kite into two isosceles triangles.

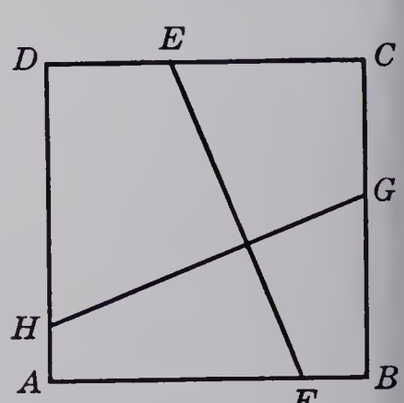
11. Through any point P within $\triangle ABC$ construct a line forming equal angles with AB and BC.

12. If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.

13. The bisector of one angle of a triangle and the bisector of an exterior angle at a second vertex form an angle equal to one half the third angle of the triangle.

14. *Given* square $ABCD$ in which E is any point on DC, F any point on AB, G any point of BC, $GH \perp EF$ and meeting AD (or its extension) in H.

Prove $EF = GH$.

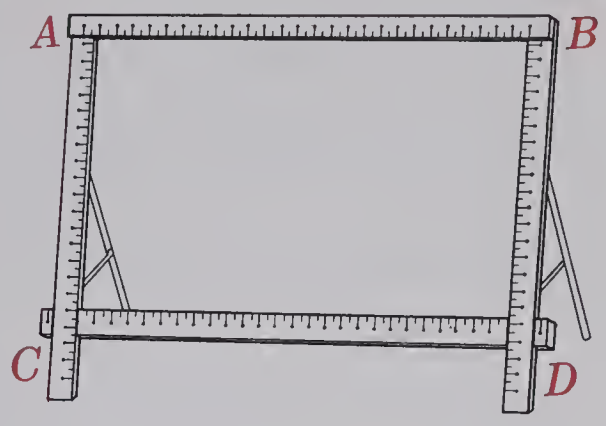


15. Given any four points, construct a square such that each side (or its extension) passes through one of the given points.

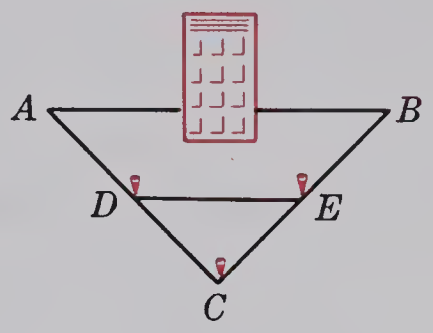
EXERCISES — Practical Problems

1. Explain how you can determine by use of cord whether or not a table top is rectangular.

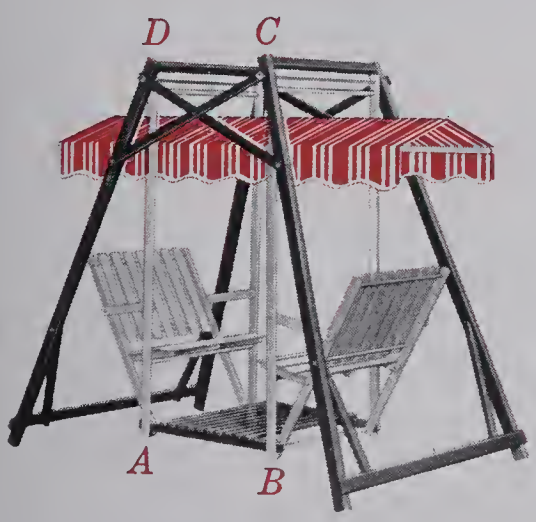
2. At the right is a diagram of a curtain stretcher. The stretcher consists of four boards which are held together by small bolts at A, B, C, and D. A curtain is fastened to the stretcher by small nails. What precautions are needed to have the curtain dry in rectangular form?

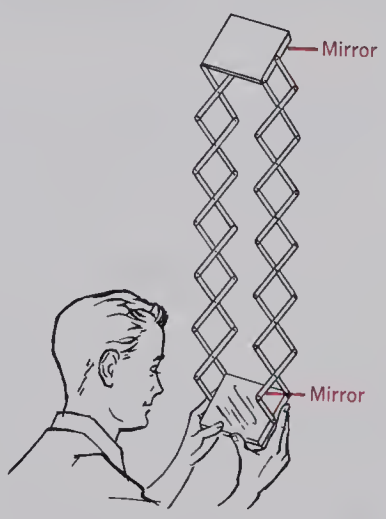


3. A boy, wishing to find the distance between two points, A and B, on opposite sides of a building, set one stake at C, another at D, the midpoint of CA, and one at E, the midpoint of CB. He then measured DE. If DE measured 36 feet, what was the length of AB?

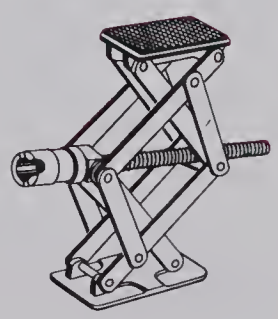


4. Explain why the floor board of a lawn swing remains parallel to the ground. Prove that the hangers, AD and BC, must be parallel if the floor board AB is parallel to the ground and equal to the distance DC between the supports of the hangers.





5. At the right above is shown a collapsible periscope with which one may see over the heads of a crowd. Explain how it is made and what geometric principles are involved in its use.



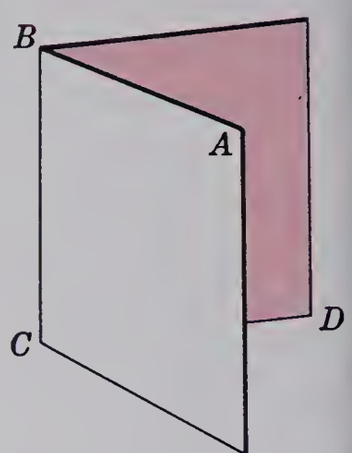
6. As the height of this jack changes, the top always remains parallel to the base. Can you explain why?

SPACE GEOMETRY

[Optional]

213. Dihedral Angles

Since two intersecting lines determine a plane, the angles we have studied so far are plane angles. An angle formed by two planes that meet is a *dihedral angle*. The two planes are called the *faces*, and their line of meeting is called the *edge*, of the dihedral angle. A dihedral angle may be read by naming a point in one face, the edge, and a point in the other face, as dihedral $\angle A\text{-}BC\text{-}D$. When there is no doubt as to the meaning, a dihedral angle may be read by naming its edge, as dihedral $\angle BC$.



214. Perpendicular Planes

Two *planes are perpendicular* to each other if one of them forms two equal adjacent dihedral angles with the other.

EXERCISES

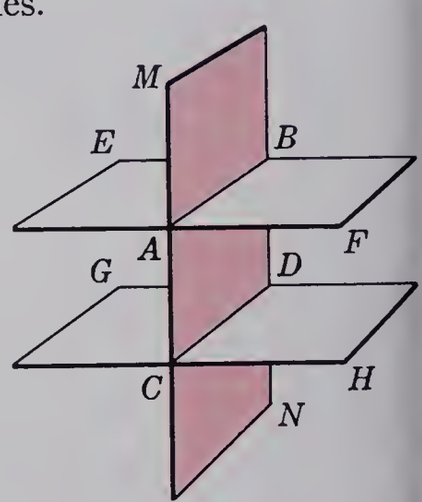
Using two rectangular pieces of cardboard, represent

1. Two perpendicular planes.

2. A right dihedral angle; an acute dihedral angle; an obtuse dihedral angle.

3. Two supplementary adjacent dihedral angles.

In the figure at the right, two planes, EF and GH, are cut by a third plane, MN.



4. Name two pairs of alternate interior dihedral angles; two pairs of alternate exterior dihedral angles.

5. Name two pairs of corresponding dihedral angles.

6. Name two pairs of vertical dihedral angles.

7. If planes *EF* and *GH* are parallel, do you think that the alternate interior dihedral angles are equal? the corresponding dihedral angles are equal?

Using pencils, rectangular pieces of cardboard, and a desk top as illustrations, answer the following questions:

8. Through a line perpendicular to a plane, how many planes perpendicular to the given plane may be drawn?

9. Through a line oblique to a plane, how many planes perpendicular to the given plane may be drawn?

10. Through a line parallel to a plane, how many planes perpendicular to the given plane may be drawn?

11. If a line is parallel to a given plane, is a plane that is parallel to the line parallel to the given plane also?

12. If two intersecting planes are perpendicular to a third plane, is their intersection perpendicular to the third plane?

13. Are two planes parallel to each other if they are parallel to a third plane?

14. Are two planes parallel to each other if they are perpendicular to the same plane?

15. Are two lines parallel if they are parallel to the same plane?

16. Are two lines parallel if they are perpendicular to the same plane?

17. If one of two parallel lines is perpendicular to a plane, is the other perpendicular to the plane?

18. Is it possible for a line to be perpendicular to each of two skew lines?

19. How many intersecting lines may be parallel to a line?

20. How many intersecting lines may be parallel to a plane?

21. How many intersecting planes may be parallel to a line?

22. Can a line be perpendicular to two lines in a plane?

23. When can a plane be perpendicular to both a given plane and a given line?

24. Through a given point outside a given plane how many planes may be drawn parallel to the given plane?

25. Are two lines parallel if they are parallel to the same line?

26. What is the shortest line segment that can be drawn from a point to a plane?

27. Are two parallel planes everywhere equidistant?

215. Reasoning with False General Statements

If, in our reasoning, we accept incorrect general statements, we may arrive at false conclusions. For example, if we accept the specific statement *A spruce is a tree*, and the general statement *All trees lose their leaves in the fall*, then we are forced to conclude: A spruce loses it leaves in the fall. In this case the incorrect general statement leads us to the false conclusion.

Unless we are careful, we may make false general statements because we *reason from special cases*. In this kind of reasoning we base our generalizations upon insufficient evidence.

Example. Mrs. Frank said that Mr. Winkler was a grafter because he was a policeman. Her general statement was, "All policemen are grafters," but she had arrived at this conclusion only by noting that two policemen had accepted tips from a storekeeper. She was judging all policemen by the two she had observed accepting tips.

Sometimes we arrive at false conclusions because we *reason by analogy*. In this kind of reasoning we assume that if two things agree in one or more ways, they agree in other ways.

Example. John said, "Frank and I make the same grades in school and have the same I.Q.'s, and look enough alike to be twins. If he can get a contract to appear on television, I can too." Why was John's reasoning poor?

EXERCISES — Optional

Discuss the reasoning used in each of the following statements.

1. Our football team will be no good this year. It lost its first game.

2. Bill's father bought a Roxie television set and had all kinds of trouble with it. I do not want a Roxie.

3. By using Reduco, Clara reduced her weight from 140 pounds to 125 pounds. I am going to be slender because I am using Reduco.

4. My parents always had difficulty with mathematics, so it is foolish for me to study it.

5. Jim's dog likes Fido dog biscuits, so I know my dog will like Fido dog biscuits.

REVIEW QUESTIONS — Chapter 7

1. What facts have you proved about parallelograms? about rectangles? about trapezoids?

2. Name the ways of proving that a quadrilateral is a parallelogram.

3. In what quadrilaterals do the diagonals bisect the angles?

4. In what quadrilaterals are the diagonals equal?

5. In what quadrilaterals do the diagonals bisect each other?

6. In what quadrilaterals are the diagonals perpendicular to each other?

7. What two facts may be stated about the diagonals of a rectangle?

8. What three facts may be stated about the diagonals of a rhombus?

9. What four facts may be stated about the diagonals of a square?

10. As the number of sides of a polygon increases, does the sum of the exterior angles increase or decrease?

11. What is the sum of the interior angles of a quadrilateral? of a triangle? of a pentagon? of a hexagon?

12. What is the sum of the exterior angles of a polygon having six sides? eight sides? ten sides?

13. What do you know about the median of a trapezoid?

14. What do you know about the line segment connecting the midpoints of two sides of a triangle?

15. What is the name of a polygon in which the sum of the interior angles equals the sum of the exterior angles?

16. What constructions have you learned to make?

17. What is the name of the quadrilateral having one, and only one, pair of parallel sides?

18. What is the perimeter of the triangle formed by joining the midpoints of the sides of a triangle whose perimeter is 192 inches?

19. What is the relation between any interior angle of a polygon and its adjacent exterior angle?

20. *Prove:* If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

21. One angle of a triangle is 20°. How large is the angle formed by the bisectors of the other two angles?

22. In $\triangle ABC$, $AC = BC$ and $\angle A = 45°$. How large is $\angle C$?

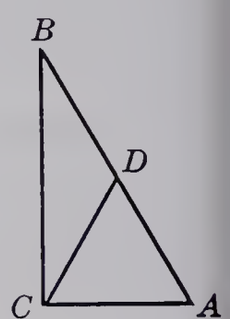
23. $\angle B$ of $\square ABCD$ is 110°. How large is $\angle C$?

24. $\angle C$ of $\square ABCD$ is four times as large as $\angle D$. How large is $\angle B$?

25. How many degrees are there in each interior angle of a regular polygon having 18 sides?

26. How many sides has a regular polygon if each exterior angle contains 24°?

In $\triangle ABC$, $AD = DB$ and $\angle C = 90°$.



Exs. 27–29

27. Find BD if $\angle B = 30°$ and $AC = 10$ inches.

28. Find CA if $CD = 8$ inches and $\angle A = 60°$.

29. Find CD if $AB = 16$ inches and $\angle A = 75°$.

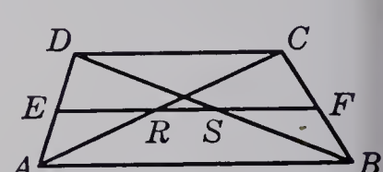
30. The sum of two angles of a triangle is 108° and their difference is 16°. How many degrees are there in each angle of the triangle?

31. In $\square ABCD$, $\angle A = 30°$, $BC = 18$ inches, and $DC = 20$ inches. Find the altitude upon AB as the base.

32. The diagonal AC of rhombus $ABCD$ is $\frac{1}{4}$ the perimeter. How large is $\angle C$?

The median EF of trapezoid $ABCD$ intersects AC in R and BD in S.

33. If $DC = 8$ and $AB = 12$, find EF; ER; SF; RS.



34. If $EF = 24$ and $AB = 30$, find DC; ER; SF; RS.

35. If $ER = 5$ and $RF = 7$, find DC; AB; SF; RS.

36. One angle of a parallelogram is 4 times as large as another. How large is each angle of the parallelogram?

37. In rhombus $ABCD$, $\angle A = 60°$ and $AB = 10$ inches. Find BD.

38. In $\square ABCD$, $\angle B = 150°$ and $AD = 12$ inches. Find the altitude from D to AB.

39. How many diagonals has a decagon?

40. Find the number of diagonals of a polygon having 4 sides; 6 sides; n sides.

SUMMARY of Principal Bases for Proof

216. Two triangles are congruent

If they are formed by a diagonal of a parallelogram.

217. Two line segments are equal

a. If they are opposite sides of a parallelogram.
b. If they are segments of parallels included between parallels.
c. If they are segments of a diagonal of a parallelogram formed by the intersection of the other diagonal.
d. If they are intercepted on a transversal by parallel lines which intercept equal segments on another transversal.
e. If they are segments of one side of a triangle formed by a line which bisects a second side and is parallel to the third side.

218. One line segment is equal to one half another

a. If one joins the midpoints of two sides of a triangle and the other is the third side of the triangle.
b. If one is the median of a trapezoid and the other is equal to the sum of the bases of the trapezoid.

219. Two angles are equal

If they are opposite angles of a parallelogram.

220. Two lines are perpendicular

a. If they are diagonals of a rhombus.
b. If they are adjacent sides of a rectangle.

221. Two lines are parallel

a. If they are opposite sides of a parallelogram.
b. If one is a line connecting the midpoints of two sides of a triangle and the other is the third side of the triangle.
c. If they are the bases of a trapezoid.
d. If one is the median of a trapezoid and the other is one of the bases.

222. Two angles are supplementary

If they are two consecutive angles of a parallelogram.

223. A quadrilateral is a parallelogram

a. If the opposite sides are parallel.
b. If the opposite sides are equal.

c. If two sides are equal and parallel.
d. If the diagonals bisect each other.
e. If the opposite angles are equal.

224. Construction

To divide a given line segment into any number of equal parts.

WORD LIST

Can you spell and use each of the following words?

complementary	equilateral	octagon	polygon
concave	exterior	opposite	quadrilateral
convex	heptagon	parallel	rectangle
decagon	hexagon	parallelogram	rhombus
diagonal	intercept	pentadecagon	supplementary
equiangular	interior	pentagon	trapezoid
equidistant	nonagon	perimeter	vertices

TEST 14 — Multiple-Choice Statements

[10 Minutes]

On your paper write the word or group of words printed in boldface type which makes each statement true:

1. An octagon is a polygon having
7 sides **5 sides** **8 sides** **10 sides.**

2. A trapezoid **is** **is not** a special kind of parallelogram.

3. The sum of the interior angles of a hexagon is
180° **360°** **540°** **720°** **900°.**

4. The consecutive angles of a parallelogram are
right angles **equal angles** **supplementary.**

5. A **parallelogram** **equilateral triangle** **rhombus** is a regular polygon.

6. The segments of two parallel lines included between
two parallels **two perpendiculars** **two oblique lines** are equal.

7. The sum of the exterior angles of a pentagon is
4 right angles **540°** **3 straight angles** **900°.**

8. The line segment joining the midpoints of two sides of a triangle is equal to **one third** **one half** **two thirds** the third side.

TEST 15 — Constructions

[25 Minutes]

Make the following constructions as accurately as you can:

1. Divide a given line segment into three equal parts.

2. Construct a square given the diagonal.

3. Construct a parallelogram given two adjacent sides and the included angle.

4. Construct a rhombus given a side and one angle.

5. Construct an equilateral triangle, given the line segment joining the midpoints of two of the sides.

TEST 16 — Applications

[25 Minutes]

1. If one angle of a parallelogram is 50°, how many degrees are there in each of the other angles?

2. The parallel sides of a trapezoid are 9 inches and 11 inches. How long is the median?

3. In a right triangle the line from the vertex of the right angle to the midpoint of the hypotenuse is 6 inches. How long is the hypotenuse?

4. How many degrees are there in each interior angle of a regular hexagon?

5. How many sides has a polygon each of whose exterior angles contains 45°?

6. The perimeter of an equilateral triangle is 18 inches. What is the perimeter of the triangle formed by joining the midpoints of its sides?

7. In a parallelogram a 30° angle is included between a 4-inch side and an 8-inch side. How long is the altitude upon the 8-inch side?

8. If the diagonals, AC and BD, of a square $ABCD$ intersect at O, and $AO = 8$ inches, how many inches are there in the sum of the diagonals?

9. How many inches are there in the perimeter of a parallelogram formed by joining the midpoints of the sides of a quadrilateral whose diagonals are 11 inches and 7 inches, respectively?

10. The sum of the interior angles of a 97-sided polygon is how many straight angles?

CHAPTER 8

Areas of Polygons

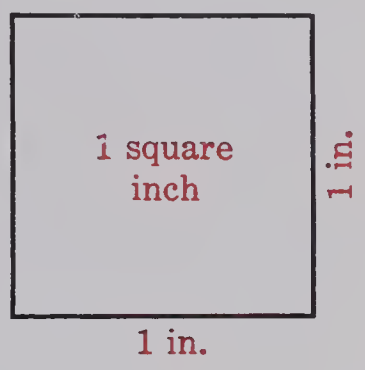
In this chapter
you will continue
the study of polygons
by studying the relationships
between their sides
and the portions of space
the sides enclose.

225. Measurement of a Surface

To measure a line segment we find how many times it contains a unit of length. Thus a line segment is 16 inches long if it contains the inch unit 16 times. To measure an angle we find how many times it contains a unit angle. For example, an angle contains 25° if it contains the 1° angle 25 times. In the same manner we measure a surface by seeing how many times it contains a unit of surface.

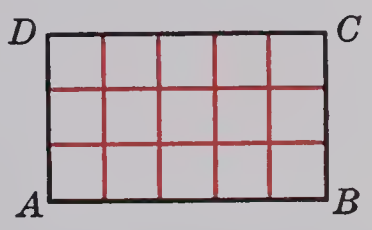
226. Unit of Surface

A *unit of surface* is a plane surface enclosed by a square each of whose sides is a certain unit of length. Thus a plane surface enclosed by a square each of whose sides is one inch is a unit of surface called a *square inch*. Other common units of surface are called the *square foot*, the *square yard*, the *square centimeter*, and the *square mile*. Can you name still other common units for measuring a surface? Why do you think the square was chosen as the best shape for enclosing a unit of surface?



227. Area of a Figure

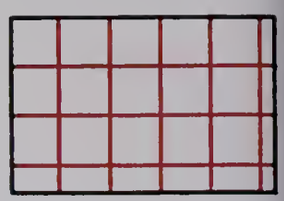
The *area of a plane figure* in terms of a unit of surface is the number of times the figure contains the unit. The rectangle $ABCD$ contains 3 rows of squares, each row containing 5 units. The area of the rectangle is 3×5, or 15 units. If each of the units is a square inch, the area of the rectangle is 15 square inches. If each of the units is a square foot, what is the area of the rectangle?



228. Area of a Rectangle

The length of the base of the rectangle above gives the number of squares in a row, and the width of the rectangle gives the number of rows. Since the number of squares in a row, multiplied by the number of rows, gives the number of squares, or square units, in the rectangle, we have:

Assumption 40. The area of a rectangle is equal to the product of its base and altitude, or $S = bh$.



Example 1. If, for the rectangle at the right, $b = 5\frac{1}{4}$ inches and $h = 3\frac{1}{2}$ inches, then $S = 3\frac{1}{2} \times 5\frac{1}{4}$, or $18\frac{3}{8}$ square inches.

Example 2. If, in a rectangle, $b = 6$ ft. and $h = \sqrt{3}$ ft., then $S = 6\sqrt{3}$ sq. ft. In theorems relating to areas, the word "rectangle" is commonly used for the words "area of a rectangle." The same is true for other plane figures.

EXERCISES — Areas of Rectangles

A

1. Find the area of a rectangular lot 50 feet wide and 135 feet long.

2. Find the cost of sodding a football field 300 feet long and 160 feet wide at 36 cents a square yard.

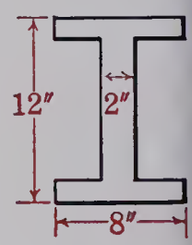
3. Find the cost of a field 50 rods wide and 80 rods long at $80 an acre.

4. What is the cost of placing clay over a tennis court 36 feet wide and 78 feet long at 40 cents a square yard?

5. Find the altitude of a rectangle whose base is 16.5 inches and whose area is 1300.3 square inches.

6. Find the area of a rectangle whose base is $2x - 1$ and whose altitude is $x + 4$.

7. Find the area of the cross section of the I-beam shown below if the width of each of the upper and lower rectangles is $1\frac{3}{8}$ inches.



8. If you double the altitude of a rectangle and leave the base the same, how is the area changed?

9. If both the altitude and the base of a rectangle are doubled, how is the area changed?

229. Ratio

The *ratio* of one quantity to another like quantity is the numerical value of the *first* quantity divided by the numerical value of the second, both expressed in terms of a common unit.

The ratio of 3 to 4 is $\frac{3}{4}$ and the ratio of 4 to 3 is $\frac{4}{3}$. The ratio of 2 pounds to 6 pounds is $\frac{2}{6}$, or $\frac{1}{3}$; and the ratio of 5 inches to 2 feet is $\frac{5}{24}$. We cannot find the ratio between two unlike quantities, such as the ratio of 3 books to 5 chairs.

The colon (:) and the division sign (÷) can also be used to denote ratio. Thus, $\frac{3}{4}$, 3 : 4, and 3 ÷ 4 denote the ratio of 3 to 4. Since ratios are really fractions, all the principles of fractions apply to them.

EXERCISES Numerical

A

1. Reduce each of the following ratios to lowest terms:

a. $\frac{4}{6}$. **b.** $\frac{a^2}{4\,a^2}$. **c.** $\frac{5\,x}{10\,x^2}$. **d.** $\frac{6\,x^2}{12\,xy}$. **e.** $8\,a^2 : 16\,a^3$. **f.** $(a^2 - b^2) : (a - b)^2$.

2. Express in lowest terms the ratio of

a. 30° to 90°.

b. A right angle to a straight angle.

c. 2 inches to 1 foot.

d. One angle of an equilateral triangle to the sum of the other two.

3. If $5\,x = 7\,y$, find the ratio of x to y; of y to x.

4. Two quantities are in the ratio of 4 : 5. If $4\,x$ represents the first quantity, what represents the second?

5. Separate 105 into two parts which are in the ratio of 2 : 5. (Let $2\,x$ = the smaller part.)

6. Find the angles of a triangle if they are in the ratio of 3 : 5 : 7.

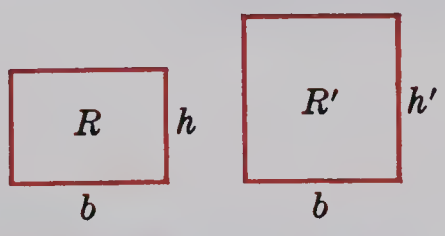
230. Relationships between Rectangles

The following three corollaries are derived from Assumption 40. All of them show relationships between rectangles.

231. Corollary I. Two rectangles having equal bases have the same ratio as their altitudes.

Given rectangles R and R', having altitudes h and h' and equal bases b.

To prove that $\frac{R}{R'} = \frac{h}{h'}$.



Proof

STATEMENTS	REASONS
1. $R = bh$ and $R' = bh'$.	1. Why?
2. $\therefore \frac{R}{R'} = \frac{bh}{bh'}$, or $\frac{R}{R'} = \frac{h}{h'}$.	2. Why?

232. Corollary II. Two rectangles having equal altitudes have the same ratio as their bases.

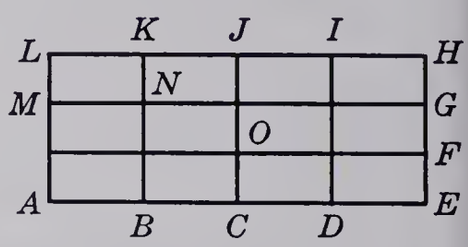
233. Corollary III. The area of a square is equal to the square of one of its sides.

EXERCISES — Areas of Rectangles

A

1. How many tiles, each 4 inches on a side, does it take to cover a floor 12 feet 4 inches long by 10 feet 8 inches wide?

2. Find the area of a square if each of its sides is $x - 4$.



3. Rectangle AH is separated into 12 rectangles. If $AB = BC = CD = DE$ and $EF = FG = GH$, find the following ratios:

a. Rectangle AN to rectangle AK.
b. Rectangle AJ to rectangle AH.
c. Rectangle AO to rectangle AF.
d. Rectangle BI to rectangle CI.
e. Rectangle BF to rectangle CF.
f. Rectangle BO to rectangle ND.

4. Show by a drawing that a square having a side $a + b$ is equal to the sum of a square having a side a, a square having a side b, and two rectangles each having the sides a and b.

234. Figures Equal in Area (Equivalent)

Figures equal in area are not necessarily congruent, but congruent figures are always equal in area. "Equal" often means "equal in area."

EXERCISES

A

1. Are two congruent triangles necessarily equal?

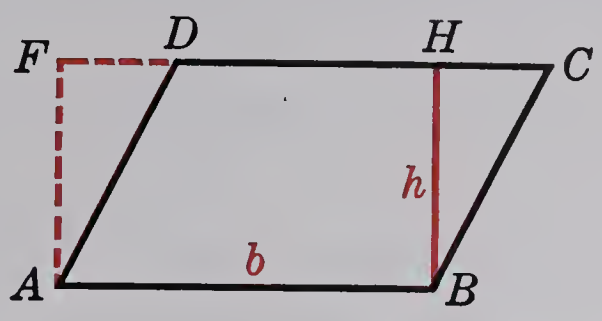
2. Are two equal triangles necessarily congruent?

3. A rectangular field and a triangular field each contain 5 acres. Are the fields congruent?

4. Why do you think the symbol for congruent ($\cong$) contains the symbol for equal ($=$)?

★ Theorem 30

235. The area of a parallelogram is equal to the product of its base and altitude, or $S = bh$.



Given $\square\ ABCD$ with base AB, denoted by b, and altitude BH, denoted by h.

To prove that the area of $ABCD = bh$.

Planning the Proof: 1. We can find the area of a polygon by §§ 228, 233.
2. We shall use § 228.

Proof

STATEMENTS	REASONS
1. $HB \perp AB$ and DC.	1. §§ 186, 159 a.
2. Through A draw $AF \parallel BH$, meeting CD extended at F.	2. § 108; Asmt. 13.
3. Then $AF \perp AB$ and FC.	3. Why?
4. $ABHF$ is a rectangle.	4. Give proof.
5. In rt. $\triangle AFD$ and BHC, $AD = BC$, and $AF = BH$.	5. Why?
6. $\therefore$ rt. $\triangle AFD \cong$ rt. $\triangle BHC$.	6. Why?
7. Trapezoid $ABHD =$ trapezoid $ABHD$.	7. Why?
8. $\therefore$ rectangle $ABHF = \square\ ABCD$.	8. Asmt. 1.
9. But the area of rectangle $ABHF = bh$.	9. § 228.
10. $\therefore$ the area of $\square\ ABCD = bh$.	10. Why?

236. Corollary I. Parallelograms having equal bases and equal altitudes are equal in area.

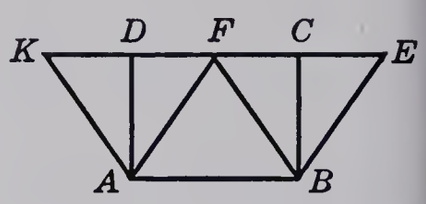
237. Corollary II. Two parallelograms having equal bases have the same ratio as their altitudes; and two parallelograms having equal altitudes have the same ratio as their bases.

EXERCISES — Areas of Parallelograms

A

1. Find the area of a parallelogram whose base is 18 inches and whose altitude is 12 inches.

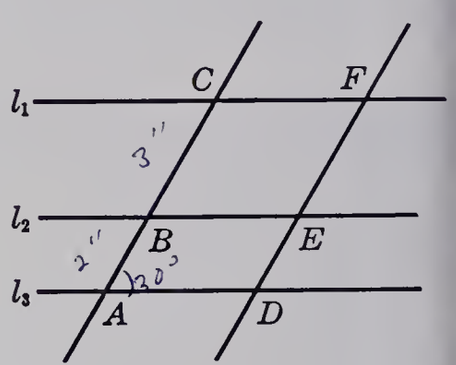
2. The base of a parallelogram is 17.4 and the altitude is 11.4. Find the area.



3. Find the area of a parallelogram whose base is 48 inches and whose altitude is 33 inches.

4. In the figure above, which is the largest, rectangle $ABCD$, parallelogram $ABEF$, or parallelogram $ABFK$?

5. The area of a parallelogram is 8430 square feet and the altitude is 150 feet. Find the base.



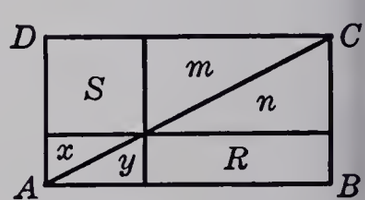
6. In the figure at the right $l_1 \parallel l_2 \parallel l_3$, $ABC \parallel DEF$, $\angle CAD = 30°$, $AB = 2$ in., and $BC = 3$ in. Find the ratio of area $ADEB$ to area $ADFC$.

7. How many tiles each 1 inch wide and 2 inches long will be required for a fireplace hearth 2 feet wide and 5 feet 4 inches long, allowing 10 per cent of the surface for cement?

8. The ratio between the base and altitude of a parallelogram is 2 : 3 and their sum is 40 inches. Find the area.

9. Find the area of rectangle $ABCD$ if the base AB is 12 inches, the diagonal BD is 14 inches, and $\angle ABD = 30°$.

C

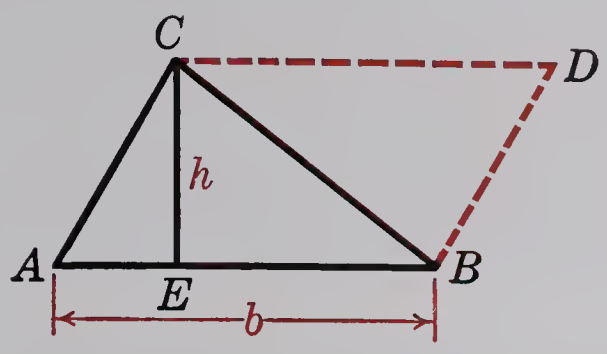


10. $ABCD$ is a rectangle with diagonal AC. R is a rectangle, and S is a square.
Prove that $S = R$.

11. In quadrilateral $ABCD$, $AB = AD$ and $BC = CD$. Find the area of the quadrilateral formed by joining in order the midpoints of $ABCD$ if $AC = 12$ and $BD = 8$.

★ Theorem 31

238. The area of a triangle is equal to one half the product of its base and altitude.



Given the $\triangle ABC$ having its base AB denoted by b and its altitude CE denoted by h.

To prove that the area of $\triangle ABC = \frac{1}{2}bh$.

Planning the Proof: 1. We can find the areas of polygons by §§ 228, 233, 235.
2. We shall use § 235.

Proof

STATEMENTS	REASONS
1. Through C draw a line $\parallel AB$, and through B draw a line $\parallel AC$.	1. Why possible?
2. These lines meet at some point D.	2. Give proof.
3. Then $ABDC$ is a $\square$ having base b and altitude h.	3. Why?
4. $\triangle ABC = \frac{1}{2}\square ABDC$.	4. Why?
5. Area of $\square ABDC = bh$.	5. Why?
6. Then area of $\frac{1}{2}\square ABDC = \frac{1}{2}bh$.	6. Why?
7. $\therefore$ area of $\triangle ABC = \frac{1}{2}bh$.	7. Why?

239. Corollary I. Two triangles having equal bases have the same ratio as their altitudes.

240. Corollary II. Two triangles having equal altitudes have the same ratio as their bases.

241. Corollary III. The areas of two triangles have the same ratio as the products of their bases and altitudes.

242. Corollary IV. The area of a rhombus is equal to one half the product of its diagonals.

243. Corollary V. If a triangle and a parallelogram have equal bases and altitudes, the area of the triangle is half the area of the parallelogram.

EXERCISES — Areas

A

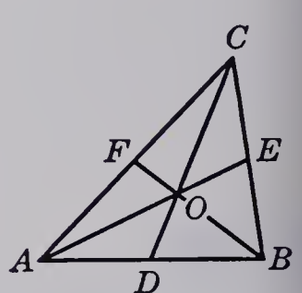
1. Find the area of a triangle if its base is 17 and its altitude is 12.

2. What is the area of a triangle whose base is 21.4 and whose altitude is 14.8?

3. Find the base of a triangle if its area is 576 square feet and its altitude is 9 feet.

4. Find the area of a rhombus if its diagonals are 32 feet and 40 feet.

5. The area of a rhombus is 2352 square inches and one diagonal is 56 inches. How long is the other diagonal?



6. The area of $\triangle ABC$ is 108 square inches and D, E, and F are the midpoints of the three sides. Find the area of $\triangle ABO$; of $\triangle BOC$; of $\triangle AOC$; of $\triangle ODB$; of $\triangle BOE$; of $\triangle AOF$.

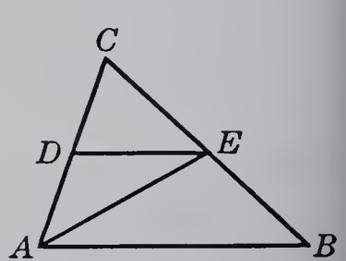
7. Prove that two triangles having equal bases and equal altitudes are equal.

B

8. $ABCD$ is a parallelogram whose diagonals intersect in E. Prove that $\triangle ABE = \triangle BEC$.

9. In $\triangle ABC$, D is the midpoint of AC and E is the midpoint of BC. Find these ratios:

$$\frac{\triangle DEC}{\triangle DEA};\ \frac{\triangle CDE}{\triangle AEC};\ \frac{\triangle ACE}{\triangle ABC};\ \frac{\triangle DCE}{\triangle ABC}.$$



10. Find the area of $\square ABCD$ if $AB = 19$, $AD = 16$, and $\angle C = 30°$.

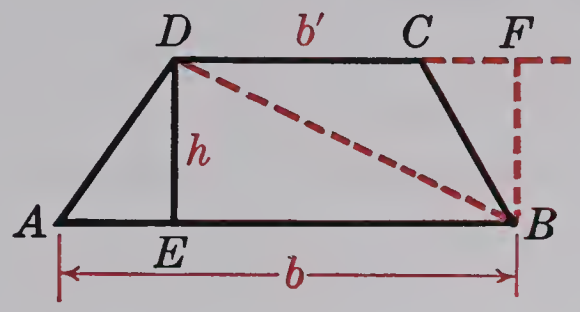
C

11. In $\triangle ABC$, $AB = 16$ in., $BC = 14$ in., and $AC = 18$ in. The altitude from B to AC is 15 in. Find the altitude from C to AB.

12. In $\triangle ABC$, AE and BD are two medians intersecting in F. Prove that $\triangle ABF =$ quadrilateral $DFEC$.

★★ Theorem 32

244. The area of a trapezoid is equal to half the product of its altitude and the sum of its bases, or $S = \frac{1}{2}h(b + b')$.



Given the trapezoid $ABCD$ having altitude DE denoted by h, base AB denoted by b, and base DC denoted by b'.

To prove that $S = \frac{1}{2}h(b + b')$.

Planning the Proof: 1. We can find the areas of polygons by §§ 228, 233, 235, 238, 242, and 243.
2. We shall use § 238.

Proof

STATEMENTS	REASONS
1. Draw DB.	1. Why possible?
2. Draw $BF \perp DC$.	2. Why possible?
3. $BF = ED = h$.	3. Why?
4. Area of $\triangle ABD = \frac{1}{2}bh$.	4. Why?
5. Area of $\triangle DBC = \frac{1}{2}b'h$.	5. Give proof.
6. $\triangle ABD + \triangle DBC = \frac{1}{2}bh + \frac{1}{2}b'h = \frac{1}{2}h(b + b')$.	6. Why?
7. But $\triangle ABD + \triangle DBC =$ trapezoid $ABCD$.	7. Why?
8. $\therefore S = \frac{1}{2}h(b + b')$.	8. Why?

NOTE. To remember this theorem, think of the trapezoid as two triangles.

EXERCISES — Areas of Trapezoids

A

1. What is the area of a trapezoid whose altitude is 18 inches and whose bases are 9 inches and 13 inches respectively?

2. Find the area of a trapezoid whose bases are 24.3 and 16.9 respectively and whose altitude is 15.7.

3. The area of a trapezoid is 4256 square inches and the bases are 68 inches and 84 inches respectively. Find the altitude.

4. The area of a trapezoid is 504 square inches. One of the bases is 24 inches and the altitude is 18 inches. Find the other base.

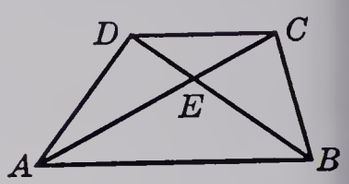
5. A diagonal divides a trapezoid into two triangles which have the same ratio as the bases of the trapezoid. Why?

6. The area of a trapezoid is 651 square feet and the bases are 27 and 35 feet respectively. Find the altitude.

7. Find the height of a trapezoid if its bases are 8 inches and 20 inches respectively and its area is 96 square inches.

8. Find the area of a trapezoid if the sum of its bases is 35 and its altitude is 17.

9. *Given* trapezoid $ABCD$ with bases AB and CD and diagonals AC and BD intersecting in E.



Prove that $\triangle ADE = \triangle BEC$.

B

10. The altitude of a trapezoid is ab, one base is $2a - b$, and the other base is $2a + b$. Find its area.

11. The area of a trapezoid is 3105 square inches, one base is 155 inches, and the altitude is 34.5 inches. Find the other base.

12. The bases of a trapezoid are 25 and 28 respectively. Find its altitude if its area is equal to that of a triangle whose base is 26.5 and whose altitude is 10.

13. In trapezoid $ABCD$, base AB is 24, base DC is 10, AD is 12, and $\angle A$ is 30°. Find the area of the trapezoid.

14. What is the area of the isosceles trapezoid whose bases are 8 inches and 18 inches respectively, if the nonparallel sides make angles of 45° with the lower base?

15. Prove that the triangle having the midpoint of one of the non-parallel sides of a trapezoid as its vertex and the other of the non-parallel sides as its base is equal to one half the trapezoid.

16. Prove that the area of a trapezoid is equal to the product of one of its non-parallel sides and the perpendicular to it from the midpoint of the other.

17. Prove that the area of a trapezoid is equal to the product of its altitude and its median.

245. Radicals

As you probably remember, a *radical* is an indicated root of a number, as $\sqrt{5}$, $\sqrt[3]{b}$, and $\sqrt{\frac{1}{2}}$. A radical is expressed in its *simplest form* when the *radicand* (the quantity under the radical sign) is a whole number and as small as possible.

Two important principles of radicals are:

Principle I. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

Principle II. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

Example 1. Simplify $\sqrt{49}$.

Solution. $\sqrt{49} = 7$.

Example 2. Simplify $\sqrt{50}$.

Solution. $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$.

Example 3. Simplify $\sqrt{\frac{3}{8}}$.

Solution. $\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}}$. Then $\sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4} = \frac{1}{4}\sqrt{6}$.

EXERCISES

A

Simplify the following radicals:

1. $\sqrt{8}$.
2. $\sqrt{25}$.
3. $\sqrt{32}$.
4. $\sqrt{60}$.
5. $\sqrt{72}$.
6. $\sqrt{81}$.
7. $\sqrt{18}$.
8. $\sqrt{20}$.
9. $2\sqrt{16}$.
10. $3\sqrt{8}$.
11. $5\sqrt{75}$.
12. $\sqrt{16}$.
13. $\sqrt{\frac{1}{2}}$.
14. $\sqrt{\frac{1}{3}}$.
15. $\sqrt{\frac{2}{3}}$.
16. $\sqrt{\frac{9}{25}}$.
17. $\sqrt{\frac{3}{5}}$.
18. $\sqrt{\frac{1}{7}}$.
19. $\sqrt{\frac{1}{a}}$.
20. $\sqrt{\frac{1}{a^3}}$.

246. How to Find the Square Root of a Number

The following examples will help you to recall the method for finding the square root of a number.

Example 1. Find the positive square root of 841.

Solution. We first separate the number into groups of two figures each, beginning at the decimal point.

$$\begin{array}{r|l} & 2\ 9. \\ \hline & 8\ 41. \\ 2 & 4 \\ \hline 49 & 4\ 41 \\ & 4\ 41 \\ \hline \end{array}$$

$2 \times 2 = 4$.

Example 2. $\sqrt{165} = ?$ (to the nearest hundredth)

Solution.

$$\begin{array}{rl}
 & 1\ 2.\ 8\ 4\ 5+ \text{ or } 12.85 \\
\hline
 & 1\ 65.\ 00\ 00\ 00 \\
 & 1 \mid 1 \\
2 \times 1 = 2. & 22 \mid 65 \\
 & \quad 44 \\
\hline
2 \times 12 = 24. & 248 \mid 21\ 00 \\
 & \quad 19\ 84 \\
\hline
2 \times 128 = 256. & 2564 \mid 1\ 16\ 00 \\
 & \quad 1\ 02\ 56 \\
\hline
2 \times 1284 = 2568. & 25685 \mid 13\ 44\ 00 \\
 & \quad 12\ 84\ 25 \\
\hline
\end{array}$$

Notice that the square root in Example 2 is an approximate number. Though the computation could have been carried to any number of decimal places, only two were desired. To get the two places as accurate as possible the computation was carried to three decimal places and then rounded back to two.

In many exercises involving square root, time can be saved by referring to the table of square roots on page 575.

247. Quadratic Equations

A *quadratic equation* is one which when simplified contains the second power, but no higher power, of the unknown. We shall now briefly review the algebraic methods of solving quadratic equations in preparation for work to follow.

248. Incomplete Quadratic Equations

An *incomplete quadratic* equation is a quadratic equation in which the first power of the unknown is absent. $7x^2 - 175 = 0$ and $3m^2 + m^2 = 256$ are incomplete quadratic equations.

To solve an incomplete quadratic equation:

1. Solve the equation for the square of the unknown.
2. Find the square root of each member.

Example 1. Solve $4x^2 - 144 = 0$.

Solution. $4x^2 - 144 = 0.$

$4x^2 = 144.$

$x^2 = 36.$

R_2, $x = \pm 6.$

Example 2. Solve $x^2 + 4x^2 = 64$.

Solution. $x^2 + 4x^2 = 64.$

$5x^2 = 64.$

$x^2 = \frac{64}{5}.$

R_2, $x = \pm\sqrt{\frac{64}{5}} = \pm\sqrt{\frac{64 \times 5}{25}}$

$= \pm\frac{8}{5}\sqrt{5}.$

EXERCISES

A

Solve:

1. $x^2 = 25.$
2. $4x^2 = 100.$
3. $x^2 + 25 = 75.$
4. $3y^2 - 192 = 0.$
5. $x^2 - \frac{x^2}{4} = 100.$
6. $4x^2 - \frac{1}{9} = 0.$
7. $\frac{2x^2 - 7}{5} - \frac{x^2 - 4}{3} + \frac{8}{5} = 0.$
8. $\frac{x^2 - 1}{4} + \frac{3x^2 + 3}{10} = 5.$
9. $(x - 2)^2 + 4x = 16.$

249. Complete Quadratic Equations

A *complete quadratic equation* is one that contains both the second and first powers of the unknown. We shall now review three methods of solving a complete quadratic equation.

a. *Solution by Factoring.* The factoring method is the simplest of the three, when it can be used.

To solve an equation by factoring:

1. If necessary, transform it so that the right member is zero.
2. Factor the left member and equate to zero.
3. Equate each factor containing the unknown to zero.
4. Solve the resulting equations.

Example. Solve $2x^2 - 4x = 6$.

Solution. Making the right member zero, $2x^2 - 4x - 6 = 0.$

Factoring the left member, $2(x - 3)(x + 1) = 0.$

Setting the factors containing x equal to zero, we have $x - 3 = 0$ and $x + 1 = 0$. Solving these two equations, $x = 3$ and $x = -1$.

EXERCISES

A

1. $x^2 - x - 6 = 0$.
2. $x^2 + x = 30$.
3. $2x^2 + 5x = 3$.
4. $3x^2 + 14x = 5$.
5. $x^2 - 25 = 0$.
6. $y^2 - 2y = 24$.
7. $x^2 - 8x = 0$.
8. $3x^2 = 15x$.
9. $12x^2 + 11x = 5$.
10. $6x^2 - x - 1 = 0$.
11. $\frac{y^2}{3} - \frac{y}{2} = 5\frac{5}{6}$.

b. *Solution by Completing the Square.* Any complete quadratic equation can be solved by completing the square.

To solve a quadratic equation by completing the square:

1. Write the equation in the form $x^2 + bx = c$.
2. Add to each member the square of half of b.
3. Find the square root of each member, placing the $\pm$ sign before the square root of the right member.
4. Solve the two resulting equations.

Example 1. Solve $2x^2 - 12x = 32$.

Solution. $2x^2 - 12x = 32$.

D_2, $x^2 - 6x = 16$.

Adding to each member the square of half of -6, which is 9,

$x^2 - 6x + 9 = 25$.

R_2, $x - 3 = \pm 5$.

Solving the two equations, $x = 3 \pm 5$; $x = 8$ or -2.

Example 2. Solve $2x^2 - 5x = -1$.

Solution. $2x^2 - 5x = -1$.

D_2, $x^2 - \frac{5}{2}x = -\frac{1}{2}$.

$[\frac{1}{2}$ of $-\frac{5}{2} = -\frac{5}{4}$ and $(-\frac{5}{4})^2 = \frac{25}{16}.]$

$A_{\frac{25}{16}}$, $x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{17}{16}$. $[\frac{25}{16} - \frac{1}{2} = \frac{17}{16}.]$

R_2, $x - \frac{5}{4} = \pm\frac{1}{4}\sqrt{17}$; $x = \frac{5}{4} \pm \frac{1}{4}\sqrt{17}$, or $\frac{1}{4}(5 \pm \sqrt{17})$.

c. *Solution by Formula.* Any complete quadratic equation can be written in the form $ax^2 + bx + c = 0$. From this we obtain the formula for the roots: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example. Solve $3x^2 - 4x = 2$.

Solution. Changing the equation into the form $ax^2 + bx + c = 0$, $3x^2 - 4x - 2 = 0$. In this equation $a = 3$, $b = -4$, and $c = -2$. Substituting these values of a, b, and c in the formula, we get

$$x = \frac{4 \pm \sqrt{16 - 4(3)(-2)}}{6}$$

$$= \frac{4 \pm \sqrt{40}}{6} = \frac{4 \pm 2\sqrt{10}}{6} = \frac{2 \pm \sqrt{10}}{3}.$$

EXERCISES

A

Solve by completing the square. Express roots containing radicals correct to the nearest tenth.

1. $x^2 + 2x = 8$.
2. $x^2 + 16x = 36$.
3. $4x^2 + 20x = 11$.
4. $x^2 - 6x - 7 = 0$.
5. $x^2 - 40x = 41$.
6. $m^2 + 8m - 9 = 0$.
7. $x^2 + 4x = 7$.
8. $2x^2 + 5x + 1 = 0$.
9. $y^2 + 3y - 1 = 0$.

Solve by formula. Express roots containing radicals in their simplest radical forms.

10. $x^2 + 10x + 24 = 0$.
11. $x^2 - 11x + 30 = 0$.
12. $x^2 - 2x = 40$.
13. $2x^2 + x = 3$.
14. $3y^2 + y = 10$.
15. $6h^2 - h = 12$.
16. $p^2 - p = 5$.
17. $x^2 + 6x + 1 = 0$.
18. $x^2 - x = 1$.

Solve by any method. Express roots containing radicals correct to the nearest tenth.

19. $x^2 - x = 90$.
20. $2x^2 - 5x = 12$.
21. $x^2 - 42x = 400$.
22. $y^2 + 2y = 10$.
23. $m^2 - 3m - 88 = 0$.
24. $p^2 - 4p = 11$.

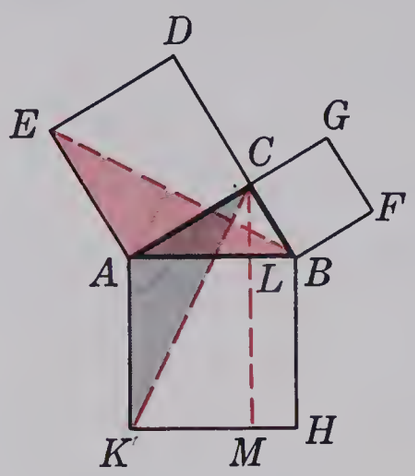
25. Find the number whose square is equal to the sum of itself and 110.

26. The sum of two numbers is 15 and their product is 54. Find the numbers.

27. The area of a rectangle is 130 sq. in. Find the base if it exceeds the altitude by 3 in.

★ Theorem 33 (The Pythagorean Theorem)

250. The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the legs.



Given the rt. $\triangle ABC$ having $\angle ACB$ the rt. $\angle$, and squares constructed upon the sides AB, BC, and AC.

To prove that square $AKHB$ = square $ACDE$ + square $BFGC$.

Planning the Proof: 1. We can compare areas of polygons by §§ 231, 232, 236, 237, 239, 240, 241, 243.
2. We shall use § 243 and Asmt. 1.

Proof

STATEMENTS	REASONS
1. Draw $CM \parallel AK$.	1. Why possible?
2. Draw CK and BE.	2. Why possible?
In $\triangle KAC$ and BAE,	
3. $AK = AB$, $AC = AE$, and	3. Why?
4. $\angle KAC = \angle BAE$.	4. Give proof.
5. $\therefore \triangle KAC \cong \triangle BAE$.	5. Why?
6. DCB is a st. line.	6. Give proof.
7. $\triangle BAE$ and square $ACDE$ have the same base AE and altitudes = AC.	7. § 193.
8. $\therefore \triangle BAE = \frac{1}{2}$ square $ACDE$.	8. § 243.
9. $\triangle KAC$ and rectangle $KMLA$ have the same base AK and altitudes = AL.	9. § 193.
10. $\therefore \triangle KAC = \frac{1}{2}$ rectangle $KMLA$.	10. Why?
11. From (5), (8), and (10), $\frac{1}{2}$ rectangle $KMLA = \frac{1}{2}$ square $ACDE$.	11. Why?
12. $\therefore$ rectangle $KMLA$ = square $ACDE$.	12. Why?

13. In like manner, rectangle $MHBL$ = square $BFGC$.	13. Steps 1–12.
14. $\therefore KMLA + MHBL = ACDE + BFGC$.	14. Why?
15. $\therefore$ square $KHBA$ = square $ACDE$ + square $BFGC$.	15. Give assumptions.

251. Proofs of the Pythagorean Theorem

Many proofs of the Pythagorean Theorem have been presented. The first proof of the theorem is attributed to Pythagoras about 525 B.C., but his method of demonstration is unknown. The proof of § 250 is attributed to Euclid. The proof given in § 430 is algebraic in character, while Euclid's proof is purely geometric. The statements of the theorems in § 430 and § 250 are also different. In § 430 the squares of the line segments are compared, while in § 250 the areas of the squares are compared. Below is an outline of the demonstration of the theorem as devised by President Garfield.

Given $\triangle ABC$ with hypotenuse AB.
To prove that $c^2 = a^2 + b^2$.

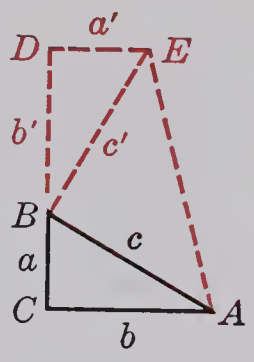
Produce CB to D, making $b' = b$. At D construct $ED \perp BD$, making $a' = a$. Draw BE and AE. The area S of the trapezoid $CAED$ is given by the formula

$$S = \tfrac{1}{2}(a + b')(b + a') = \tfrac{1}{2}(a^2 + 2\,ab + b^2)$$
$$= \tfrac{1}{2}\,a^2 + ab + \tfrac{1}{2}\,b^2.$$

Considering the ◬ of the trapezoid,

$$S = \tfrac{1}{2}\,ab + \tfrac{1}{2}\,c^2 + \tfrac{1}{2}\,ab = ab + \tfrac{1}{2}\,c^2.$$
$$\therefore ab + \tfrac{1}{2}\,c^2 = \tfrac{1}{2}\,a^2 + ab + \tfrac{1}{2}\,b^2.$$

Solving, $c^2 = a^2 + b^2$.



EXERCISES — Pythagorean Theorem

A

In the figure at the top of the next page, AB is the hypotenuse of rt. $\triangle ABC$, each side of which is a side of a square, and CS is $\perp DE$.

1. Find the area of rectangle $SEBR$ if $BF = 8$.
2. If $KH = 8$ and $CG = 15$, what is the area of $ADEB$?

3. If $BC = 7$ and $AC = 9$, what is the area of $SEBR$? of $DSRA$?

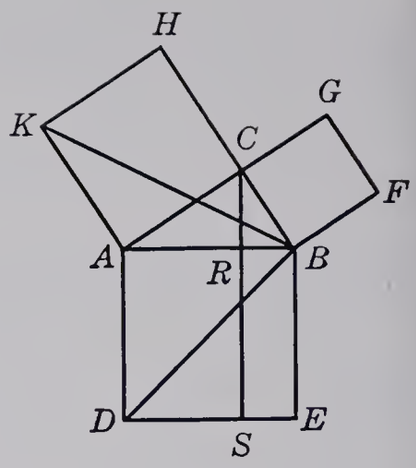
4. Find AR if $AD = 30$ inches and the area of $\triangle KAB$ is 360 square inches.

5. Find the area of $ACHK$ if BC is 12 inches and RS is 20 inches.

6. Find DS if $FG = 12$ and $DE = 16$.

7. If $AK = 36$ and $RS = 39$, what is the area of $BFGC$?

8. Find the area of $ACHK$ if $AR = 12$ and $RB = 4$.



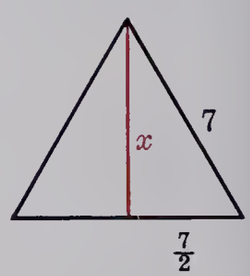
EXERCISES — Right Triangles

A

Example 1. Find the altitude of an equilateral triangle whose side is 7.

Solution. Let $x =$ the altitude. Then x bisects the base.

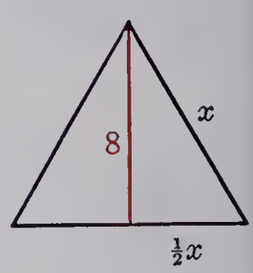
Then
$$x^2 + (\tfrac{7}{2})^2 = 7^2$$
$$x^2 + \tfrac{49}{4} = 49$$
$$x^2 = \tfrac{147}{4}$$
$$x = \tfrac{7}{2}\sqrt{3}.$$



NOTE. If 7 is the exact length of the side, then $\frac{7}{2}\sqrt{3}$ is the exact length of the altitude. In many exercises of this type the decimal form of the solution is desired, even though it is an approximate solution. To find the decimal value, we substitute 1.732 for $\sqrt{3}$ in $x = \frac{7}{2}\sqrt{3}$, obtaining 6.06.

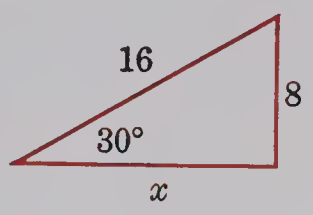
Example 2. The altitude of an equilateral triangle is 8. What is the length of a side?

Solution.
$$x^2 = (\tfrac{1}{2}x)^2 + 8^2.$$
$$x^2 = \tfrac{1}{4}x^2 + 64.$$
$$4x^2 = x^2 + 256.$$
$$3x^2 = 256.$$
$$x^2 = \tfrac{256}{3}.$$
$$x = \sqrt{\tfrac{256}{3}}$$
$$= \tfrac{16}{3}\sqrt{3}, \text{ or } 9.24.$$



Example 3 In a 30°-60° triangle the short leg is 8. Find the long leg.

Solution. By § 136 the hypotenuse = 16. Let x = the long leg.



$$x^2 + 64 = 256.$$

Solving, $x = \sqrt{192} = 8\sqrt{3}$, or 13.86.

1. If c is the hypotenuse and a and b are the legs of a right triangle,

a. Find c if $a = 15$ and $b = 20$.

b. Find a if $c = 13$ and $b = 12$.

c. Find b if $a = 33$ and $c = 44$.

d. Find c if $a = 14.4$ and $b = 6$.

2. Find the diagonal of a square whose perimeter is 40.

3. Find correct to the nearest tenth the side of a square whose diagonal is 10.0 inches.

4. A ladder 20 feet long reaches a window sill 16 feet from the ground. How far from the house is the foot of the ladder?

5. The diagonal of a square is $5\sqrt{2}$. Find a side.

6. Find the short leg of a 30°-60° triangle whose long leg is 20.

7. Find the side of a square whose diagonal is $3\sqrt{3}$.

8. Find the altitude of an equilateral triangle having a side equal to $8\sqrt{2}$.

9. Find the diagonal of a rectangle whose length is 45 inches and whose width is 24 inches.

B

10. Find the side of an equilateral triangle whose altitude is 96 inches.

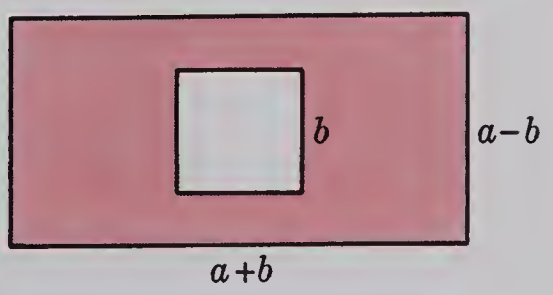
11. Prove that a median of a triangle bisects the triangle.

12. P is any point in the diagonal AC of $\square ABCD$. Prove that $\triangle ABP = \triangle ADP$.

SUGGESTION. Consider AP the base of each triangle.

13. What is the area of a tract of land in the shape of a right triangle if its hypotenuse is 52 feet and one of the other sides is 40 feet?

14. Write a formula for the area of the shaded surface bounded by the rectangle and square.



15. Find the area of an isosceles triangle whose base is b and whose other two sides are each equal to s.

C

16. Show that the altitude of an equilateral triangle having a side s is given by the formula $h = \frac{s}{2}\sqrt{3}$.

17. The difference between the base and altitude of a rectangle is 91 and a diagonal is 221. Find the base and altitude.

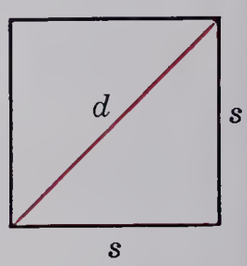
18. The sides of a triangle are 10, 13, and 13 respectively. Find the altitude drawn to a side 13. Find the area of the triangle.

19. The sides of a right triangle are 3 inches and 4 inches. Find the two segments of the hypotenuse formed by the altitude to the hypotenuse.

252. Important Formulas

In the next two sections we shall develop important formulas relating to the square and equilateral triangle. You should remember these formulas, for they will be useful in solving many problems.

253. Theorem 34. The diagonal of a square having a side s is given by the formula $d = s\sqrt{2}$.



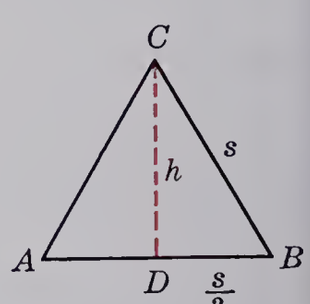
Proof:

$d^2 = s^2 + s^2$. Why?
$d^2 = 2\,s^2$. Why?
$d = s\sqrt{2}$. Why?

254. Theorem 35. The altitude and area of an equilateral triangle having a side s are given by the formulas $h = \frac{s}{2}\sqrt{3}$ and $A = \frac{s^2}{4}\sqrt{3}$.

Proof:

$h^2 + \frac{s^2}{4} = s^2$. Why?
$4\,h^2 + s^2 = 4\,s^2$. Why?
$4\,h^2 = 3\,s^2$. Why?
$h^2 = \frac{3\,s^2}{4}$. Why?
$h = \frac{s}{2}\sqrt{3}$. Why?



Let A stand for the area.

Then $A = \frac{1}{2} \times s \times \frac{s}{2}\sqrt{3} = \frac{s^2}{4}\sqrt{3}$.

EXERCISES — Equilateral Polygons

A

Find the diagonal of a square whose side is

1. 6. 2. $\sqrt{2}$. 3. 8. 4. $2\sqrt{3}$.

Find the altitude of an equilateral triangle whose side is

5. 10. 6. 18. 7. 15. 8. $14\sqrt{2}$.

Find the area of an equilateral triangle whose side is

9. 8. 10. 12. 11. 7. 12. $\sqrt{5}$.

B

Example. Find the side of an equilateral triangle with area $36\sqrt{3}$.

Solution. Let s = the length of each side of the $\triangle$.

Then $\frac{s^2}{4}\sqrt{3} = 36\sqrt{3}$; $s^2\sqrt{3} = 144\sqrt{3}$; $s^2 = 144$; $s = 12$.

Find the side of the equilateral triangle whose area is

13. $9\sqrt{3}$. 15. $\frac{1}{4}\sqrt{3}$. 17. 16. 19. 64.
14. $16\sqrt{3}$. 16. $\sqrt{3}$. 18. 20. 20. 100.

21. Find the area of a regular hexagon having a side of 8 inches.

22. Find the side of an equilateral triangle whose altitude is 15 inches.

23. Show that the side of an equilateral triangle having altitude h is $\frac{2\sqrt{3}\,h}{3}$.

24. Find the area of an equilateral triangle whose altitude is 9 inches.

25. Show that the area of an equilateral triangle having altitude h is $\frac{h^2\sqrt{3}}{3}$.

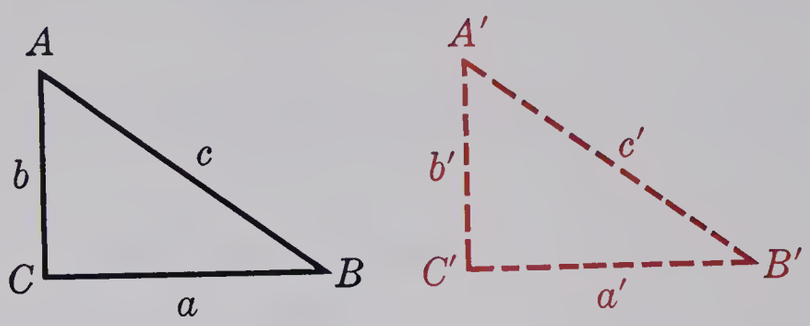
26. Find the side of a regular hexagon whose area is $54\sqrt{3}$ square feet.

27. In equilateral $\triangle ABC$, O is a point on altitude CD such that $CO = \frac{2}{3}CD$. Draw AO and BO and show that triangles AOB, BOC, and COA have equal areas.

28. The area of an equilateral triangle is $60\sqrt{3}$. Find its altitude.

Theorem 36

255. If the sum of the squares of two sides of a triangle is equal to the square of the third side, the triangle is a right triangle.



Given $\triangle ABC$, the longest side AB denoted by c, the side BC denoted by a, the side AC denoted by b, and $c^2 = a^2 + b^2$.

To prove that ABC is a right triangle.

Planning the Proof: 1. We can prove a triangle a right triangle by § 57.
2. We shall use § 57.

Proof

STATEMENTS	REASONS
1. Construct rt. $\triangle A'B'C'$ with leg $A'C' = AC$ and leg $C'B' = CB$.	1. §§ 169, 173.
2. $c'^2 = a'^2 + b'^2$.	2. Why?
3. But $a = a'$ and $b = b'$.	3. Why?
4. Then $a^2 = a'^2$ and $b^2 = b'^2$.	4. Why?
5. From (2), $c'^2 = a^2 + b^2$.	5. Why?
6. But $c^2 = a^2 + b^2$.	6. Why?
7. Then $c^2 = c'^2$.	7. Why?
8. And $c = c'$.	8. Why?
9. Then $\triangle ABC \cong \triangle A'B'C'$.	9. Why?
10. $\angle C = \angle C'$.	10. Why?
11. $\angle C'$ is a rt. $\angle$.	11. Why?
12. Then $\angle C$ is a rt. $\angle$.	12. Why?
13. Then $\triangle ABC$ is a rt. $\triangle$.	13. Why?

EXERCISES

A

1. What is the converse of Theorem 36?
2. Can 5, 12, and 13 be the sides of a right triangle?

256. Hero's Formula

A formula which first appeared in the writings of Hero, a mathematician who lived in Alexandria about 125 B.C., is useful for finding the area of a triangle whose three sides are known. The formula, without proof, follows.

257. Theorem 37 (Hero's Formula). **The area of a triangle whose sides are a, b, and c is given by the formula**

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \tfrac{1}{2}(a+b+c).$$

Example. Find the area of a triangle whose sides are 8, 15, and 17.

Solution. $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(8+15+17) = 20.$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{20(20-8)(20-15)(20-17)} = 60.$$

258. Corollary. **In any triangle whose sides are a, b, and c, the altitude h on the side c is given by the formula**

$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISES — Areas of Triangles

B

Find the area of a triangle whose sides are

1. 13, 20, 21.
2. 13, 14, 15.
3. 25, 17, 28.
4. 29, 36, 25.
5. 5, 6, 7.
6. 5, 12, 15.
7. 200, 300, and 400.
8. $a+b$, $a-b$, $2c$.
9. $m+n$, $m+p$, $n+p$.
10. Using Hero's Formula, show that the area of an equilateral triangle having side s is $\frac{1}{4}s^2\sqrt{3}$.
11. The sides of a triangle are 36, 29, and 25. Find the altitude to the side 36; to the side 29; to the side 25.

NOTE TO THE TEACHER. §§ 526, 527 and §§ 538, 540 may be studied now.

259. Computation with Approximate Numbers

In the exercises you have been studying, the numbers in the given data have been exact. For example, if you were asked to find the area of a rectangle with base 6.4 inches and altitude 4.1 inches, it was intended

that you consider 6.4 and 4.1 as exact numbers. Substituting 6.4 and 4.1 for b and h in the formula $S = bh$ gives 26.24 as the exact area in square inches.

Had 6.4 inches and 4.1 inches been the results of measurements, the exact area could not have been found because exact measurements cannot be made. When we say that the measured length of the base is 6.4 inches, we simply mean that the true length is between 6.35 inches and 6.45 inches. When we say that the measured length of the altitude is 4.1 inches, we mean that the true length of the altitude is between 4.05 inches and 4.15 inches.

Now, notice what happens when we try to find the area of the rectangle using these values. Substituting the lower limits (6.35 and 4.05) for b and h in $S = bh$ gives $S = 6.35 \times 4.05$, or 25.7175 square inches. Substituting the upper limits (6.45 and 4.15) for b and h in the formula gives $S = 6.45 \times 4.15$, or 26.7675 square inches. Thus, instead of being able to give the exact area of the rectangle, we can only say that the exact area lies somewhere between 25.7175 square inches and 26.7675 square inches.

Similarly, when 6.4 inches and 4.1 inches are exact, the perimeter of the rectangle is exactly $6.4 + 4.1 + 6.4 + 4.1$, or 21.0 inches. If 6.4 inches and 4.1 inches are the results of measurements, we can only say that the true perimeter lies between 20.80 inches and 21.20 inches. Do you see why?

Even though we cannot arrive at exact solutions when we are computing with approximate numbers, we can, by following certain procedures, arrive at reasonably accurate approximate solutions. You may have studied these procedures in an earlier course in mathematics. They are outlined below.

> To add (or subtract) approximate numbers, round them off so that they have the same number of decimal places. The result will then have the same number of decimal places as the numbers that are added (or subtracted).

Example. By measurement, the sides of a triangle have been found to be 2.75 in., 5.0 in., and 6.5 in. What is the perimeter of the triangle?

Solution. Since the perimeter of a triangle is the sum of its sides, we round off 2.75 to 2.8, then find $2.8 + 5.0 + 6.5$, or 14.3. The perimeter of the triangle is approximately 14.3 in.

The product of two approximate numbers should not contain more significant digits than either the multiplier or the multiplicand. When one of the numbers is exact, the product should not contain more significant digits than the approximate number.

The quotient of two approximate numbers should not contain more significant digits than either the divisor or the dividend. When one of the numbers is exact, the quotient should not contain more significant digits than the approximate number.

When one of two approximate numbers to be multiplied or divided contains several more significant digits than the other, it is permissible to round it off to contain only one more digit than the other.

NOTE. In general, all the digits of a number are significant. The only exception is in the case when zero is used as a place-holder following a decimal point, as in .015 where there are only two significant digits. In a number such as 640 we cannot know the number of significant figures unless we know the unit used in making the measurement.

Example. Find the area of the parallelogram whose base and altitude have been found by measurement to be 3.62 ft. and .80 ft., respectively.

Solution. Since 3.62 contains *three* significant digits and .80 contains *two*, the product should contain only *two*. The product $3.62 \times .80 = 2.8960$. Rounding off 2.8960 to *two* significant digits gives the area of the parallelogram as 2.9 sq. ft.

To find the square root of an approximate number, carry the computation to one more digit than the number of significant digits in the original number, then round back one digit.

Example. Find the square root of the approximate number 42.0.

Solution. Since 42.0 contains *three* significant digits, its square root should contain *three* significant digits. Carrying out the computation to *four* digits gives 6.481. Rounding this to *three* significant digits gives 6.48.

In the exercises of this book you should consider the given numbers as exact unless you are given information to show that they are approximate. Remember that such forms as 1.41 for $\sqrt{2}$ and 3.14 for π are approximate numbers.

EXERCISES — Computation with Approximate Numbers

A

In the following exercises consider all numbers as having been obtained by measurement.

1. A room measures 17.5 ft. by 11.5 ft. What is the area of the floor?

2. A back yard is 22 feet by 34 feet. What is the length of the longest rope that can be stretched across it?

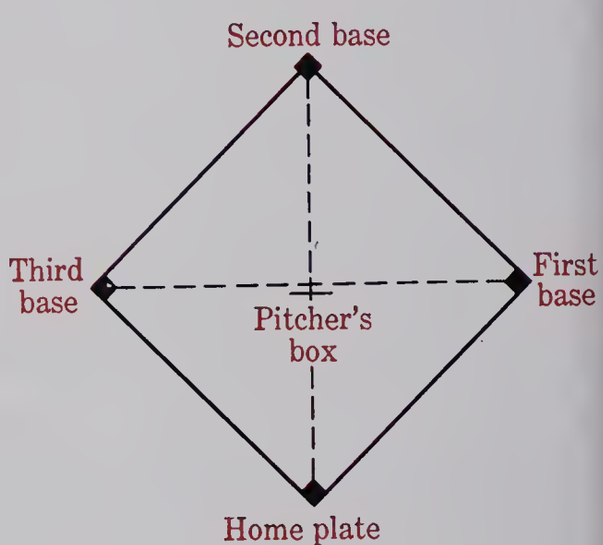
3. A living room is 12 feet wide and 24 feet long. Which would be the cheaper, and how much, to buy a rug for the room for $108 or to cover the floor with carpet at $3.25 per square yard?

4. A square field contains 1847 square rods. How many rods long is a side of the field?

5. An 18-foot ladder resting against a house just reaches a window 16 feet above the ground. How far is the foot of the ladder from the house?

B

6. A baseball field is a square each side of which is 90 feet long. What is the distance from first base to third base?

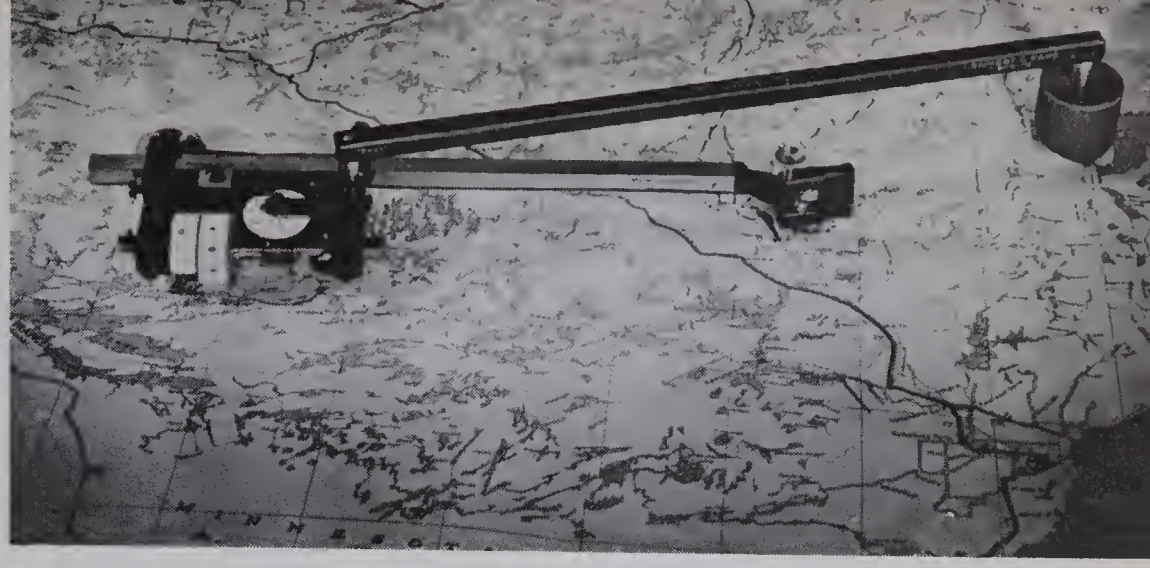


7. The pitcher's box is 60 feet from the home plate. How far is it from second base? from third base?

8. The shortstop fields a batted ball at a point one third of the distance from second base to third base. How far must he throw the ball to make an out at first base?

9. A metal triangle has its sides 17.2 in., 12.3 in., and 8.5 in. long. What will be the area of the upper surface of the triangle?

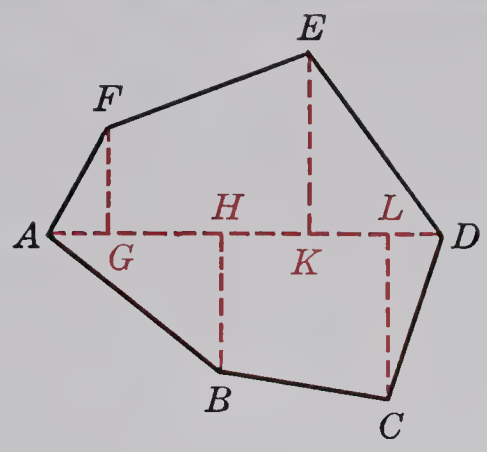
10. A ladder 50 feet long is placed so that its foot is 8 feet from a building. How far must the foot of the ladder be moved from the building to lower the top of the ladder 6 feet?



A planimeter, an instrument for measuring the area of a plane figure

260. Areas of Polygons

The area of a polygon may be found by separating it into triangles and trapezoids as shown. Any diagonal AD is drawn. Then perpendiculars to AD are drawn from B, C, E, and F, forming triangles and trapezoids.



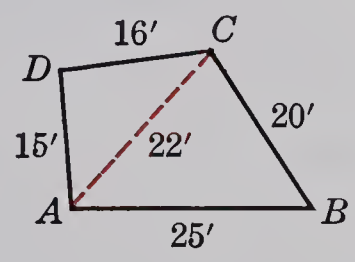
EXERCISES — Areas

C

Consider that the numbers in Exercises 1–7 were obtained by measurement.

1. In the figure of § 260, if $AG = 2$, $GH = 2$, $HL = 5$, $LK = 2$, $KD = 4$, $FG = 3\frac{1}{2}$, $EK = 5$, $HB = 4\frac{1}{2}$, and $LC = 6$, find the area of polygon $ABCDEF$.

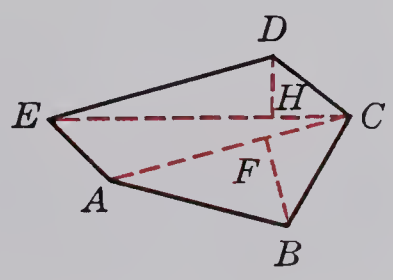
2. Find the area of the quadrilateral $ABCD$ in the figure at the right.



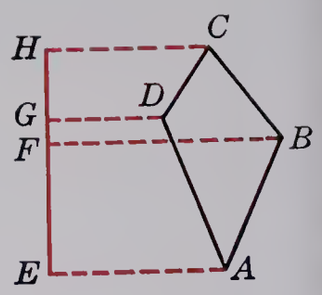
3. Find the area of quadrilateral $ABCD$ if $AB = 20$ feet, $\angle B = 90°$, diagonal $AC = 24$ feet, $\angle ACD = 90°$, and $CD = 10$ feet.

4. Find the area of quadrilateral $RSTU$ if $RU = 20$ inches, $TU = 16$ inches, $\angle RTU = 90°$, $RS = 11$ inches, and $ST = 5$ inches.

5. A farmer had a pentagon-shaped field of wheat which produced 324 bushels. A sketch of the field is shown at the right. He made the following measurements: $AC = 60$ rods; $EC = 70$ rods; $AE = 18$ rods; $\perp BF = 22$ rods; and $\perp DH = 14$ rods. Find the average number of bushels of wheat per acre produced by the field.

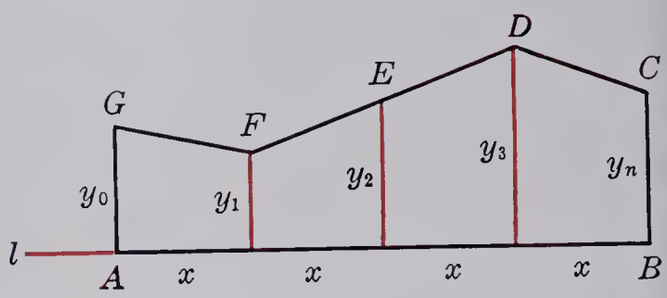


6. In this figure the dashed lines are perpendicular to EH, $EA = 60$, $FB = 80$, $GD = 40$, $HC = 56$, $EF = 45$, $FG = 8$, and $GH = 25$. Find the area of $ABCD$.



7. Find the area in square miles of the state of Utah. (Obtain your measurements from a map of Utah.)

8. In the figure at the right y_0, y_1, y_2, y_3, and y_n are $\perp$ to line l and cut off equal segments, x, on it. Denote the area $ABCDEFG$ by S and show that

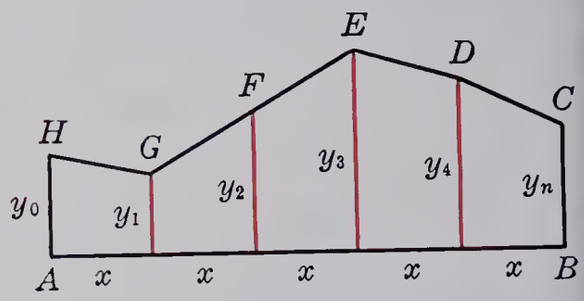


$$S = \frac{x}{2}(y_0 + 2\,y_1 + 2\,y_2 + 2\,y_3 + y_n).$$

9. Show that the formula of Ex. 8 may be written as

$$S = x\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3\right).$$

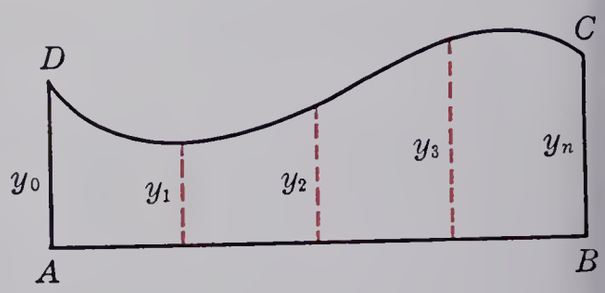
10. Show that the formula of Ex. 9 may be extended to include any number of trapezoids, thus becoming



$$S = x\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \cdots + y_{n-1}\right).$$

261. The Trapezoid Formula

The formula of Exercise 10 above, known as the trapezoid formula, is sometimes used to find the approximate area of a figure bounded by an arc of a curve, a straight line, and perpendiculars to the straight line from the end points of the arc. The figure $ABCD$ shown at the right is such a figure. To find its area we divide AB into any number of equal segments of length x, and at the points of division draw perpendiculars (known as *offsets*) to AB, extending them to meet the curve. The figure thus obtained approximates



that of Ex. 10. By substituting the measured lengths of x and y_0, y_1, $\cdots$, y_n in the formula of Ex. 10 we can determine the approximate area of $ABCD$.

Example. Using the measurements shown, find the area of the figure at the right.

Solution. Note that in the formula y_0 and y_n denote the first and last perpendiculars, while y_1, y_2, $\cdots$ denote the intermediate ones. In the drawing above $y_0 = 20'$, $y_n = 10'$, $y_1 = 30'$, $y_2 = 25'$, and $x = 20'$. Substituting,

$$S = x\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \cdots + y_{n-1}\right)$$

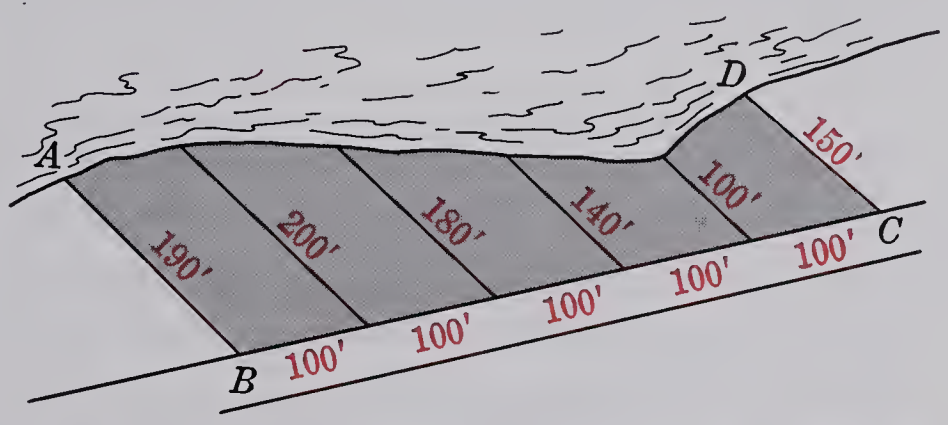
gives $$S = 20\left(\frac{20 + 10}{2} + 30 + 25\right), \text{ or } S = 1400$$

Thus, the area is approximately 1400 sq. ft.

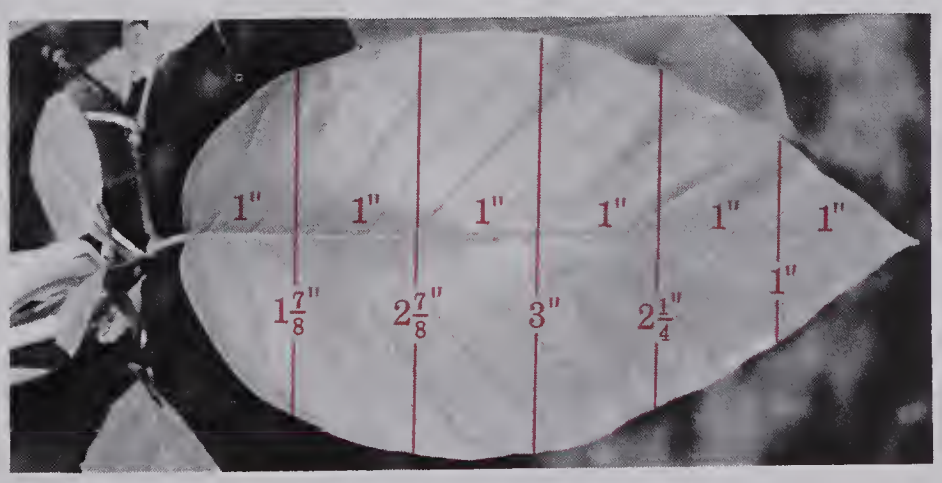
EXERCISES — Trapezoid Formula

C

1. A real-estate man wants to know the area $ABCD$ of some land bounded by the shore AD of a lake, a road BC, and the line segments AB and DC perpendicular to the road. He made the measurements shown. What is the area of the tract?



2. A botanist wishing to know the area of the surface of a leaf made the measurements shown. Find the area of the leaf. $1\frac{7}{8}''$, $2\frac{7}{8}''$, $\cdots$ indicate the width of the leaf at the points indicated.



Rutherford Platt

EXERCISES — Miscellaneous

A

1. Find the side of a square whose area equals that of a triangle with base 144 inches and altitude 72 inches.

2. The area of a rhombus is 2304. Find its diagonals if one is half the other.

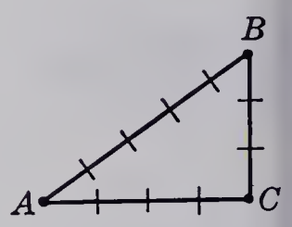
3. The legs of a right triangle are 15 and 20. Find the area of an equilateral triangle which has one of its sides equal to the hypotenuse of the right triangle.

4. If the base of a triangle is doubled and the altitude remains unchanged, how is the area changed?

5. If both base and altitude of a triangle are trebled, how is the area changed?

6. If the area of a triangle remains the same when the altitude is doubled, how is the base changed?

7. About 2000 B.C. the Egyptians knew that a right triangle could be formed by stretching around three pegs a cord measured off into 3, 4, and 5 units. Prove that a triangle with sides 3, 4, and 5 is a right triangle.



B

8. The sides of a triangle are 16 inches, 30 inches, and 34 inches respectively. Find the altitude drawn to the longest side.

SUGGESTION. Let $x =$ the altitude. Then $17x =$ the area of the triangle. Find the area by Hero's Formula and equate the areas. Or use § 258.

9. *Prove:* Two triangles are equal in area if two sides of one are equal respectively to two sides of the other and the included angles are supplementary.

10. Write a formula for the area of a kite in terms of its diagonals, d and d'.

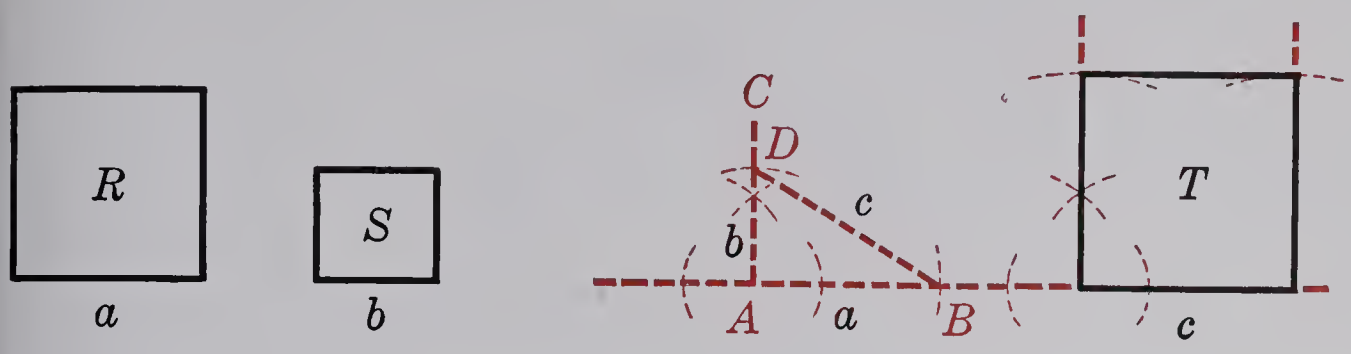
11. Write a formula for the area of a rhombus in terms of a side, s, when one angle contains 120°.

C

12. Find the side of an equilateral triangle that is equal in area to a square 18 inches on a side.

Problem 11

262. To construct a square equal to the sum of two given squares.



Given the squares R and S having sides a and b respectively.

To construct a square $T = R + S$.

Construction

STATEMENTS	REASONS
1. Construct $AB = a$.	1. Why possible?
2. Construct $AC \perp AB$.	2. Why possible?
3. On AC lay off $AD = b$.	3. Why possible?
4. Draw DB.	4. Why possible?
5. Construct square T with a side $= BD(c)$.	5. Why possible?

Then square $T =$ *square* $R +$ *square* S.

Proof: (The proof is left to you.)

EXERCISES Constructions

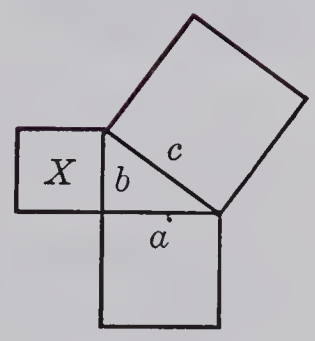
B

1. Construct a square twice as large as a given square.

2. Construct a square equal to the sum of three given squares.

3. Construct a square equal to the difference of two given squares.

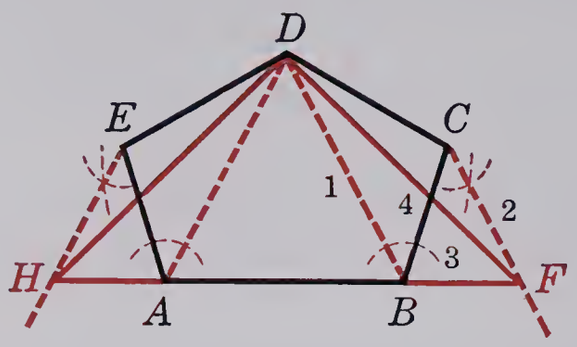
SUGGESTION. a and c are known. How can you find b, the side of the square X?



4. Construct a square three times as large as a given square.

Problem 12

263. To construct a triangle equal to a given polygon.



Given the polygon $ABCDE$.

To construct a $\triangle = ABCDE$.

Construction

STATEMENTS	REASONS
1. Draw diagonal DB, forming $\triangle BCD$.	1. Why possible?
2. Construct a line through $C \parallel BD$.	2. Why possible?
3. Produce AB to meet this line at F.	3. Why possible?
4. Draw DF.	4. Why possible?
5. In like manner, construct HD.	5. Statements 1–4.

Then $\triangle HDF = ABCDE$.

Proof: (The proof is left to the student.)

SUGGESTIONS. $\triangle BFD = \triangle BCD$ since they have the same base BD and equal altitudes. Why are the altitudes equal? $\triangle BCD$ is replaced by $\triangle BFD$ of equal area. In this way the number of sides of the polygon is reduced by one. In like manner, $\triangle AED$ is replaced by $\triangle AHD$ of equal area, and the number of sides of the polygon $AFDE$ is reduced by one.

EXERCISES — Constructions

A

1. Construct a triangle equal in area to a given square.
2. Construct a right triangle equal in area to a given triangle.
3. Construct a right triangle equal in area to a given square.
4. Construct a triangle equal in area to a given hexagon.
5. Construct a quadrilateral equal in area to a given pentagon.

NOTE TO THE TEACHER. If desired, §§ 541–547 may be studied now.

EXERCISES — Review

A

1. Find the area of a rectangle whose dimensions are

a. 18.2 and 16.

b. $4\frac{2}{3}$ and $7\frac{5}{8}$.

c. 4.18 and 9.32.

d. $x + 6$ and $4x - 3$.

e. $2x - 3y$ and $4xy$.

f. $x^2 + x + 1$ and $x^2 - x + 1$.

2. The area of a rectangle is $a^2 - ab - 2b^2$ and the base is $a - 2b$. Find the altitude.

3. Find the base of a parallelogram

a. If its area is 360.43 and its altitude is 13.3.

b. If its area is 10 acres and its altitude is 15 rods.

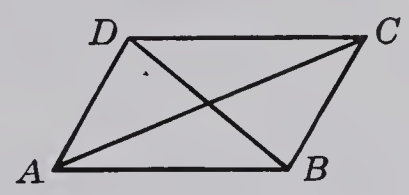
c. If its area is $4x^3 - 8x^2 + 4x$ and its altitude is $4x$.

d. If its area is $x^2 + x - 30$ and its altitude is $x - 5$.

4. Find the area of a cement walk 5 feet wide which surrounds a rectangular plot of grass 115 feet long and 82 feet wide.

5. Find the side of a square equal in area to a trapezoid whose bases are 34 feet and 62 feet and whose altitude is 27 feet.

6. $ABCD$ is a parallelogram. Prove that $\triangle ABD = \triangle ABC$.



7. Find the area of a rhombus whose diagonals are 18 feet and 24 feet respectively.

8. Each side of a rhombus is 13 inches and one diagonal is 10 inches. Find the area.

9. The area of a rhombus is 384 square inches and one diagonal is 16 inches. Find the other diagonal.

10. Prove that the area of a square is equal to one half the square of a diagonal.

11. The area of a square is 60 square inches. Find its diagonal.

12. Find the area of an equilateral triangle each of whose sides is

a. 8 inches.

b. 10 inches.

c. 14 inches.

d. $3\sqrt{3}$.

e. 1.732.

f. $\frac{1}{2}\sqrt{3}$.

13. A rectangular field whose adjacent sides have the ratio 2 : 5 contains 1440 square rods. Find the dimensions of the field.

14. The sides of a rectangle whose area is 240 square feet have the ratio 3 : 5. Find the diagonal of the rectangle.

15. Find the base of a triangle whose area is $x^2 - x - 20$ and whose altitude is $x + 4$.

B

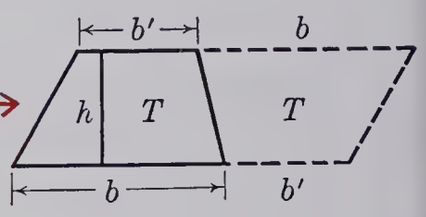
16. Find the area of an isosceles trapezoid whose bases are 20 inches and 30 inches respectively and whose nonparallel sides are each 17 inches.

17. Find the area of a trapezoid whose bases are 16 inches and 24 inches respectively if the angles at one base are each 60°.

18. Find the side of a square which is equal to a trapezoid whose bases are 40 inches and 56 inches respectively and whose altitude is $21\frac{1}{3}$ inches.

19. Prove that the line segment joining the midpoints of the bases of a trapezoid divides the trapezoid into two equal trapezoids.

20. Find the area of trapezoid $ABCD$ in which base AB is 50 inches, BC is 20 inches, $\angle B$ is 30°, and $\angle A$ is 45°.



21. Prove Theorem 32 (§ 244), using this figure.

22. The area of a trapezoid is 390 square inches and the altitude is 15 inches. Find the bases if their difference is 12 inches.

23. The area of an isosceles right triangle is 72 square inches. Find the length of each side.

24. Find the area of a rhombus whose longer diagonal is 48 inches and whose perimeter is 120 inches.

25. Find the side of a square equal in area to the sum of two squares whose areas are 121 and 135 square inches respectively.

26. Find the longest side of an isosceles right triangle if its area is 288 square inches.

27. The area of a trapezoid is 51 square inches. Its altitude is 8.5 inches and one base is 4 inches. Find the other base.

28. Find the altitude of an equilateral triangle whose area is $12.25\sqrt{3}$.

29. The altitude of a rectangle is $\frac{3}{5}$ of its length. Find its dimensions if its area is 2535.

30. The ratio of the width to the length of a rectangular city lot is 3 : 8. Find the dimensions of the lot if its area is 9600 square feet.

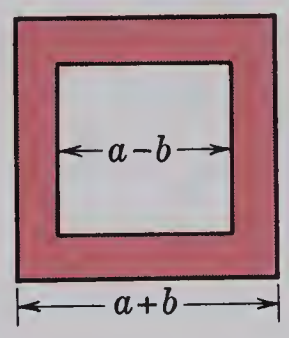
31. The area of a rhombus is 150 square inches and the diagonals have the ratio 3 : 4. Find the length of each side of the rhombus.

32. The sides of a triangle are 18 inches, 20 inches, and 24 inches. Find the altitude upon the longest side.

33. The sides of a triangle are 35, 84, and 91 respectively. Find the altitude drawn to the side 91.

34. One angle of a parallelogram is 30°. If the adjacent sides are 12 feet and 16 feet respectively, what is the area of the parallelogram?

35. The shaded surface between the two squares is of uniform width. Find the area and width of the shaded surface.

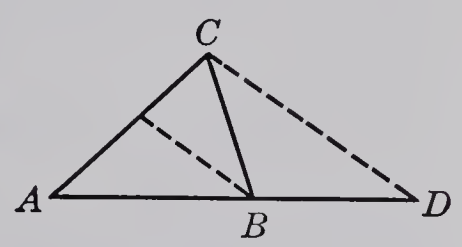


36. The area of a rectangle whose base is 18 feet is 432 square feet. Find the diagonal.

C

37. Transform $\triangle ABC$ below into a triangle having base AD.

38. A diagonal of a rectangle is 25 feet. Find the area of the rectangle if its width is 9 feet.



39. Find the perimeter of a rectangular field if its area is 11.25 acres and the length is twice the width.

40. The difference between two adjacent sides of a parallelogram is 4 feet, and the included angle contains 60°. What are the lengths of its sides if its area is $126\sqrt{3}$ square feet?

41. One angle of a parallelogram contains 60°. If the adjacent sides are 12 feet and 16 feet respectively, what is the area of the parallelogram?

42. Find the area of an isosceles triangle whose perimeter is p and whose base is b.

43. Find the area of an equilateral triangle whose altitude is h.

44. The difference between a side and the altitude of an equilateral triangle is $2(2-\sqrt{3})$ feet. Find the area of the triangle.

45. The shortest side of a 30°-60° right triangle is 15. Find the area.

46. The area of a 30°-60° right triangle is 17.32. Find the lengths of the three sides.

47. Using § 238, prove that the product of the legs of a right triangle is equal to the product of the hypotenuse and the altitude upon it.

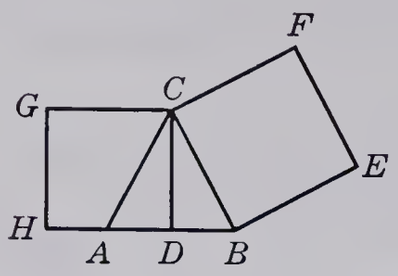


Fig. 1

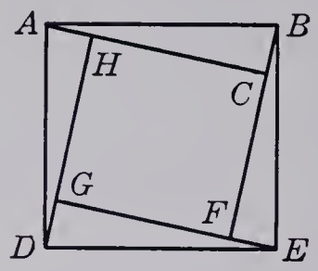
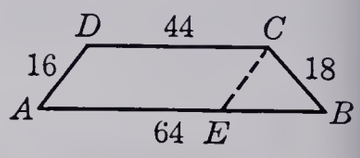


Fig. 2

48. In Fig. 1, ABC is an equilateral triangle with altitude CD. If the area of square BF is 24 square inches, what is the area of square DG?

49. Using Fig. 2, see if you can prove the Pythagorean Theorem.

50. Find the area of the trapezoid $ABCD$.



Suggestions. Draw $CE \parallel AD$. Find the area of $\triangle CEB$. Find the altitude of $\triangle CEB$ and then find the area of the parallelogram.

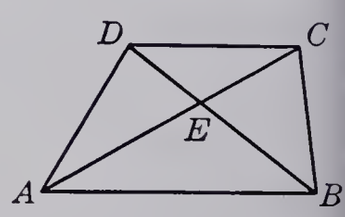
51. The sum of the diagonals of a rhombus is 69 inches, and one diagonal is 21 inches longer than the other. Find the area of the rhombus.

52. The area of a rhombus is 165 square inches, and one diagonal exceeds the other by 7 inches. Find the two diagonals.

53. In trapezoid $ABCD$, $\angle A = 60°$, $\angle B = 90°$, $AD = 12$ inches, and $BD = 24$ inches. Find the area of $ABCD$.

54. The bases of an isosceles trapezoid are 15 inches and 21 inches. If each of the nonparallel sides is 5 inches, what is the area of the trapezoid?

55. Find the diagonal of a rectangle which has an altitude of 16 feet and an area of 480 square feet.



56. Given the trapezoid $ABCD$, one base AB, area $\triangle ABC = 126$, area $\triangle ABE = 81$. Find the area of $\triangle AED$.

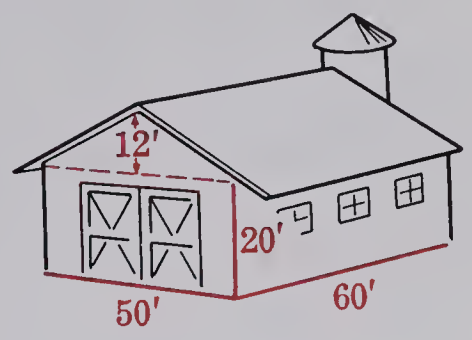
57. In $\triangle ABC$ medians BD and CE intersect in point G. Prove that $\triangle BGC =$ quadrilateral $AEGD$.

58. In $\triangle ABC$, side $AC = 13$ and altitude $AD = 12$. Find the area of the triangle if the perimeter is 38.

59. The legs of a right triangle are 6 ft. and 8 ft. Find the length of the altitude to the hypotenuse.

EXERCISES — Applications

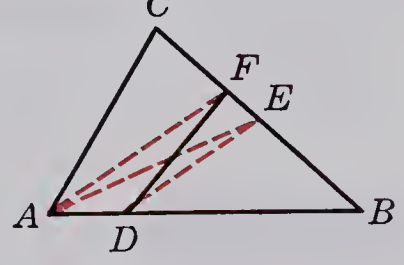
1. A barn is 50 feet wide, 60 feet long, and 20 feet high to the eaves. The rise of the roof is 12 feet. Estimate the cost of applying two coats of paint to the barn if one gallon of paint will cover 400 square feet. The paint costs \$4.50 a gallon, and the cost of labor is twice the cost of the paint. Find the cost of roofing for the barn at \$6 a square (100 square feet) if the roof extends beyond the siding 2 feet at each end and 18 inches at each side. The dimensions were obtained by measurement.



2. Find the cost of sheet roofing for the barn (Ex. 1) if the roof extends 1 foot beyond the siding on all four sides and if the roofing costs \$17 a square. (1 square = 100 square feet.)

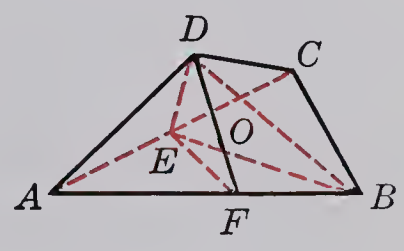
3. A tree 90 feet high is broken off 40 feet from the ground. How far from the foot of the tree will the top strike the ground?

4. A farmer wished to divide a triangular pasture field ABC into two fields having equal areas by building a fence from D, a point in AB. How did he do it?



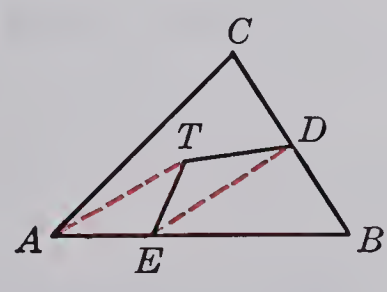
SUGGESTIONS. E is the midpoint of BC. Prove that $\triangle ADE = \triangle DEF$.

5. A farmer had a farm in the shape of the quadrilateral $ABCD$ and wished to separate it into two farms equal in area by a fence DF from D to a point in AB. How did he do it?



SUGGESTIONS. $AE = EC$. $EF \parallel DB$.
$\triangle OFB = \triangle EOD$.

6. A farmer wished to run one fence from a tree T to the side CB of a field and another fence from T to the side AB and thus separate the field ABC into two fields of equal area. He ran a fence from T to D, the midpoint of BC. He then drew DE parallel to TA and ran a fence from T to E. Can you prove that polygon $EBDT$ = polygon $AETDC$?



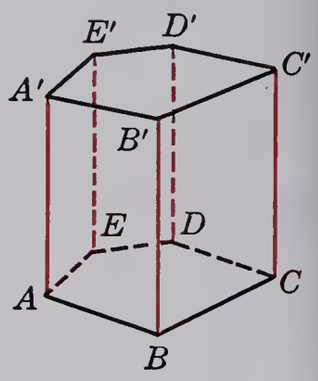
SUGGESTIONS. Draw AD and compare $\triangle ABD$ with $\triangle ADC$. How does AD divide $AEDT$?

SPACE GEOMETRY

[Optional]

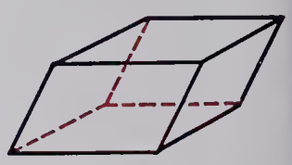
264. Prisms and Parallelepipeds

This figure represents a prism. The top and bottom polygons are called its *bases*. The bases lie in parallel planes. The other faces are called *lateral faces* and intersect in parallel line segments called *edges*. In a *right prism* the lateral faces and edges are perpendicular to the bases. A *regular prism* is a right prism whose bases are regular polygons. Prisms are triangular, quadrangular, pentagonal, etc., according as their bases are triangles, quadrilaterals, pentagons, etc.



A parallelepiped is a prism whose bases are parallelograms. A *rectangular parallelepiped* is a right prism whose bases are rectangles. It is sometimes called a *rectangular solid*.

A *cube* is a parallelepiped all of whose faces are squares.



Parallelepiped

EXERCISES

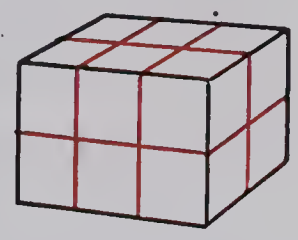
1. Name several objects that are rectangular parallelepipeds; prisms.
2. How many faces has a parallelepiped? a triangular prism?
3. How many edges has a parallelepiped? a quadrangular prism?
4. How many vertices has a parallelepiped? a pentagonal prism?
5. Prove that the lateral faces of a prism are parallelograms.
6. Prove that the lateral edges of a prism are equal.
7. Prove that the lateral faces of a right prism are rectangles.
8. Prove that the bases of a triangular prism are equal.
9. A room is 24 feet long, 12 feet wide, and 8 feet high. Find the distance from a lower corner to the opposite upper corner.
10. Prove that the sum of the squares of the edges of a rectangular parallelepiped is equal to the sum of the squares of the four diagonals.

265. Area of a Prism

The *lateral area* of a prism is the sum of the areas of the lateral faces. The area of a prism means its total area and is the sum of its lateral area and the areas of its two bases.

266. Volume of a Solid

The volume of a solid is the number of cubic units of measure contained in the solid. A cubic unit of measure is a cube whose edge is one linear unit. This solid contains 12 of these cubic units of measure.



267. Assumption. **The volume of a right prism is the product of its base and altitude.**

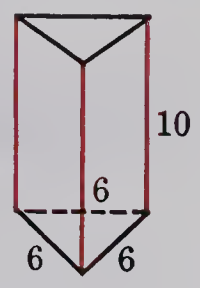
Example. Find the volume of a regular triangular prism if its lateral edge is 10 inches and one edge of a base is 6 inches.

Solution. 1. Area of one base $= \frac{s^2}{4}\sqrt{3} = 9\sqrt{3}$.

2. Volume = base × height

$= 9\sqrt{3} \times 10$

$= 90\sqrt{3}$, or 155.9 cubic inches.



EXERCISES

1. Find the area of a cube 6 inches on an edge. Find its volume.

2. Find the lateral area of a regular triangular prism if each of its edges is 4 inches. Find its volume.

3. Find the area and volume of a right prism whose lateral edge is 12 inches and whose base is a right triangle with legs of 8 inches and 6 inches respectively.

4. A classroom is 24 feet wide, 36 feet long, and 12 feet high. What is its volume in cubic yards? What is its lateral area in square yards?

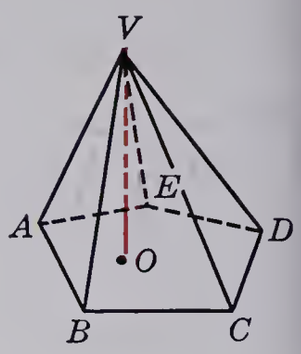
5. Find the area and volume of a rectangular parallelepiped having an altitude of 16 inches and a base 10 inches long by 5 inches wide.

6. Find the lateral area of a regular hexagonal prism 2 feet high if one edge of a base is 6 inches.

7. How many gallons are needed to fill a pool 140 feet long and 50 feet wide to a depth of 6 feet? (231 cubic inches = 1 gallon.)

268. Pyramids

The figure below represents a *pyramid*. In it the polygon $ABCDE$ is called the *base* of the pyramid. The lateral faces are enclosed by triangles meeting in a common vertex V. The intersections of the lateral faces are the *lateral edges*. The *altitude* of a pyramid is the perpendicular from the vertex to the plane of the base. Pyramids are called triangular, quadrangular, etc., in the same manner as prisms.



A *regular pyramid* is a pyramid whose base is a regular polygon and whose altitude is perpendicular to the plane of its base at the point which is equidistant from the vertices of the base.

The *slant height* of a regular pyramid is the altitude of any of the lateral faces.

EXERCISES

1. Prove that the lateral edges of a regular pyramid are equal.

2. Prove that the lateral faces of a regular pyramid are enclosed by congruent isosceles triangles.

3. How many edges has a triangular pyramid? a hexagonal pyramid? an octagonal pyramid?

4. Find the slant height of a regular quadrangular pyramid if one edge of the base is 16 inches and a lateral edge is 17 inches. Find the altitude.

5. Find the slant height of a regular triangular pyramid if one edge of the base is 6 inches and a lateral edge is 5 inches. Find the altitude.

6. Find the lateral area of a regular pentagonal pyramid if the slant height is 7 inches and one edge of the base is 5 inches.

7. Find the lateral area of a hexagonal pyramid whose lateral edges are 7 inches each and whose base has sides of 6 inches each.

8. Find the area of a regular triangular pyramid if each edge is 8 inches.

9. Find the area of a regular hexagonal pyramid if each edge of the base is 10 inches and each lateral edge is 13 inches.

269. Assumption. The volume of a pyramid is equal to one third the product of its base and altitude.

EXERCISES

1. Find the volume of a regular pyramid whose altitude is 3 feet and whose base is a square 2 feet on a side.

2. Find the volume of a regular quadrangular pyramid whose slant height is 26 feet and whose base is 20 feet on a side.

3. Find the volume of a regular triangular pyramid if the altitude is 20 inches and a side of the base is 8 inches.

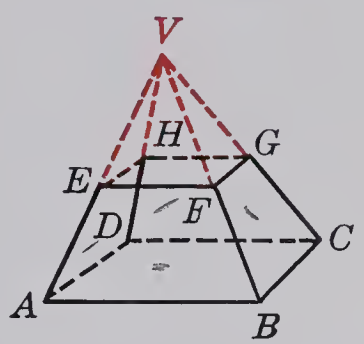
4. Find the volume and area of a triangular pyramid if each lateral edge is 36 inches and the base is 12 inches on a side.

5. A prism and a pyramid have equal bases and equal altitudes. How do their volumes compare?

6. Why do only regular pyramids have slant heights?

270. Frustum of a Pyramid

A *frustum of a pyramid* is the part of a pyramid formed between the plane of the base of a pyramid and a plane parallel to the base. In the figure, $ABCD\text{-}EFGH$ is a frustum of a pyramid.

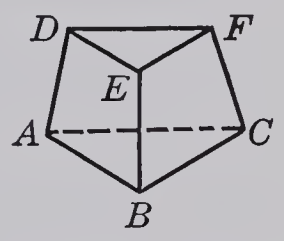


EXERCISES

1. $ABC\text{-}DEF$ is the frustum of a pyramid.

a. *Prove* that $AB \parallel DE$, $BC \parallel EF$, and $AC \parallel DF$.

b. *Prove* that $ABED$, $BCFE$, and $ACFD$ are trapezoids.

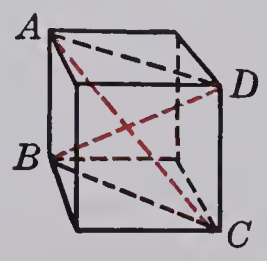


2. Find the lateral area of the frustum of a regular pyramid whose lower base is a square having a side of 8 inches and whose upper base is a square having a side of 6 inches, the slant height of the frustum being 12 inches.

3. In the figure at the right AC and BD are any two diagonals of a cube.

a. *Prove* that $AC = BD$.

b. *Prove* that AC and BD bisect each other.

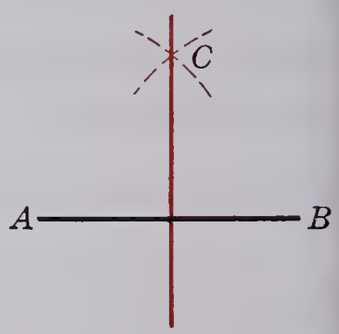


271. Determining, Overdetermining, and Underdetermining

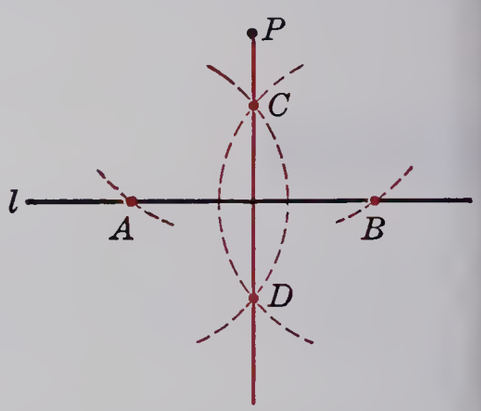
Turn to page 56 and read Assumption 11 once more. Do you see that the assumption consists of two parts: (1) a line can be drawn through two points and (2) only one line can be drawn through two points? We sometimes use another form of Assumption 11 which states (1) and (2) briefly by saying: Two points *determine* a line.

Similarly, we often say that three points not in a straight line determine a triangle. The word "determine" in this case means that (1) there is a triangle which has the three points as vertices and (2) there is only one such triangle.

Beginning geometry students sometimes underdetermine figures. For example, perhaps someone in your geometry class has made the error of underdetermining the perpendicular bisector of a line segment as shown at the right. If so, the student has assumed that one point determines a line. How many lines can be drawn through C? Which one of these lines is the perpendicular bisector of AB? Copy the part of construction shown and correct the error by completing the construction.



Sometimes geometry students overdetermine figures. While this error does not always lead to incorrect results, it does show muddled thinking. For example, in constructing a perpendicular to a line from a point outside the line, as shown at the right, the student who made the construction neglected the fact that only two points are needed to determine a line. How many did he use?



There are situations other than those involved in making constructions in which the words "determined," "overdetermined," and "underdetermined" need to be considered. For example, in the equation $2x = 8$, the value of x is determined, 4 being the only value of x that satisfies the equation; in the equation $x + y = 6$, the values of x and y are underdetermined since there is an unlimited number of pairs of values that satisfy the equation; and in the pair of equations $x + 3 = 9$ and $x + 1 = 7$, the value of x is overdetermined since only one of the equations is needed to find the value of x.

EXERCISES — Determination

A

Which of the words *determined*, *overdetermined*, and *underdetermined* would you apply to Exercises 1–3?

1. To bisect an angle find two points each equidistant from the sides of the angle, then draw the line through the vertex and these points.

2. If the opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

3. Angles opposite equal sides are equal.

In each of the following exercises tell whether the values of x and y are determined, overdetermined, or underdetermined.

4. $2x + 3 = 11$

5. $x + 3y = 13$

6. $x + y = 9$
$x - y = 3$

7. $3x - 5 = 7$
$2x + 1 = 9$

REVIEW QUESTIONS — Chapter 8

1. Give formulas for the following:
a. Area of a rectangle.
b. Area of a parallelogram.
c. Area of a rhombus.
d. Area of a square. Give two.
e. Area of a triangle. Give two.
f. Area of an equilateral triangle in terms of a side.
g. Area of a trapezoid.
h. Relation between the sides of a right triangle and the hypotenuse.
i. Altitude of an equilateral triangle in terms of a side.

2. Can two squares with different perimeters have the same area? Can two triangles with different perimeters have the same area?

3. If two parallelograms have equal bases, to what is the ratio of their areas equal?

4. If two polygons are equal in area, do they have the same shape?

5. What is the name of the triangle whose altitude is one of its sides?

6. Is a triangle divided into two equal triangles by a bisector of one of its angles? by an altitude? by a median?

7. Who was Pythagoras? Euclid? Archimedes?

8. Can two rectangles have equal areas and different perimeters?

9. Can two rectangles have equal perimeters and different areas?

10. What part of a triangle is cut off by a line which is parallel to the base and bisects a side?

11. If both dimensions of a rectangle are doubled, how is its area changed?

12. If two triangles have equal bases, how do their areas compare?

13. How can you find the height of a triangle when its base and area are known?

14. How can you find a side of a square when its perimeter is given? when its diagonal is given?

15. How can you find a leg of a right triangle when the hypotenuse and the other leg are given?

16. State the converse of the Pythagorean Theorem.

17. Give a rule for finding the height of a trapezoid when its bases and area are known.

SUMMARY of Bases for Proof

272. Equal ratios

a. Two rectangles, parallelograms, or triangles having equal bases have the same ratio as their altitudes.

b. Two rectangles, parallelograms, or triangles having equal altitudes have the same ratio as their bases.

c. Two triangles have the same ratio as the products of their bases and altitudes.

273. Areas

a. A *rectangle* is equal to the product of its base and altitude.

b. A *parallelogram* is equal to the product of its base and altitude.

c. A *triangle* is equal to one half the product of its base and altitude.

d. A *triangle* whose sides are a, b, and c and whose semiperimeter is s is equal to $\sqrt{s(s-a)(s-b)(s-c)}$.

e. A *trapezoid* is equal to half the product of its altitude and the sum of its bases.

f. An *equilateral triangle* whose side is s is equal to $\frac{s^2}{4}\sqrt{3}$.

g. A *rhombus* is equal to one half the product of its diagonals.

h. A *square* is equal to the square of a side.

i. The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the legs.

274. Polygons equal in area

a. Two rectangles, parallelograms, or triangles are equal in area if they have equal bases and equal altitudes.

b. If a triangle and parallelogram have equal bases and equal altitudes, the area of the triangle equals half the area of the parallelogram.

275. Constructions

a. To construct a square equal to the sum of two given squares.

b. To construct a triangle equal to a given polygon.

WORD LIST

area	mathematics	proposition	solution
equivalent	numerator	Pythagoras	trapezoid

TEST 17 — True-False Statements

[12 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number. Do not guess.

1. The area of a triangle is equal to the product of its base and altitude. F

2. If the dimensions of a rectangle are doubled, the area is doubled. F

3. The area of a trapezoid is equal to the product of the sum of its bases and its altitude. F

4. The area of a rhombus is equal to one half the product of the diagonals. T

5. The product of the legs of a right triangle is equal to the product of the hypotenuse and the altitude upon it. T

F 6. The hypotenuse of a right triangle whose legs are a and b is $\sqrt{ab}$.

T 7. Two parallelograms having equal bases have the same ratio as their altitudes.

F 8. Triangles that have equal bases and equal altitudes are congruent.

F 9. In the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, $s = a + b + c$.

T 10. The area of a trapezoid is equal to the product of its median and altitude.

F 11. If a side of a triangle is doubled and its altitude remains the same, its area is doubled.

T 12. If a side of a square is doubled, its perimeter is doubled.

F 13. If the perimeter of a square is trebled, its area is trebled.

F 14. Pythagoras is a theorem.

F 15. The altitude of a triangle divides the triangle into two equal triangles.

T 16. If the midpoints of the sides of a triangle are joined by straight-line segments, the triangle formed by the segments is one fourth the original triangle.

T 17. The triangle whose sides are 16, 30, and 34 is a right triangle.

TEST 18 — Construction

[12 Minutes]

Make the following geometric constructions accurately:

1. Construct a quadrilateral equal to a given pentagon.

2. Construct a triangle equal to a given square.

3. Construct a square equal to the difference of two given squares.

TEST 19 — Applications

[30 Minutes]

1. The base of a parallelogram is 11 feet and the altitude is 8 feet. Find the area.

2. The base of a triangle is 6 feet and the altitude is 8 feet. Find the area.

3. The hypotenuse of a right triangle is 13 feet and one leg is 5 feet. Find the area.

4. Find the area of an equilateral triangle whose side is 10 feet.

5. The bases of a trapezoid are 14 feet and 8 feet respectively and the area is 66 square feet. Find the altitude.

6. Find the area of a square whose diagonal is 12 feet.

7. The base of a rectangle is 22 and its area is 550. Find the length of one of its diagonals.

8. Find the side of a square equal to a trapezoid whose bases are 32 and 18 and whose altitude is 9.

9. Find the area of a rhombus if a side is equal to 12 and one angle contains 30°.

10. The area of a trapezoid is 144 square inches. One base is double the other and the altitude is 8 inches. Find the bases.

11. Find the area of a triangle whose sides are 6, 8, and 12.

12. Find the area of a rhombus if the diagonals are 10 and 16.

13. Find the altitude of an equilateral triangle if a side is equal to 12.

14. The bases of an isosceles trapezoid are 12 and 18. The equal nonparallel sides are each 5. Find the area of the trapezoid.

15. The area of a rectangle is 320 and the base is five times its altitude. What is its altitude?

TEST 20 — Drawing the Figure for a Theorem

[13 Minutes]

Draw a figure for each of the following theorems:

1. The bisectors of the four angles of a parallelogram which is not a rhombus form a rectangle.

2. The line joining the midpoints of the nonparallel sides of a trapezoid bisects the diagonals.

3. If perpendiculars are drawn from the end points of the shorter base of an isosceles trapezoid to the other base, two congruent triangles are formed.

4. If perpendiculars are drawn from the four vertices of a parallelogram to any line outside the parallelogram, the sum of the perpendiculars from one pair of opposite vertices equals the sum of the perpendiculars from the other pair.

5. The line joining the midpoints of two opposite sides of a parallelogram is parallel to the other two sides.

CHAPTER 9

Circles, Angles, and Arcs

In this chapter
we shall make a study of the circle
and the lines and angles
associated with it.

The circle has many properties which no other plane figure possesses. For example, it is symmetric with respect to its center and is symmetric with respect to any of its diameters. Of all the plane geometric figures, the circle is the only one which can be rotated about a point without changing its position. It is perhaps the most appealing of all simple geometric figures. Used in decoration it readily harmonizes in composition with other geometric figures. It is frequently used in the designs of public buildings, churches, and gardens.

Have you ever considered how useful the circle is? Without the circle there would be, for example, no watches, clocks, bicycles, automobiles, or steamships.

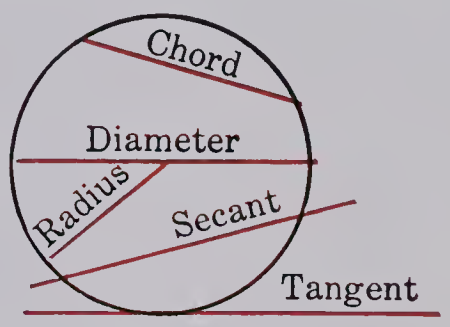
276. Circle

A *circle* (⊙) is a closed plane curve all points of which are equidistant from a point within called the *center*. *Congruent* or *equal circles* are circles that can be made to coincide. If two circles coincide, their centers coincide.

277. Lines Related to the Circle

A *radius* (plural, *radii*) of a circle is a line segment connecting the center with any point on the circle.

A *chord* is a line segment connecting any two points on the circle.



A *diameter* is a chord passing through the center of the circle. A 10-inch circle means a circle having a diameter of 10 inches.

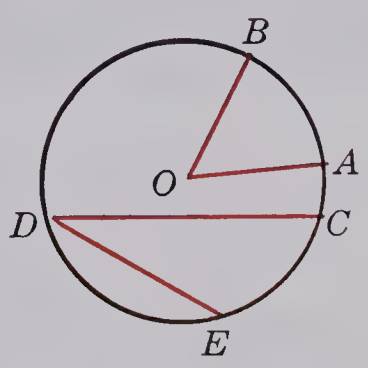
A *secant* is a line which intersects a circle in two points.

A *tangent* is a line touching a circle at one point, and only one. This point is called the *point of tangency* or *point of contact*.

The *line of centers* of two circles is the straight line determined by the centers of the circles.

278. Angles Related to the Circle

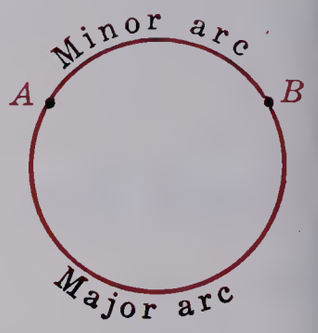
A *central angle*, as $\angle AOB$ in the figure, is an angle whose vertex is the center of the circle and whose sides are radii. An *inscribed angle*, as $\angle CDE$, is an angle whose vertex is on the circle and whose sides are chords.



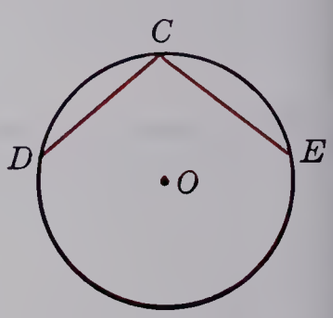
279. Arc

An *arc* ($\frown$) of a circle is the part of a circle included between any two of its points. An arc is usually named by its end points or by a small letter near it.

A *semicircle* is an arc that is half of a circle and a *quadrant* is an arc that is one fourth of a circle. A *minor arc* is one that is less than a semicircle and a *major arc* is one that is larger than a semicircle. Any two points of a circle divide the circle into two arcs. If these two arcs are unequal, one is a major arc and the other is a minor arc. From now on when two capital letters are used to name an arc the minor arc is the one intended, unless stated otherwise. Thus $\overset{\frown}{AB}$ means the minor arc AB. Every chord of a circle has two arcs. If these arcs are unequal, we shall agree that "a chord and its arc" means "a chord and its minor arc."

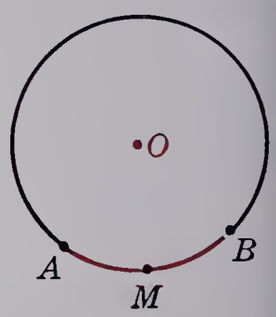


An angle is *inscribed in an arc* if its vertex is on the arc and its sides are chords joining the vertex to the end points of the arc. $\angle DCE$ is inscribed in $\overset{\frown}{DCE}$.



Equal arcs are arcs which can be made to coincide. It follows that they are arcs of the same circle or of equal circles.

The *midpoint of an arc* is the point which divides the arc into two equal parts. The *center of an arc* is the center of the circle of which the arc is a part. In the figure, M is the midpoint and O is the center of $\overset{\frown}{AB}$.



280. Circumference

The *circumference* of a circle is its length expressed in linear units (as feet and inches).

281. Assumptions Relating to a Circle

From definitions and a study of the circle we may state three assumptions which relate to a circle.

2500 BC

FIRST WHEEL SECTION OF LOG

1400 BC

EGYPTIAN CHARIOT WHEEL

60 AD

ROMAN CHARIOT WHEEL

1750

WAGON WHEEL

1906

AUTOMOBILE WHEEL

TODAY

AUTOMOBILE WHEEL AND TIRE

Courtesy of The Budd Company

These pictures of wheels suggest the history of overland transportation from the time of early man down to today. In the spoked wheel we find practical illustrations of central angles and arcs.

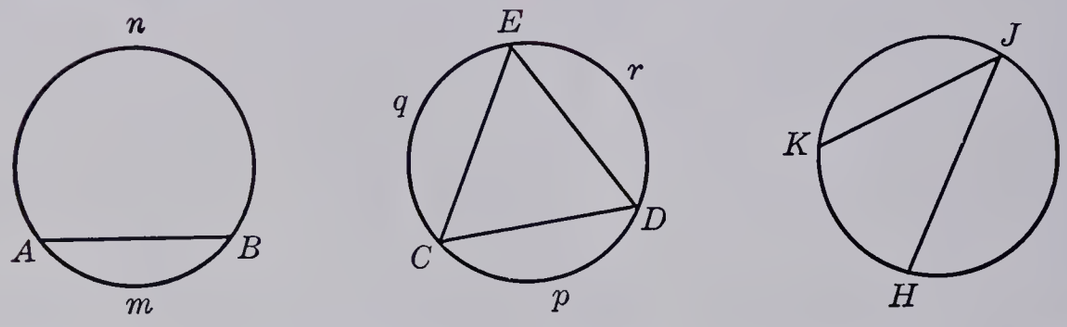
*a. **Assumption 41.** Circles having equal radii are equal; and conversely.

b. **Assumption 42.** A point is within, on, or outside a circle if its distance from the center is less than, equal to, or greater than the radius; and conversely.

c. **Assumption 43.** Two minor arcs, or two major arcs, coincide if their end points and centers coincide; and conversely.

EXERCISES Oral

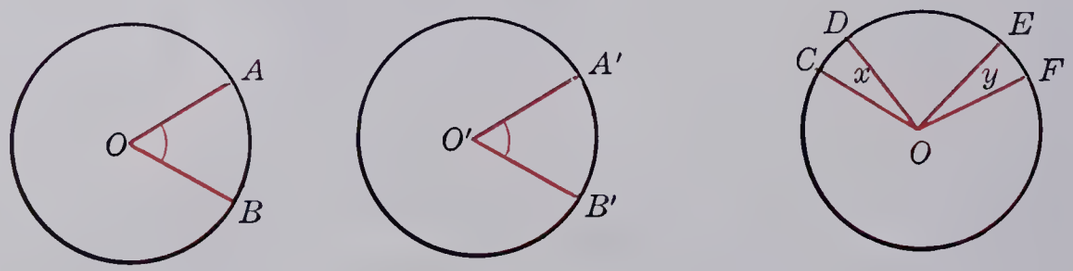
A



1. Name the minor arcs of the circles above; the major arcs.

2. Complete: $\angle$ __?__ is inscribed in $\overset{\frown}{CDE}$; $\angle$ __?__ is inscribed in $\overset{\frown}{CED}$; $\angle J$ is inscribed in __?__.

282. Assumption 44. In a circle or in equal circles equal central angles have equal arcs.



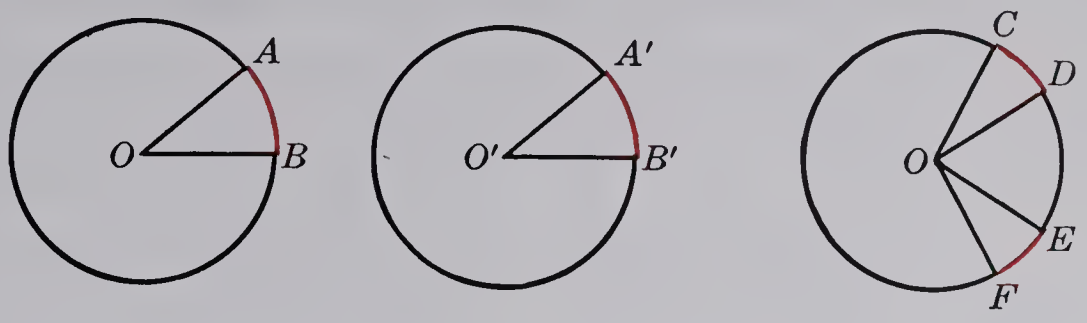
This assumption states that if $\odot O = \odot O'$ and $\angle O = \angle O'$, then $\overset{\frown}{AB} = \overset{\frown}{A'B'}$. It also states that if $\angle x = \angle y$, $\overset{\frown}{CD} = \overset{\frown}{EF}$.

283. Corollary. A diameter of a circle bisects the circle.

(What kind of central angles are formed by a diameter?)

*Other assumptions on the circle were given on pages 56 and 57.

284. Assumption 45. In a circle or in equal circles equal arcs have equal central angles.



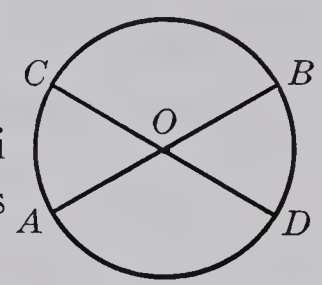
This assumption states that if $\odot O = \odot O'$ and $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, then $\angle O = \angle O'$. It also states that if $\overset{\frown}{CD} = \overset{\frown}{EF}$, $\angle COD = \angle EOF$.

285. Corollary. If a chord bisects a circle, it is a diameter.

EXERCISES

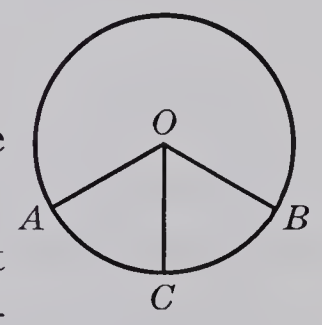
A

1. *Given* circle O with diameters AB and CD.
Prove $\overset{\frown}{AC} = \overset{\frown}{DB}$.



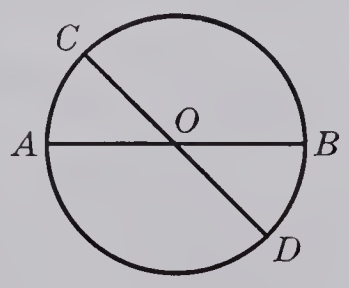
2. If a circle is divided into six equal arcs and radii are drawn to the points of division, how many degrees are there in each of the six central angles?

3. *Given* $\odot O$, radii AO, CO, and BO, and C the midpoint of $\overset{\frown}{AB}$.
Prove that CO bisects $\angle AOB$.



4. *Prove:* The bisector of a central angle of a circle bisects the intercepted arc.

5. On paper locate three points, A, B, and C, not in the same straight line. Locate several points equidistant from A and B (see §§ 139 and 171). Using these points as centers, construct circles passing through A and B. In like manner, construct several circles passing through B and C. Can a circle be drawn through A, B, and C if these points do not lie on a straight line? Why?



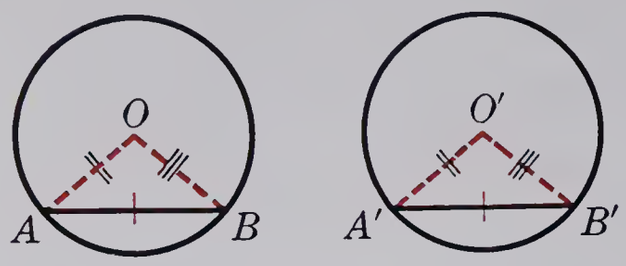
6. *Given* diameter AB of $\odot O$, radii CO and OD, and $\overset{\frown}{AC} = \overset{\frown}{BD}$.

a. *Prove* that $\overset{\frown}{BC} = \overset{\frown}{AD}$.

b. *Prove* that COD is a diameter.

★ Theorem 38

286. In a circle or in equal circles equal chords have equal arcs.



Given $\odot O = \odot O'$ and chord $AB =$ chord $A'B'$.

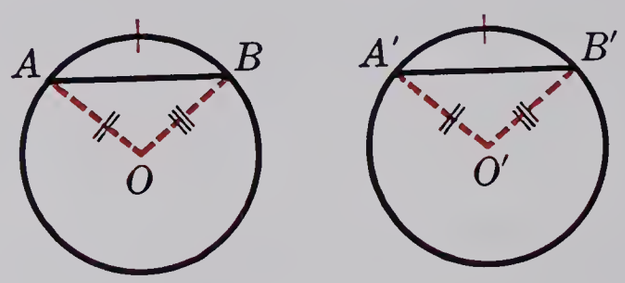
To prove that $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Planning the Proof: 1. We can prove arcs = by §§ 281 c, 282.
2. You should use § 282.

Proof: (The proof is left to the student.)

★ Theorem 39

287. In a circle or in equal circles equal arcs have equal chords.



Given $\odot O = \odot O'$ and $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

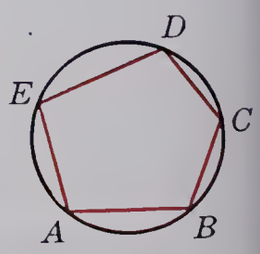
To prove that $AB = A'B'$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 157, 217, 281 a.
2. You should use § 97 a.

Proof: (The proof is left to the student.)

288. Inscribed Polygon

A polygon is *inscribed in a circle* if its vertices lie on the circle. The circle is then said to be *circumscribed about the polygon.* $ABCDE$ is an inscribed polygon.

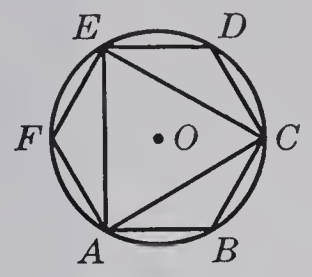


EXERCISES — Angles and Arcs

A

1. Name the ways you have had of proving arcs equal; of proving chords equal; of proving central angles equal. Have you located these ways on pages 567, 564, and 565?

2. If an equilateral polygon is inscribed in a circle, what is true of the arcs? Why?

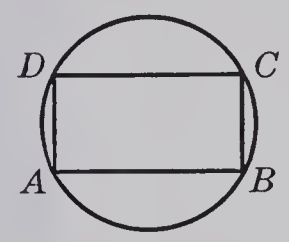


3. *Given* equilateral hexagon $ABCDEF$ and $\triangle ACE$, inscribed in $\odot O$.

Prove as many facts as you can without using any auxiliary lines.

4. *Given* $\overset{\frown}{AB} = \overset{\frown}{DC}$ and $\overset{\frown}{AD} = \overset{\frown}{BC}$.

Prove that $ABCD$ is a $\square$.



5. If a circle contains 360°, how many degrees are in each arc of a circle circumscribed about an equilateral triangle? a square? a regular hexagon? a regular octagon?

6. If an arc is doubled, is its chord doubled? its central angle?

7. What is the difference between the "midpoint of an arc" and the "center of an arc"?

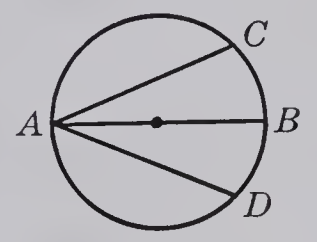
B

8. *Given* the diameter AB bisecting $\overset{\frown}{CD}$.

Prove that $AC = AD$.

9. *Given* the diameter AB and $AC = AD$.

Prove that $\overset{\frown}{BC} = \overset{\frown}{BD}$.

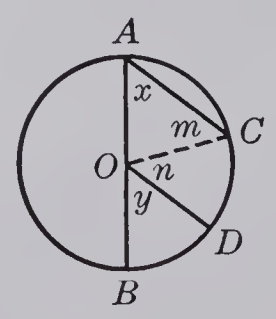


10. The line joining the midpoints of the two arcs of a chord is the perpendicular bisector of the chord.

C

11. *Given* diameter AB of $\odot O$, chord AC, and radius $OD \parallel AC$.

Prove that $\overset{\frown}{BD} = \overset{\frown}{CD}$.



12. If the end points of two perpendicular diameters are joined in order, the figure formed is a square.

Leonardo da Vinci's famous painting of Mona Lisa hangs in the Louvre Museum, Paris

289. Geometry in the Arts

It has been known for years that nature has used geometric forms in the construction of crystals and in the framework of plant and animal life. Much of the beauty that is found in nature is due to some geometric pattern or to the use of numbers associated with geometry. It seems likely that as time passes, man will discover many more applications of geometry in nature.

Since the artist, sculptor, and architect often find their themes in nature, they too either deliberately or instinctively use geometric forms and proportions. Some of these applications of geometry to the arts are readily seen, while others are latent, being so hidden that they are not easily detected.

In paintings the geometric figures are usually latent and have to be discovered. Some of the early painters who deliberately based their works on geometric principles were Fra Bartolommeo, Raphael, Michelangelo, and Leonardo da Vinci.

Leonardo da Vinci obtained the "Mona Lisa" smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes. The outline of the top of the head is the arc of another circle exactly twice as large as the first. In the same artist's "Last Supper" the visible part of Christ conforms to sides of an equilateral triangle.

The geometry in architecture is both latent and visible. Almost any building of any consequence is a harmonious arrangement of geometric forms. One of the most famous buildings of all time is pictured below. It is the Parthenon, the largest of the group of buildings on the Acropolis in Athens. Built in the years 447–432 B.C., the Parthenon is noted for its perfection of form. Modern studies have shown that the principle of dynamic symmetry must have been known to its designer, Ictinus.

Dynamic symmetry is the term applied to designs which are based upon the division of a rectangle into similar rectangles, of which you will read more in a later chapter. Many modern artists use this principle.

The plane figures most commonly used in architecture are the circle, rectangle, square, and equilateral triangle. The Romans used these figures

Davis Pratt from Rapho-Guillumette

Modern technology creates many forms which are not only useful but beautiful. The anode pictured at the upper right is part of an instrument for generating microwave power. In its form it suggests the symmetry and beauty of a rose window in a great cathedral

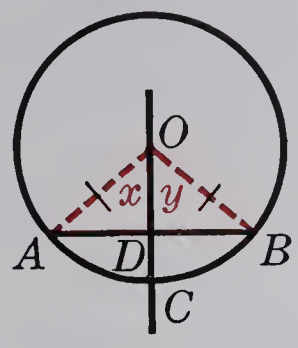
in determining the proportions of triumphal arches and the Italians in constructing Gothic cathedrals. An excellent modern example of Gothic architecture is the National Cathedral at Washington, D.C.

Sculpture, more than painting, is based on geometry. It makes more use of geometry when it is used with architecture than when it is used alone. The sculptor makes frequent use of the circle, the regular polygon, and the proportions of the human figure. As in paintings, the geometry is often latent in sculpture.

After making a most careful study of the "Apollo Belvedere," the great art critic Winckelmann said that even its seemingly careless beautiful lines show the action of the most exact mathematics. He found that the charm of the statue was due to its scientific proportions.

★ Theorem 40

290. If a line through the center of a circle is perpendicular to a chord, it bisects the chord and its arc.



Given $\odot O$ with the line $OC \perp AB$.

To prove that $AD = DB$ and $\overset{\frown}{AC} = \overset{\frown}{CB}$.

Planning the Proof: 1. We can prove: (a) line segments = by §§ 97, 157, 217, 281 a, 287; (b) arcs = by §§ 281 c, 282, 286.
2. We shall use (a) § 97 a; (b) § 282.

Proof

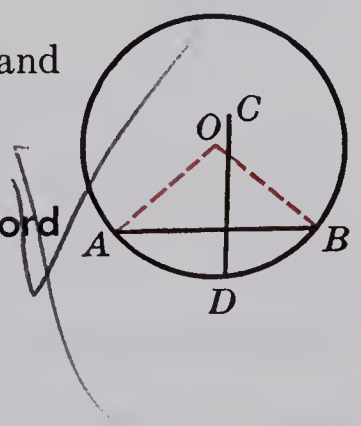
STATEMENTS	REASONS
1. Draw OA and OB.	1. Why possible?
2. $\measuredangle ADO$ and BDO are rt. $\measuredangle$.	2. Why?
3. Rt. $\triangle AOD \cong$ rt. $\triangle BOD$.	3. Give full proof.
4. $AD = DB$.	4. Why?
5. $\angle x = \angle y$.	5. Why?
6. $\therefore \overset{\frown}{AC} = \overset{\frown}{CB}$.	6. Why?

NOTE. The proof of Theorem 40 is based upon the assumption that the chord AB is not a diameter. Why is it not necessary to prove that OC bisects AB when AB is a diameter?

Theorem 41

291. If a line through the center of a circle bisects a chord that is not a diameter, it is perpendicular to the chord.

SUGGESTION. Draw radii to the end points of the chord and use § 159 *b*.



292. Corollary. The perpendicular bisector of a chord passes through the center of the circle.

SUGGESTIONS. $OA = OB$. Why? Use § 139.

EXERCISES

A

1. *Given an arc of a circle, find the center of the circle.*

SUGGESTION. Draw any two nonparallel chords and use § 292.

2. Are two arcs equal if they have equal lengths?

3. Name all the ways you have had of proving lines perpendicular.

4. *Given* $\triangle ABC$ inscribed in $\odot O$ and $\overset{\frown}{AC} = \overset{\frown}{BC}$.
Prove that $\angle A = \angle B$.

5. *Given* $\triangle ABC$ inscribed in $\odot O$ and $\angle A = \angle B$.
Prove that $\triangle ABC$ is isosceles.

6. The line joining the midpoints of a chord and its arc is perpendicular to the chord.

7. If a radius bisects an arc, it is the perpendicular bisector of the chord of the arc.

8. *Prove*: The perpendicular bisector of one of two parallel chords bisects the other chord.

9. *Prove:* If a diameter of a circle bisects each of two chords which are not diameters, the chords are parallel to each other.

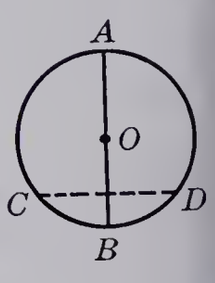
10. *Given* $\odot O$ with chord AB = chord AC, radius OA, F the midpoint of AC, and H the midpoint of AB.
Prove: $\angle OAB = \angle OAC$.

B

11. *Prove:* The line joining the midpoints of a chord and its arc passes through the center of the circle.

12. *Prove:* If two chords of a circle are drawn from a point of the circle and make equal angles with the radius drawn to the point, the chords are equal.

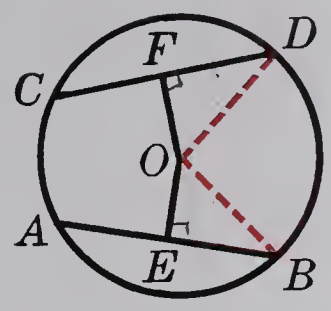
13. *Prove:* If a diameter bisects one of two parallel chords (which are not diameters), it bisects the other.



14. *Prove:* Two points of a circle each equidistant from two other points of the circle determine a diameter of the circle.

Theorem 42

293. In a circle or in equal circles equal chords are equidistant from the center.



Given $\odot O$ with $AB = CD$, $OE \perp AB$, and $OF \perp CD$.
To prove that $OE = OF$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 157, 217, 281 a, 287, 290.
2. We shall use § 97 a.

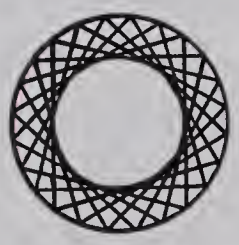
Proof

STATEMENTS	REASONS
1. Draw OB and OD.	1. Why possible?
2. $OE \perp AB$ and $OF \perp CD$.	2. Why?
3. $\therefore EB = \frac{1}{2} AB$ and $FD = \frac{1}{2} CD$.	3. § 290.
4. But $AB = CD$.	4. Why?
5. $\therefore \frac{1}{2} AB = \frac{1}{2} CD$ and $EB = FD$.	5. Asmts. 4, 6.
6. $OD = OB$.	6. Why?
7. Rt. $\triangle OEB \cong$ rt. $\triangle OFD$.	7. Why?
8. $\therefore OE = OF$.	8. Why?

EXERCISES — Chords

A

1. In this circle equal chords are drawn. Are they equidistant from the center? Why?



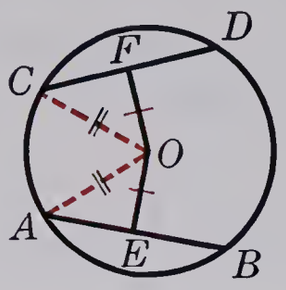
2. The sides of an inscribed square are equidistant from the center.

3. If two equal chords which are not diameters intersect a diameter at the same point, they make equal angles with it.

4. The perpendicular bisectors of the sides of an inscribed polygon meet in a point.

★★ Theorem 43

294. In a circle or in equal circles chords equidistant from the center are equal.



Given $\odot O$, the chords AB and CD, $OE \perp AB$, $OF \perp CD$, and $OE = OF$.
To prove that $AB = CD$.

Planning the Proof: 1. We can prove line segments = by §§ 97, 157, 217, 281 a, 287, 290, 293.
2. We shall use § 97 a.

Proof

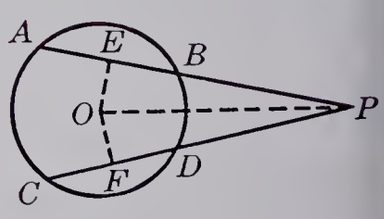
STATEMENTS	REASONS
1. Draw OA and OC.	1. Why possible?
2. $OE \perp AB$ and $OF \perp CD$.	2. Why?
3. Rt. $\triangle AEO \cong$ rt. $\triangle CFO$.	3. Give full proof.
4. $\therefore AE = CF$.	4. Why?
5. $AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$.	5. § 290.
6. $\therefore \frac{1}{2} AB = \frac{1}{2} CD$ and $AB = CD$.	6. Why?

EXERCISES — Chords

B

1. Prove informally: If two chords of a circle are unequal, they are not equally distant from the center.

2. If two chords intersect a diameter at the same point and make equal angles with it, they are equal.



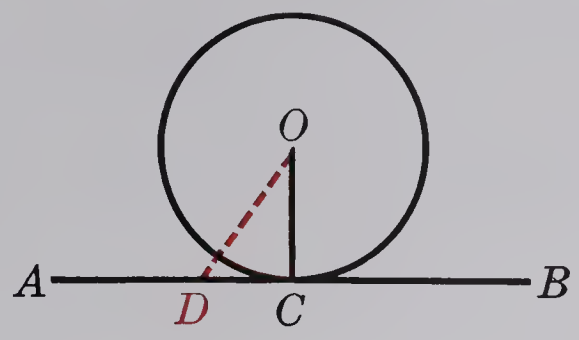
3. *Given* equal chords AB and CD extended to meet at P.

Prove that $PA = PC$.

4. If two equal chords of a circle intersect, the segments of one are respectively equal to the segments of the other.

★Theorem 44

295. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.



Given AB tangent to $\odot O$ at C and OC a radius.

To prove that $AB \perp OC$.

Planning the Proof: 1. We can prove lines $\perp$ by §§ 51, 159, 220, 291.
2. We shall use § 51 b.

Proof

STATEMENTS	REASONS
1. From D, any point on AB except C, draw DO.	1. Why possible?
2. AB is tangent to $\odot O$ at C.	2. Why?
3. D is outside $\odot O$.	3. Def. of a tangent.
4. $\therefore OD > OC$, or OC is the shortest line segment from O to AB.	4. § 281 b.
5. $\therefore OC \perp AB$ or $AB \perp OC$.	5. § 51 b.

296. Corollary I. If a line is perpendicular to a tangent at the point of contact, it passes through the center of the circle.

SUGGESTION. Draw a radius to the point of contact and show that it coincides with the given perpendicular.

297. Corollary II. If two circles are tangent to the same line at the same point, the line of centers passes through the point of contact.

SUGGESTION. Show by § 296 that a perpendicular to the tangent at the point of contact passes through the centers of both circles.

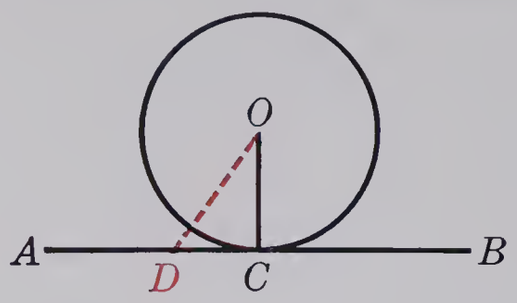
EXERCISES

1. State the converse of Theorem 44.

2. A point P is 17 inches from the center of a circle whose radius is 8 inches and PA is tangent to the circle at A. Find the length of PA.

★ Theorem 45

298. If a line is perpendicular to a radius at its point on a circle, it is a tangent to the circle.



Given $\odot O$ with $AB \perp OC$ at C.

To prove that AB is tangent to $\odot O$.

Planning the Proof: 1. We can prove a line tangent to a $\odot$ by § 277 (Def. of a tangent).
2. We shall use § 277 (Def. of a tangent).

Proof

STATEMENTS	REASONS
1. From D, any point in AB except C, draw DO.	1. Why possible?
2. $OC \perp AB$.	2. Given.
3. OD is not $\perp AB$.	3. § 94.
4. $\therefore OD > OC$.	4. Asmt. 21.
5. $\therefore D$ lies outside $\odot O$.	5. § 281 b.
6. $\therefore AB$ is tangent to $\odot O$.	6. § 277.

NOTE. Proving that any point D in AB except C lies outside the circle proves that every point in AB except C lies outside the circle. Why?

299. Tangent and Secant from an External Point

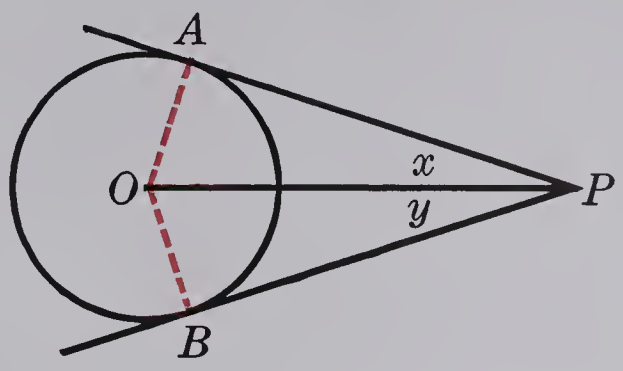
A *tangent from an external point* to a circle is the line segment between the external point and the point of contact. In the figure of Theorem 45, the line segment AC is the tangent from point A to the circle O. A *secant from an external point* to a circle is the line segment between the external point and the more remote point of intersection.

EXERCISE

Prove: If a radius is perpendicular to a tangent to a circle, it passes through the point of contact.

★Theorem 46

300. The tangents to a circle from a point are equal and make equal angles with the line joining the point to the center of the circle.



Given PA and PB tangents to $\odot O$, and PO joining the point to the center O.

To prove that $PA = PB$ and that $\angle x = \angle y$.

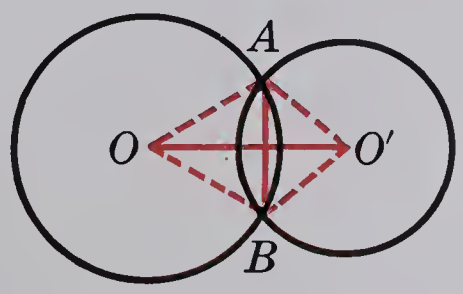
Planning the Proof: 1. You can prove (a) line segments = by §§ 97, 157, 217, 281 a, 287, 290, 293, 294; (b) $\measuredangle$s = by §§ 50, 98, 158, 219, 284

2. You should use (a) § 97 a; (b) § 98 b.

Proof: (The proof is left to the student.)

Theorem 47

301. If two circles intersect, the line joining their centers is the perpendicular bisector of the common chord.



(The demonstration is left to the student.)

SUGGESTION. Use § 140.

EXERCISES

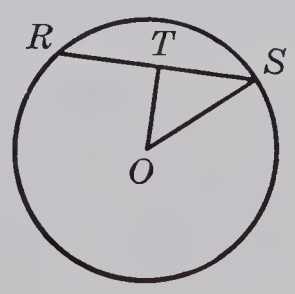
A

1. In the circle at the right $OT \perp RS$.

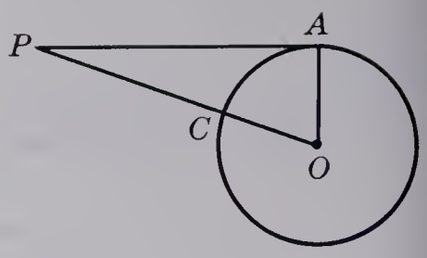
a. Find RS when $OT = 5$ and $SO = 7$.

b. Find OS when $OT = 12$ and $RS = 20$.

c. Find RS when $\angle TOS = 45°$ and $OT = 6$.



2. PA is a tangent from point P to $\odot O$, and PO is a straight line segment.

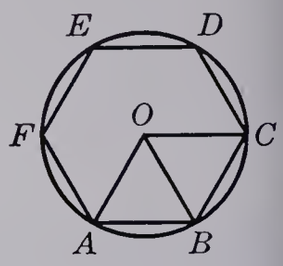


a. Find PA when AO is 5 and PO is 13.

b. Find AO when PA is 30 and PO is 34.

c. Find PO if AO is 4.2 and PA is 5.6.

3. $ABCDEF$ is a regular polygon inscribed in a circle whose radius is 10 inches.



a. How large is $\angle B$?

b. How many degrees are there in $\angle AOB$?

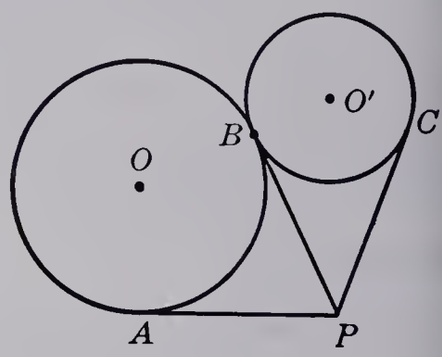
c. How many degrees are there in $\angle ABO$?

d. How long is AB?

4. In a circle the distances from the center to two equal chords are represented by $x^2 + 5x$ and $2x + 10$. Find these distances.

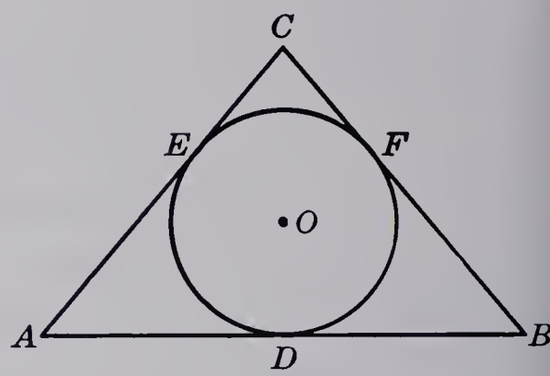
5. AB is a 12-inch chord in a circle whose diameter is 20 inches. How far is the chord from the center of the circle?

6. A pentagon is circumscribed about a circle whose radius is 8 inches. Find the area of the polygon if its perimeter is 62 inches.



7. In this figure PA and PB are tangents to $\odot O$ and PB and PC are tangents to $\odot O'$. If $PA = 14''$, find PB and PC.

8. Rectangle $ABCD$ is inscribed in $\odot O$. Prove that the sides AB and CD are equidistant from O.

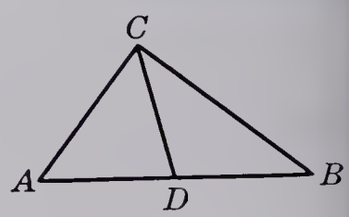


9. *Given* $\triangle ABC$ with sides AB, BC, and AC tangent to $\odot O$ at points D, F, and E respectively; $AC = BC$. *Prove* $AD = DB$.

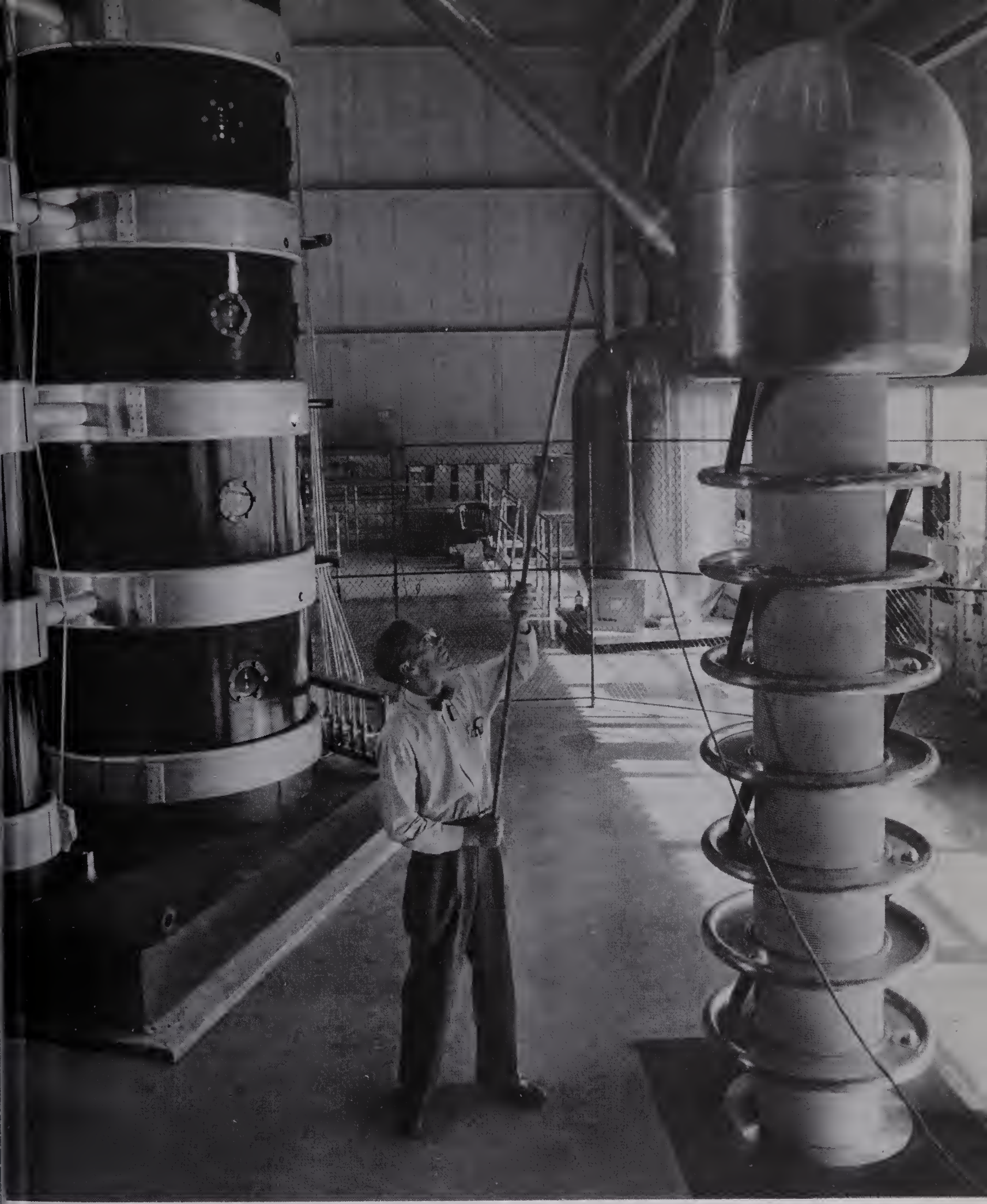
B

10. Two tangents to a circle meet at an angle of 60°. If the radius of the circle is 8 inches, find the length of the tangents.

11. A chord 10 in. long is 1 in. from the midpoint of its arc. Find the radius of the circle.



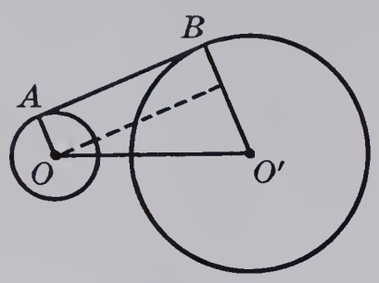
12. In $\triangle ABC$, $AB = 51''$, $AC = 24''$, and $BC = 45''$. Find the length of the median CD.



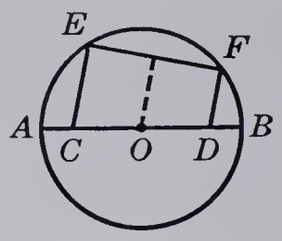
United States Atomic Energy Commission

This picture shows the top part of the famous Van de Graaff atom-smasher at Oak Ridge National Laboratory. A complex device occupying three floors, it helps scientists learn more about the atom. It is interesting to notice the geometric forms used in technological equipment

13. AB is tangent to $\odot O$ and $\odot O'$ at A and B respectively. OA is 6 inches, $O'B$ is 16 inches, and OO' is 26 inches. Find the length of AB.



Ex. 13



Ex. 14

14. In the circle O, AB is a diameter and EF is a chord. $CE \perp EF$ and $DF \perp EF$. Prove that $AC = DB$.

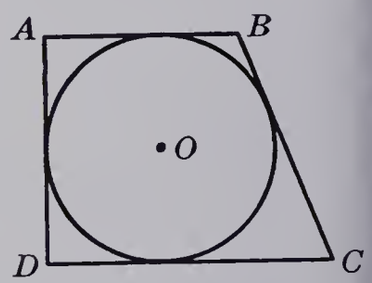
C

15. The foreman in a street-railway shop discovers that a set of wheels has a flat surface 1 inch long. What makes the wheels flat? If the diameter of the wheels is 32 inches, how much of each wheel must be ground off to make the wheels round?

16. A quadrilateral $ABCD$ is formed by the diameter CD of a circle, two tangents at C and D respectively, and another tangent which meets the other tangents at B and A respectively. Prove that the area of $ABCD$ is equal to half the product of the opposite sides AB and CD.

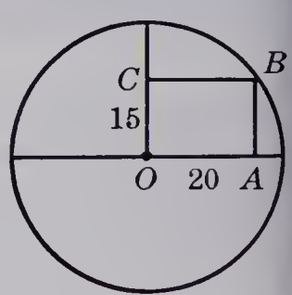
17. The diameter AB of a circle is 24 inches. Chord BC of the circle forms an angle of 30° with AB. Find the distance of BC from the center of the circle.

18. The sides of polygon $ABCD$ are tangent to $\odot O$. $AB = 14$, $DC = 20$, and $AD = 16$. Find BC.



19. Two chords of a 40-inch circle are perpendicular to each other. One chord is 32 inches long and the other is 24 inches long. How far is their intersection from the center of the circle?

20. In the figure $OABC$ is a rectangle with vertex O the center of a circle and vertex B on the circle. If $OC = 15$ and $OA = 20$, find the diameter of the circle.



21. A circle is circumscribed about triangle ABC. If AC is 60 inches long and 16 inches from the center of the circle, find the distances from the center to AB and BC when $AB = BC$.

EXERCISES — Proofs

A

1. *Given* chord AB and tangents PA and PB to $\odot O$ (Fig. 1).
Prove that $\angle PAB = \angle PBA$.

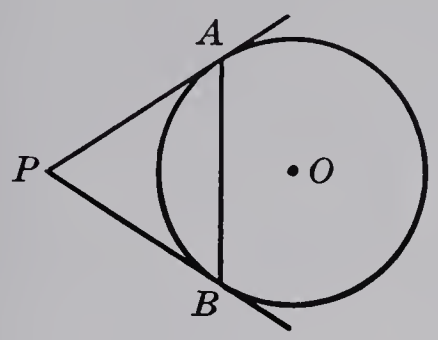


Fig. 1

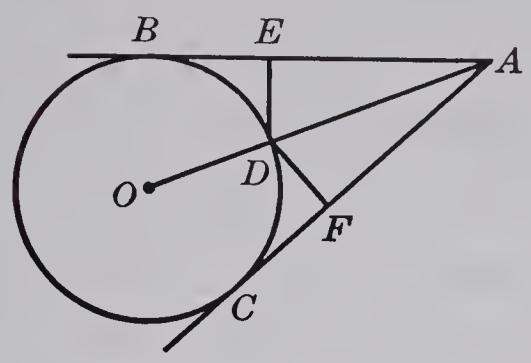


Fig. 2

2. *Prove:* If a diameter of a circle bisects one of two parallel chords which are not diameters, it bisects the other.

3. *Given* tangents AB and AC to $\odot O$ and AO intersecting the circle in D (Fig. 2).
Prove that D is equidistant from AB and AC.

B

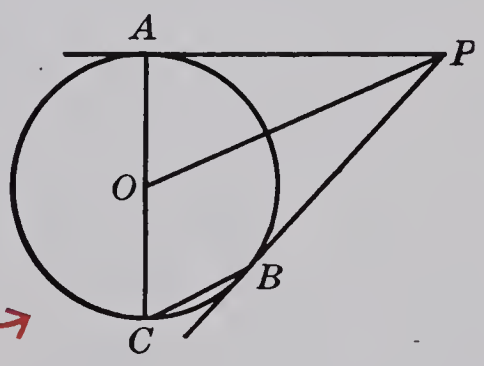
4. $\triangle ABC$ is inscribed in a circle; $\angle A = \angle C$. *Prove:* $\overset{\frown}{AB} = \overset{\frown}{BC}$.

5. *Prove:* If a point on a circle is equidistant from two radii, it bisects the arc included by the radii.

6. Tangents AB and AC are drawn to a circle O and BC is a chord. Prove that $\angle A = 2\ \angle OBC$.

C

7. *Prove:* If two tangents are drawn to a circle from a point without, the triangle formed by these tangents and a tangent to the arc included by them has a perimeter equal to the sum of the tangents.



8. *Given* $\odot O$, tangents PA and PB, diameter AC, and line segment OP.
Prove that $CB \parallel OP$.

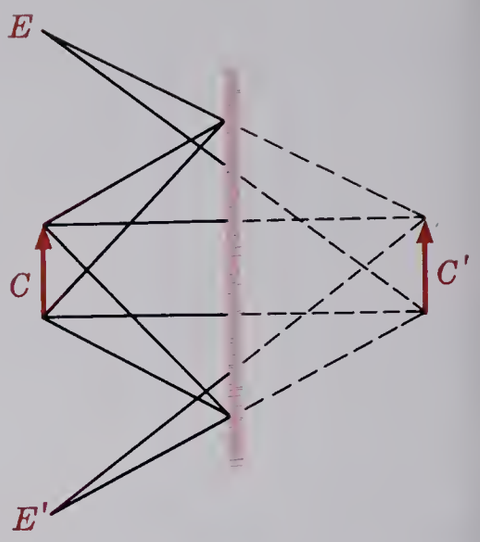
9. *Given* $\odot O$, with radius $OA \perp$ radius OB, C a point on OB, chord ACD, and the tangent at D intersecting OB produced in E.
Prove that $\triangle CDE$ is isosceles.

10. Point O is the center of a semicircle, radius $CO \perp$ diameter AB, and chord $RD \perp$ bisector of CO. Prove $\overset{\frown}{CD} = 2\ \overset{\frown}{RA}$.

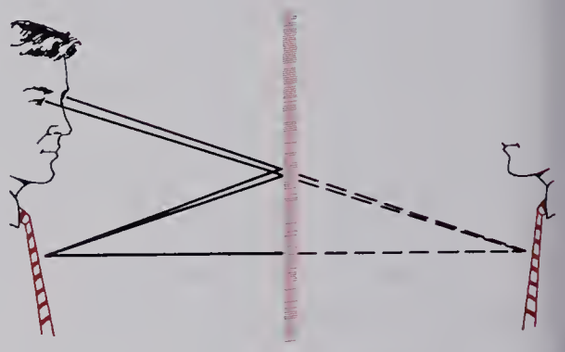
EXERCISES — Reflection of Mirrors

B

1. No matter where the light from a point strikes a plane mirror, it will, when reflected, be traveling in a line which passes through a point called its image. In the figure at the right, C represents an object, C' its image, and E and E' positions of the eye. Where is the image located in a plane mirror? When one sees an object in a plane mirror, how do the size and shape of the object compare with the size and shape of the image?



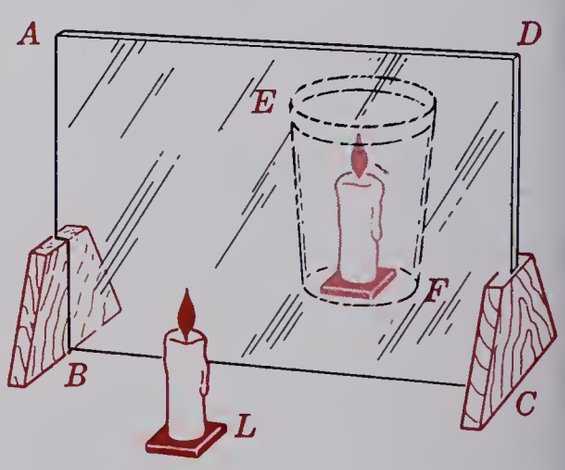
2. Study the diagram and explain why the image of the boy's tie seems to be behind the mirror. Explain why one can judge distances better by using both eyes instead of one.



3. Show that in order to see your full-length image in a vertical mirror when you are standing before it, the mirror must be at least half your height.

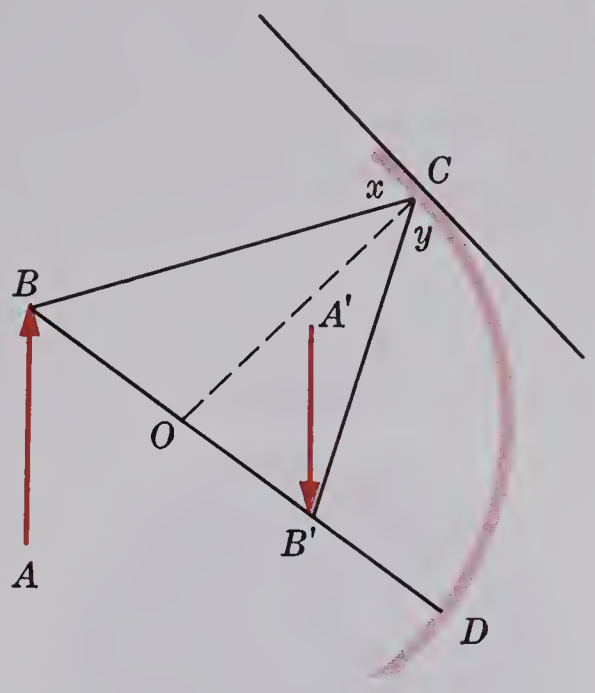
4. Look at your image in a mirror and raise your right hand. Which hand of the image appears to be lifted? Is the image direct or reversed?

5. The figure at the right shows a method of producing an optical illusion. $ABCD$ is a vertical plate glass and L is a lighted candle placed in front of it. EF is a glass of water behind the plate. In darkness it is possible to move the glass of water to a position where the candle appears to be burning in the water. What are the relative positions of the glass of water and the candle in reference to the plate glass when this illusion is produced?



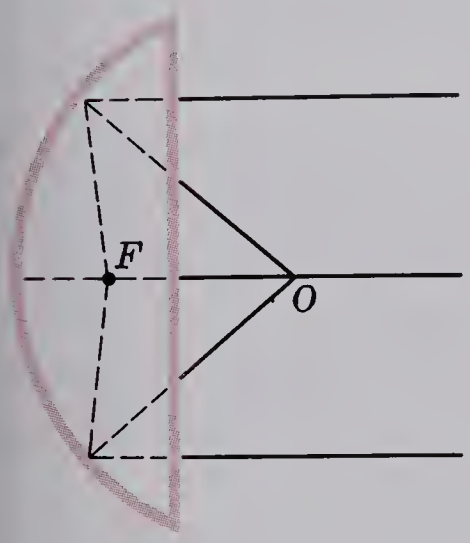
Name some other illusions which can be produced by reflections.

6. Concave mirrors are used to reflect light in a small beam. The diagram below represents a section of a concave spherical mirror, O being the center of the sphere and called the center of curvature. A ray of light BC from point B is reflected along CB' making equal angles x and y with the tangent at C. A ray of light from B through O is reflected back through DO. Why? The two reflected rays meet at B'. Why do BC and $B'C$ make equal angles with OC? B' is the image of B. The point A' is found in the same manner. Then $A'B'$ is the image of AB.

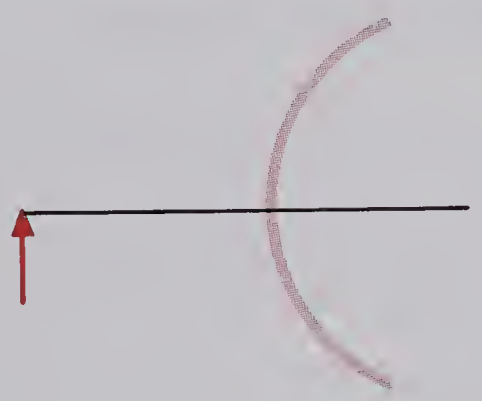


Following the plan outlined above, construct the image produced by a concave mirror when the object is beyond the center of curvature.

7. At the left below is a longitudinal section of a spherical mirror. Copy the figure and show that the rays of the light F are reflected in a beam of nearly parallel rays.



Ex. 7



Ex. 8

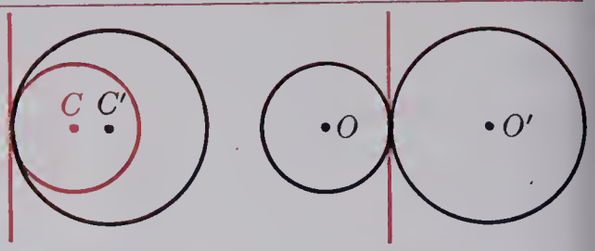
8. At the right above is the diagram of a convex spherical mirror. Construct the image of an object. Is the image upright or upside down?

9. Make a drawing showing whether or not a girl surrounded by three vertical mirrors can see the back of her head.

10. Why are the rear-vision mirrors on trucks convex instead of plane?

302. Tangent Circles

Two circles are *tangent circles* when they are tangent to the same line at the same point. The circles are *tangent internally* when they lie on the same side of the line, and *tangent externally* when they lie on opposite sides of the line. Circles C and C' are tangent internally, and circles O and O' are tangent externally.



Hamilton Watch Co.

The intermeshing gears of a watch or clock suggest tangent circles. Do they illustrate circles which are tangent internally or externally?

303. Common Tangent

A *common internal tangent* of two circles is a tangent to both circles which cuts the line segment joining the centers of the circles. A *common external tangent* of two circles is a tangent to both circles which does not cut the line segment joining the centers of the circles.

In Fig. 1 the circles have no common tangents. In Fig. 2, m and m' are common external tangents. In Fig. 3, l_1 and l_2 are common external tangents and l_3 and l_4 are common internal tangents.

Fig. 1

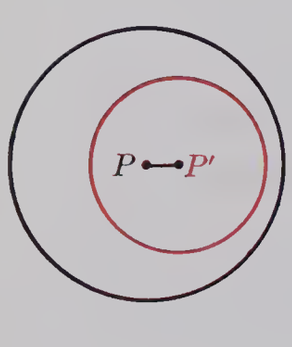


Fig. 2

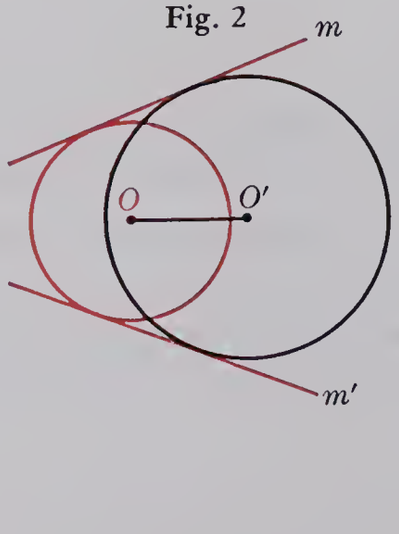
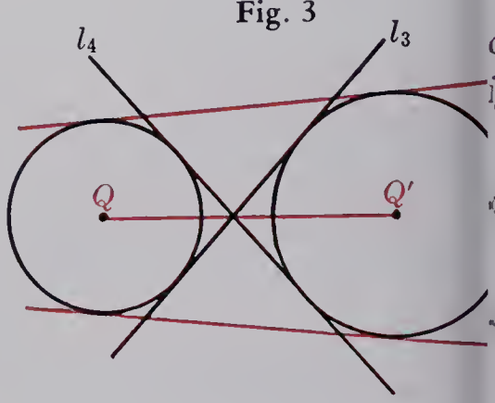


Fig. 3

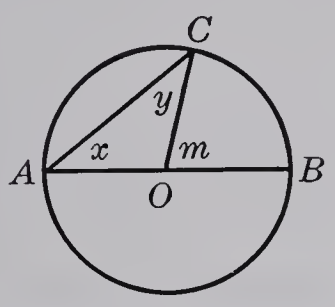


EXERCISES

A

1. Two circles may have from zero to four common tangents, depending upon their relative positions. How many common internal tangents and common external tangents do two circles have (**a**) if one circle is within the other? (**b**) if the circles are tangent internally? (**c**) if the circles intersect? (**d**) if the circles are tangent externally? (**e**) if one circle lies without the other?

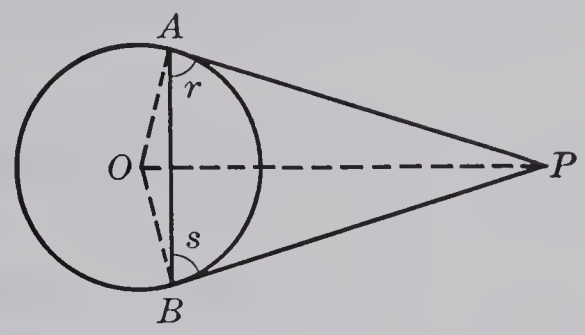
2. How many tangents may be drawn to a circle through a point within the circle? through a point on the circle? through a point without?



3. *Given* AB a diameter of $\odot O$, AC a chord, and OC a radius.

a. *Prove* that $\angle m = 2\ \angle x$.

b. *Prove* that the bisector of $\angle BOC \parallel AC$.



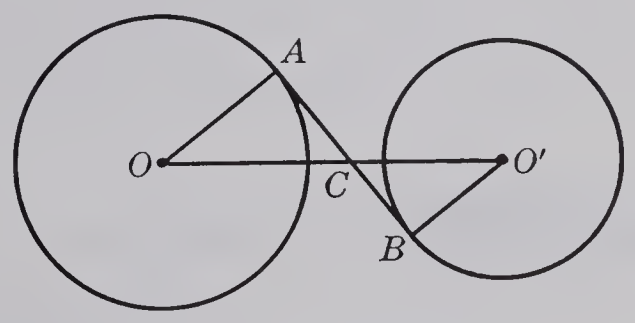
4. *Given* PA and PB tangents to $\odot O$, and AB a chord joining the points of contact.

a. *Prove* that $\angle r = \angle s$.

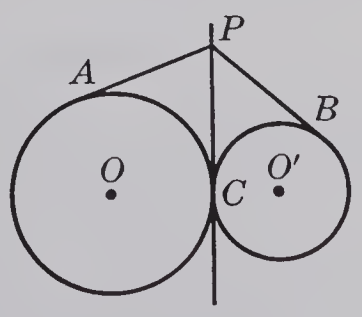
b. *Prove* that PO is the $\perp$ bisector of AB.

5. *Given* Ⓢ O and O', AB a common internal tangent, and OO' the line of centers.

Prove that $\angle O = \angle O'$.



6. If two circles are tangent externally, tangents drawn to the circles from any point in their common internal tangent are equal.

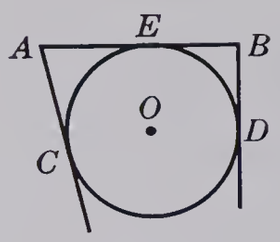


7. The tangents to a circle at the extremities of a diameter are parallel.

8. *Given:* AK and BL are tangents to a circle O at the ends of diameter AB, and KL is tangent to the circle at P.

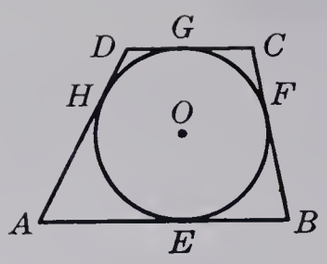
Prove that area of $BLKA = \frac{1}{2}\ AB \times KL$.

B



9. *Given* AB, AC, and BD tangents to $\odot O$.

Prove that $AB = AC + BD$.



10. *Given* AB, BC, CD, and DA tangents to $\odot O$.

Prove that $AB + DC = AD + BC$.

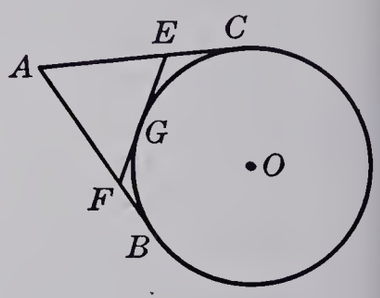
11. If two circles are tangent externally, their common internal tangent bisects either common external tangent.

C

12. If one circle is without another, their common internal tangents are equal.

13. If one circle is without another, their common external tangents are equal.

14. If the angle made by two tangents to a circle is 60°, the circle bisects the line segment joining the vertex of the angle and the center of the circle.



15. *Given* AB, AC, and EF tangents to $\odot O$; tangent EF intersects AC in E and AB in F.

Prove: As point G moves on the circle between B and C, the perimeter of $\triangle AFE$ is constant, i.e., it remains unchanged.

EXERCISES — Miscellaneous

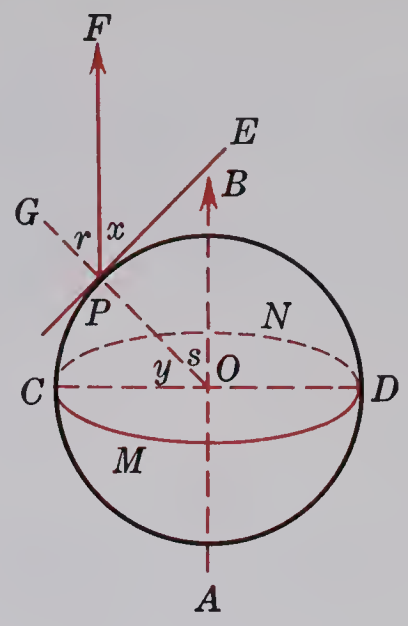
B and C

1. Two tangents, drawn from an external point to a circle, form an angle of 60°. The radius of the circle is 5 inches. Find the length of the line segment joining the external point to the center of the circle.

2. Prove that the perpendicular bisectors of the sides of an inscribed triangle pass through the same point.

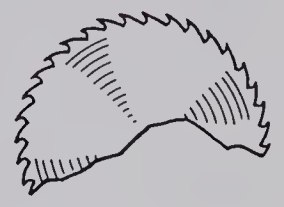
3. If two tangents drawn from an external point to a circle form an angle of 120°, the segment joining the point to the center of the circle is equal to the sum of the tangents.

4. In this figure, AB represents the axis of the earth, O the center, and $CMDN$ the equator. AB is perpendicular to CD. If P is any point on the earth's surface, OG, a line through P, is vertical to the earth's surface, and PE, a perpendicular to OG, is horizontal to the earth's surface. The axis AB points to the North Star. Because of the great distance to the North Star, the line PF, pointing to the star, may be considered parallel to AB.

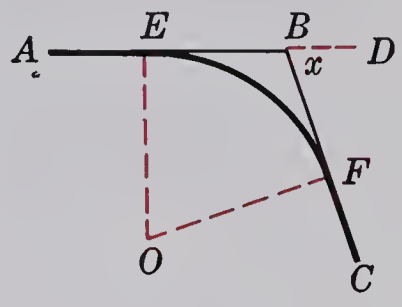


Prove that the angle of elevation of the North Star from P, or $\angle x$, is equal to the latitude of P, or $\angle y$.

5. This is a diagram of a piece of a broken circular saw. In order to purchase a new saw it is necessary to know the diameter of the original saw. Show how the diameter may be found.

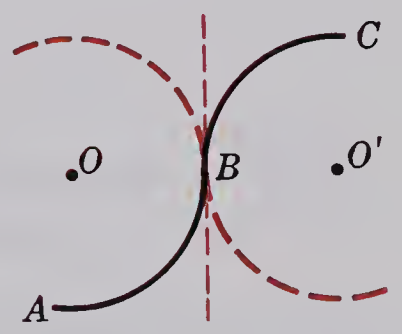


6. This figure shows the plan for laying out a simple railroad curve. The straight lines AB and BC are to be connected by an arc of a circle so that AB and BC are tangent to the circle. The lengths of BE and BF depend upon the surface of the ground and the speed at which trains are to be operated. What is the relation of EO to AD? of OF to BC? Does $BE = BF$? Prove that $\angle x = \angle O$. If trains are to be operated at a high speed, should the radius of the curve be large or small? Can $\angle ABC$ be acute?

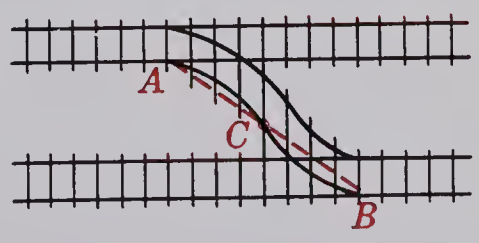


7. The arc AB of circle O and the arc BC of circle O' form a *compound curve.*

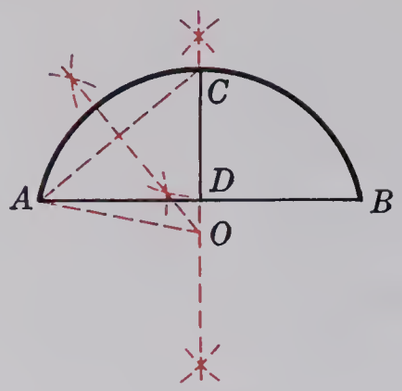
Are the circles O and O' tangent internally or externally? What do you know concerning the points O, B, and O'?

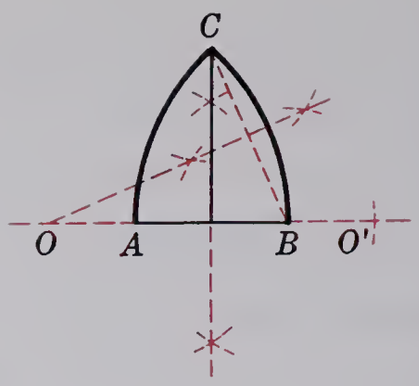


8. This diagram shows two parallel railroad tracks connected by a switch from A to B. Construct a plan for laying out the switch in the form of a compound curve composed of arcs having equal radii.



SUGGESTIONS. Draw AB. Find C, the midpoint of AB. The center of arc AC is the intersection of the $\perp$ bisector of AC and the $\perp$ to the track at A.





9. $\overset{\frown}{ACB}$ represents a *circular arch*. AB is the *span* and CD is the altitude of the arch.

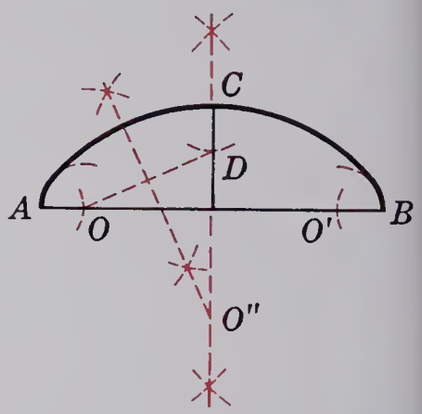
Draw the plan for a circular arch having a span of 60 feet and an altitude of 30 feet, using the scale of 1 inch = 12 feet.

10. ACB represents a *Gothic arch*.

Draw the plan for a Gothic arch having a span of 8 feet and an altitude of 12 feet, using a scale of 1 inch = 4 feet.

11. ACB represents a *basket-handle arch*.

Draw a plan for a basket-handle arch having a span of 20 feet and an altitude of 6 feet, using a scale of 1 inch = 6 feet. (Take AO a suitable length and make $CD = AO = O'B$.)

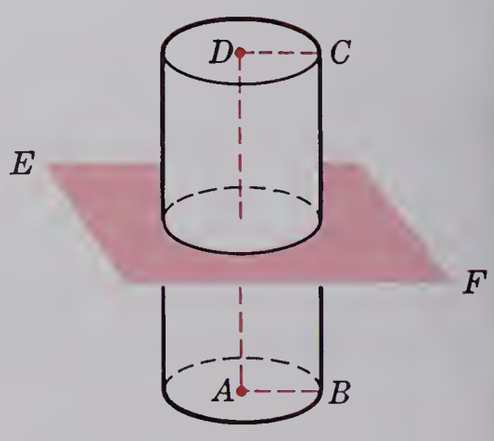


SPACE GEOMETRY

[Optional]

304. Right Circular Cylinder

The adjoining figure represents a *right circular cylinder*. The bases are circles lying in parallel planes. AD, a line segment joining the centers of the bases and perpendicular to them, is the axis of the cylinder.



EXERCISES

1. In the figure at the foot of page 312, if EF is a plane parallel to the base, what kind of figure is its intersection with the cylinder?

2. If plane EF is oblique to the base, what kind of figure is its intersection with the cylinder?

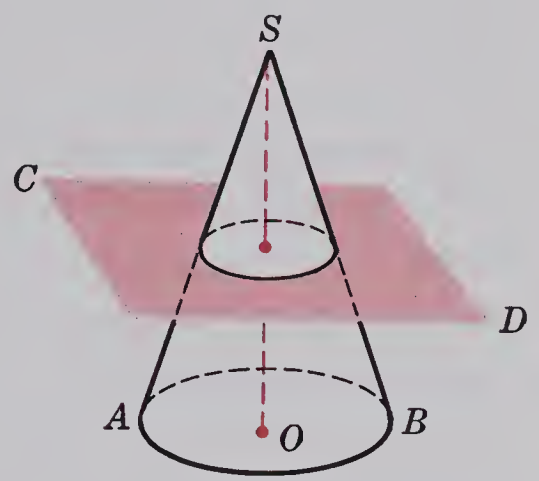
305. Right Circular Cone

The figure below represents a *right circular cone.* The base is a circle and the axis SO is perpendicular to the base at its center.

EXERCISES

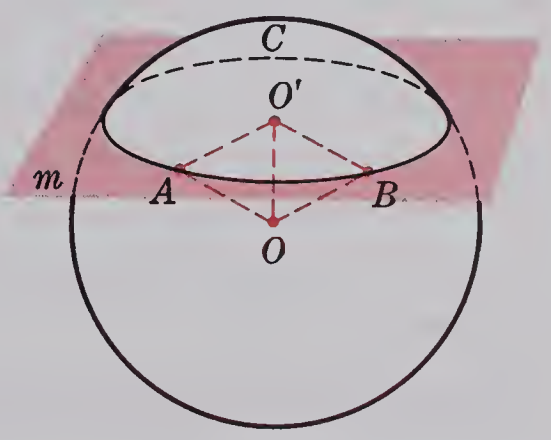
1. If CD is a plane parallel to the plane of the base, what kind of figure is its intersection with the cone?

2. If plane CD is not parallel to the plane of the base, what kind of figure is its intersection with the cone?



306. Spheres

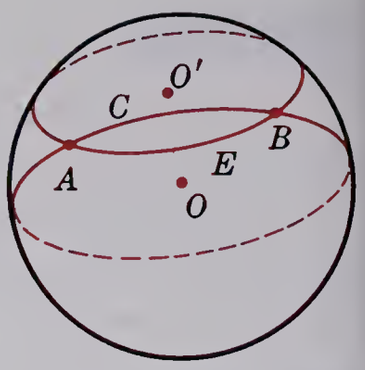
A sphere is a closed surface all points of which are equidistant from a fixed point within called the center. A *radius* of a sphere is a straight line segment joining the center to any point of the sphere. All radii of a sphere are equal.



EXERCISES

1. Prove that the intersection of a plane and a sphere is a circle.

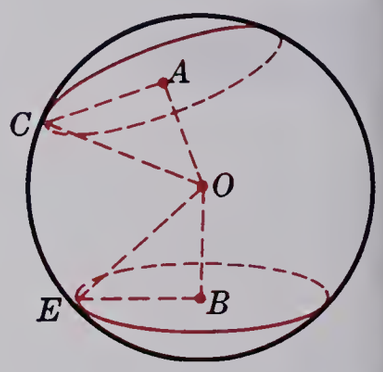
SUGGESTIONS. In the figure of § 306 take A and B any two points of the intersection and O the center of the sphere. Draw $OO' \perp m$. Draw OA, OB, $O'A$, and $O'B$. Then prove $O'A = O'B$.



2. In the figure of § 306, ABC is a *small circle* of the sphere. If a plane passes through the center of a sphere, its intersection with the sphere is called a *great circle* of the sphere. In the adjoining figure a small circle and a great circle are represented as passing through the same two points A and B. Is the arc ACB of the great circle or the arc AEB of the small circle the greater distance? Test your answer, using chalk and a baseball or a basketball.

3. If a plane is perpendicular to a radius at its point on the sphere, is it tangent to the sphere? To what theorem in plane geometry does this question correspond?

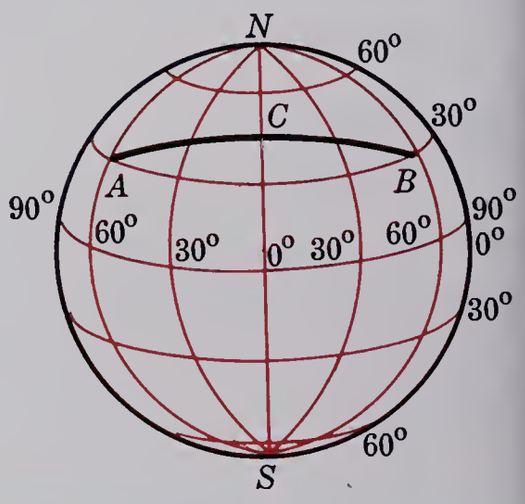
4. In how many points can a line intersect a sphere?



5. Using the figure at the right, prove that if the radii of two small circles of a sphere are equal, their planes are equidistant from the center of the sphere.

6. How many lines may be tangent to a sphere at a point on the sphere?

7. How many planes may be tangent to a sphere at a point on the sphere?



8. The meridians of the earth are great circles passing through the north and south poles. What kind of circle of the earth is the equator? What kind of circles are the other circles of latitude? ACB represents an airplane course between two places having a latitude of 30°. Why does the course not follow the circle of latitude through the places?

307. Assuming the Converse in Everyday Reasoning [Optional]

A common error in everyday reasoning is that, knowing a statement is true, we assume that the converse is true. As you know, the converse of a true statement is not always true.

EXERCISES

Optional

In each of the exercises 1–5 are two statements, each being the converse of the other. Tell which are true statements and which are false.

1. (a) If an angle is acute, it is less than a right angle.
(b) If an angle is less than a right angle, it is acute.

2. (a) If a man has a son, the man is a father.
(b) If a man is a father, he has a son.

3. (a) If a girl has pneumonia, her temperature is above normal.
(b) If a girl's temperature is above normal, she has pneumonia.

4. (a) The vertex of an isosceles triangle is equidistant from the extremities of the base.
(b) A point that is equidistant from the extremities of the base of an isosceles triangle is the vertex of the triangle.

5. (a) A student who attends school regularly gets good grades.
(b) A student who gets good grades attends school regularly.

Discuss the reasoning used in each of the following exercises:

6. Dick knew that an automobile would not run without gasoline. One morning when his automobile would not start, he said that the gas tank was empty.

7. Lucy knew that water boils at approximately 212°. When she and her family were boiling eggs on Pikes Peak, she said, "The water is boiling, so it is 212° Fahrenheit."

8. At Lincoln High School all juniors are required to take driving lessons. Frank Ross is a junior for he takes driving lessons.

REVIEW QUESTIONS — Chapter 9

1. What lines form a central angle? an inscribed angle?

2. How many chords may be drawn from a point on a circle? How many diameters?

3. How many tangents may be drawn to a circle from an external point? How many secants?

4. How many tangents can be drawn to a circle at a point on the circle? from a point inside the circle?

5. The radius of an 8-inch circle is __?__ inches.

6. What is the greatest chord in a circle?

7. An arc has how many chords? A chord has how many arcs?

8. Through what point does the perpendicular bisector of a chord pass?

9. What line segments in a circle are always equal? What line segments in a circle are not always equal?

10. What is the name of a line that has one point, and only one, in common with a circle? that has two points in common with the circle?

11. Where is the vertex of an inscribed angle?

12. What is the name of the point where a tangent touches the circle?

13. What is the distance from the center of a circle to one of its tangents?

14. How long is the chord which is perpendicular to a tangent to a circle having a radius equal to 6 inches?

15. If a chord were extended at either end or both ends, what would it become?

16. Are two circles equal if they have equal radii? equal diameters? equal chords?

17. If a diameter bisects a chord, is it perpendicular to the chord?

18. What is the difference between a common internal tangent and a common external tangent?

19. Complete: A polygon is __?__ in a circle when its vertices lie on the circle.

SUMMARY of Principal Bases for Proof

308. In a circle or in equal circles two line segments are equal

a. If they are radii.
b. If they are chords with equal arcs.
c. If they are chords equidistant from the center.
d. If they are perpendiculars from the center to equal chords.

309. Two line segments are equal

a. If they are segments of a chord which are made by a line through the center of a circle perpendicular to the chord.
b. If they are tangents to a circle from an external point.

310. Two angles of a circle or of equal circles are equal

a. If they are central angles with equal arcs.

311. Two lines are perpendicular

a. If one is a chord that does not pass through the center of a circle and the other is a line through the center bisecting the chord.

b. If one is a tangent to a circle and the other is a radius drawn to the point of contact.

312. In a circle or in equal circles two arcs are equal

a. If they are arcs of equal central angles.

b. If they are arcs of equal chords.

313. In a circle two arcs are equal

If they are parts of an arc which are made by a line through the center of the circle perpendicular to the chord of the arc.

WORD LIST

Can you spell and use each of the following words?

alternate	decagon	interior	rectangle
altitude	demonstration	isosceles	regular
chord	diagonal	median	rhombus
circumference	equiangular	minor	secant
circumscribed	equilateral	octagon	semicircle
contact	exterior	parallel	square
converse	hexagon	parallelogram	tangency
convex	hypotenuse	perimeter	tangent
corollary	inscribed	quadrant	transversal
corresponding	intercept	quadrilateral	trapezoid

TEST 21 — Completing Statements

[7 Minutes]

On your paper write one word, and only one, for each blank to make the following statements true:

1. If a line is perpendicular to a radius at its point on the circle, the line is _ _?_ _ _ _?_ _ the circle.

2. A chord passing through the center of a circle is a _ _?_ _.

3. The perpendicular bisector of a chord passes through __?__ __?__ __?__ the circle.

4. If a chord bisects a circle, it is a __?__.

5. In a circle or in equal circles chords equidistant from the center are __?__.

6. Tangents to a circle from __?__ __?__ __?__ are equal.

7. If two circles intersect, the line of centers is the perpendicular bisector of the __?__ __?__.

8. Two unequal circles with the same center have __?__ common tangents.

9. At a point on a circle there can be only __?__ tangent(s) to the circle.

10. A diameter of a circle is the longest __?__ of the circle.

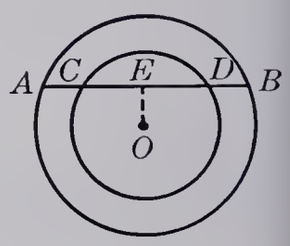
TEST 22 — Supplying Reasons

[18 Minutes]

Supply axioms, postulates, definitions, or theorems for the following statements:

1. *Given* two unequal Ⓢ having the same center O, and the chord AB of the greater ⊙ intersecting the smaller ⊙ at C and D.

Prove that $AC = DB$.

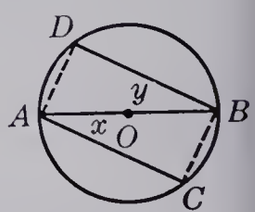


a. Draw $OE \perp AB$.
b. $AE = EB$.
c. $CE = ED$.
d. $AC = DB$.

2. *Given* ⊙ O with diameter AB, and chord AC = chord BD.

Prove that $AC \parallel BD$.

a. $\overset{\frown}{ACB} = \overset{\frown}{ADB}$.
b. $AC = BD$.
c. $\overset{\frown}{AC} = \overset{\frown}{BD}$.
d. $\overset{\frown}{BC} = \overset{\frown}{AD}$.
e. $BC = AD$.
f. $AB = AB$.
g. $\triangle ABC \cong \triangle ADB$.
h. $\angle x = \angle y$.
i. $AC \parallel BD$.



TEST 23 — Multiple-Choice Statements

[6 Minutes]

On your paper write the word or group of words in boldface type which makes each statement true:

1. The line segment joining any two points on a circle is a
diameter **chord** **secant.**

2. The greatest number of common tangents two circles may have is
one **two** **three** **four** **five.**

3. A central angle whose chord is equal to the radius contains
45° **60°** **75°** **90°.**

4. If tangents are drawn at the ends of a chord which has an arc of 100°, a
right **obtuse** **isosceles** **equilateral**
triangle is formed.

5. The sides of an inscribed angle are
two radii **two diameters** **two chords** **a chord and a tangent.**

6. In a circle two equal chords are
perpendicular **parallel** **equidistant from the center.**

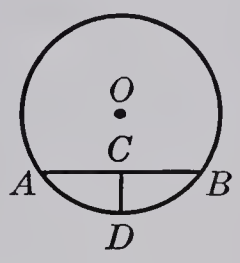
TEST 24 — Stating the *Given* and *To prove*

[10 Minutes]

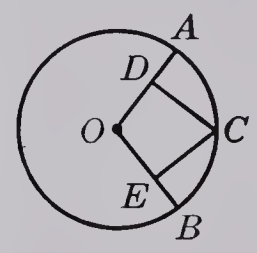
State the *Given* and *To prove* in the following:

1. The line joining the midpoints of a chord and its arc is perpendicular to the chord.

2. If perpendiculars are drawn to two radii from the midpoint of the arc intercepted by the radii, the perpendiculars are equal.



Ex. 1



Ex. 2

CHAPTER 10

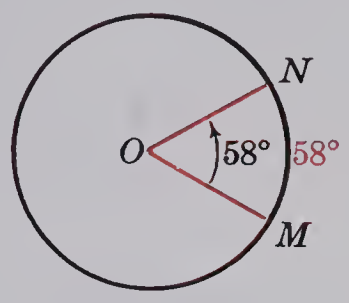
Measurement of Angles and Arcs

In this chapter
you will study the numerical relations between the arcs of a circle and the angles associated with a circle.

314. The Central Angle and Its Arc

Any quantity can be measured by finding how many times it contains some unit of measurement. As you know, the degree is a unit for measuring angles and has been defined as one ninetieth of a right angle.

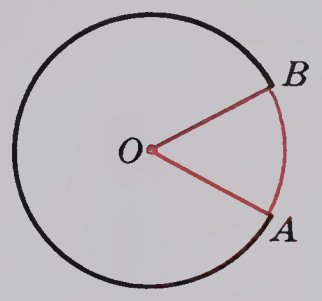
The arc which a central angle of one degree intercepts on a circle is called an *arc degree*. We shall use the arc degree as a unit for measuring arcs. The number of angle degrees about a point is 360, and the number of arc degrees in a circle is 360. Do you think that any central angle, such as $\angle MON$, contains the same number of angle degrees as its intercepted arc, $\overset{\frown}{MN}$, contains arc degrees?



We shall not attempt to prove the answer to this question. Instead, we shall agree that the following assumption, which is fundamental in the work of this chapter, is true.

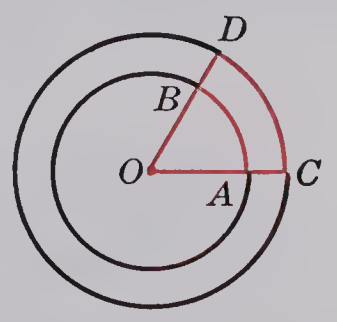
315. Assumption 46. A central angle is equal in degrees to (or has the same number of degrees as) its intercepted arc.

This means that the number of angle degrees in $\angle O$ is equal to the number of arc degrees in $\overset{\frown}{AB}$. Thus if $\angle O$ contains 50°, $\overset{\frown}{AB}$ contains 50°. The relation between $\angle O$ and $\overset{\frown}{AB}$ may be written $\angle O =$ (in degrees) $\overset{\frown}{AB}$, or $\angle O \mathrel{\overset{\circ}{=}} \overset{\frown}{AB}$. We sometimes say that $\angle O$ is measured by $\overset{\frown}{AB}$. In reading the expression $\angle O \mathrel{\overset{\circ}{=}} \overset{\frown}{AB}$, we should be careful not to say "Angle O equals arc AB," because an angle does not equal an arc. The symbol $\mathrel{\overset{\circ}{=}}$ never means "is congruent to."

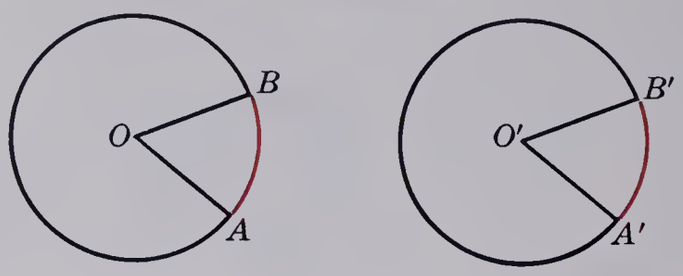


316. Comparison of Arcs

In this figure the two circles are unequal and have the same center, O. Since each of the arcs AB and CD contains the same number of arc degrees as $\angle O$ contains angle degrees (§ 315), the two arcs have the same number of arc degrees ($\overset{\frown}{AB} \mathrel{\overset{\circ}{=}} \overset{\frown}{CD}$). But the two arcs are unequal. Thus two arcs in different circles may have the same number of arc degrees and be unequal. Now we are ready to prove the following theorem.



317. Theorem. In a circle or in equal circles two arcs which have the same number of arc degrees are equal.



SUGGESTIONS. $\overset{\frown}{AB} \overset{\circ}{=} \overset{\frown}{A'B'}$. Why? $\angle O \overset{\circ}{=} \overset{\frown}{AB}$ and $\angle O' \overset{\circ}{=} \overset{\frown}{A'B'}$. Then $\angle O = \angle O'$ and $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

EXERCISES

A

1. If two arcs of a circle are equal, are their central angles equal?

2. If two arcs of unequal circles contain the same number of degrees, are their central angles equal?

3. Is it possible for two circles to have one point in common? two points in common? three points in common?

4. If a chord is equal to the radius of a circle, how many degrees are in its central angle?

5. Are the sides of an inscribed angle radii or chords?

6. As an arc of a circle increases in size, what change takes place in its central angle?

7. If an arc of a circle is halved, is its central angle halved?

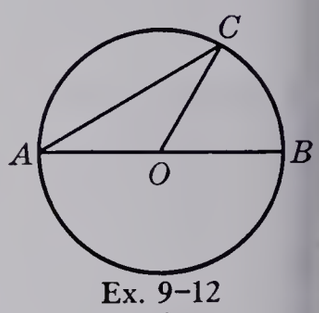
8. If a chord of a circle is doubled, is its central angle doubled?

9. In the figure, what arc contains the same number of degrees as $\angle BOC$?

10. What is the exterior angle of $\triangle AOC$?

11. Does $\angle A = \angle C$? Does $\angle A = \frac{1}{2} \angle BOC$?

12. Does $\angle BOC \overset{\circ}{=} \overset{\frown}{BC}$? Does $\angle A \overset{\circ}{=} \frac{1}{2} \overset{\frown}{BC}$?

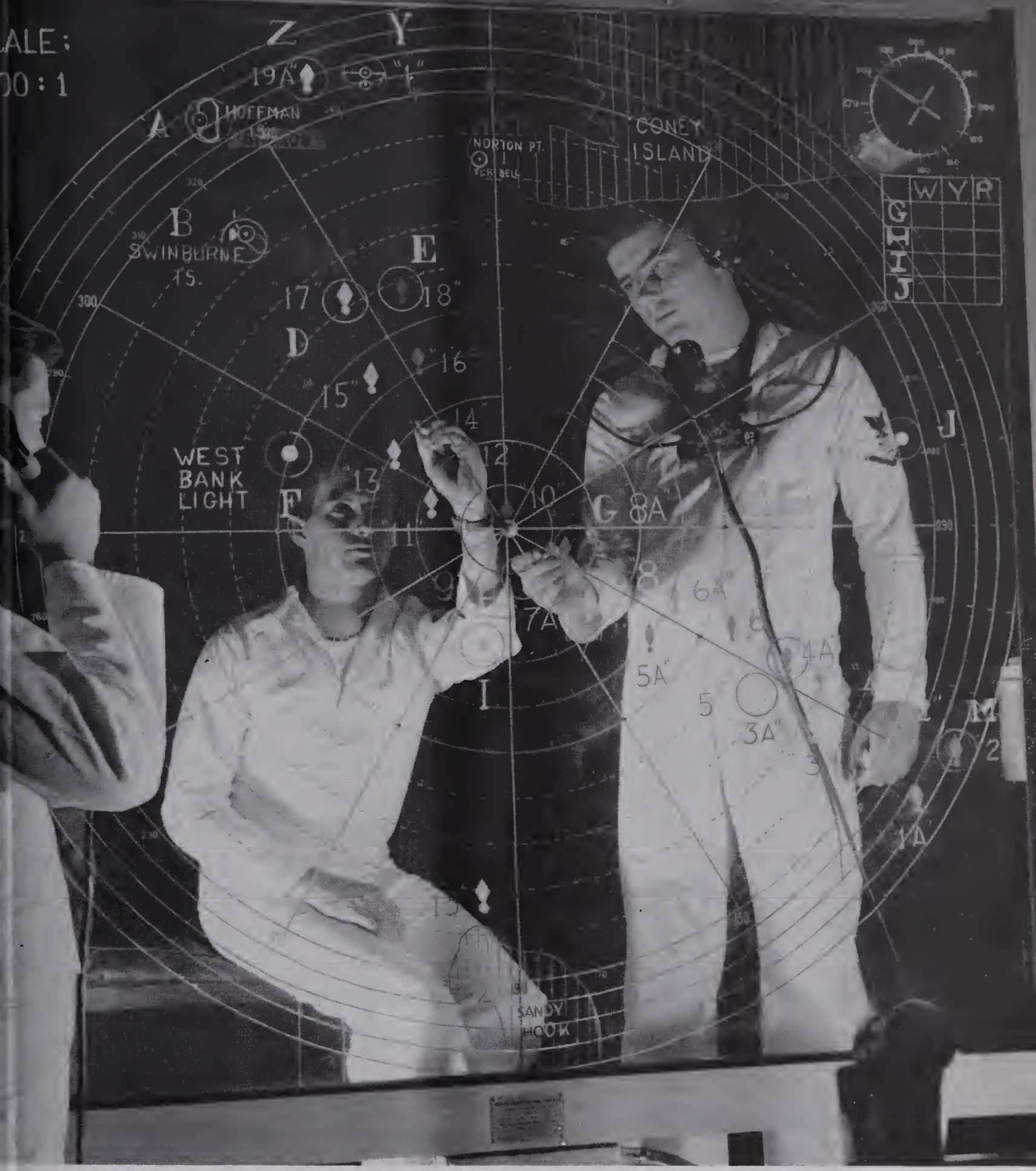


Ex. 9–12

B

13. Construct a quilt pattern using the circle as the basis of design.

14. Construct a decorative design for the back of a chair.

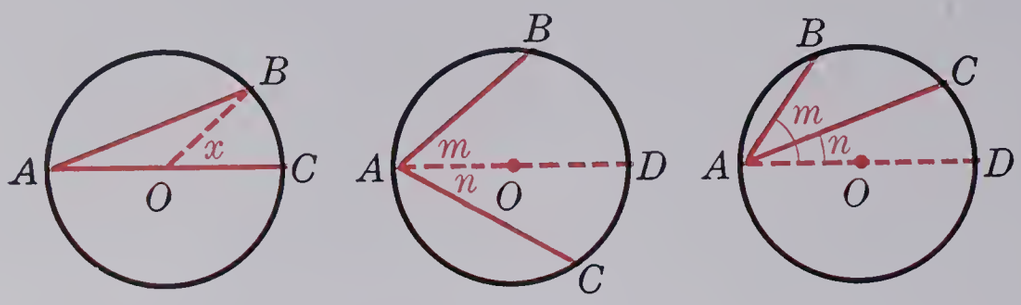


Official United States Navy photograph

These men are working on a plotting board during radar navigation of a channel as the ship proceeds to its mooring. The buoy symbols on the board enable the men to show the progress of the ship in relation to buoys and other shipping in the channel. Do you think some knowledge of geometry might be useful in this work?

★★ Theorem 48

318. An inscribed angle is equal in degrees to one half its intercepted arc.



Case I Case II Case III

Given $\angle CAB$ inscribed in $\odot O$.

To prove that $\angle CAB \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CB}$.

Planning the Proof: 1. We can compare $\measuredangle$s with their arcs by § 315.
2. We shall use § 315.

Case I. If one side of the angle is a diameter.

Proof

STATEMENTS	REASONS
1. Draw OB.	1. Why possible?
2. $OB = OA$.	2. Why?
3. $\therefore \angle A = \angle B$.	3. Why?
4. $\angle A + \angle B = \angle x$.	4. § 123.
5. $2\angle A = \angle x$.	5. Why?
6. $\angle x \stackrel{\circ}{=} \overset{\frown}{CB}$.	6. Why?
7. $\therefore 2\angle A \stackrel{\circ}{=} \overset{\frown}{CB}$.	7. Why?
8. $\therefore \angle A \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CB}$.	8. Why?

Case II. If the center of the circle lies within the angle.

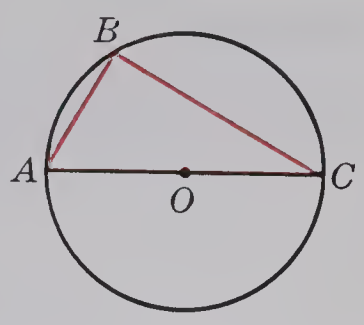
SUGGESTIONS. Draw the diameter AD. Apply Case I to $\angle m$ and to $\angle n$. Use the addition axiom.

Case III. If the center of the circle lies without the angle.

SUGGESTIONS. Draw the diameter AD. Apply Case I to $\angle m$ and to $\angle n$. Use the subtraction axiom.

As you study the relations between angles and their arcs, you should note the sequence of the theorems. Note that Asmt. 46 is used to prove Theorem 48, that Theorem 48 is used to prove Theorems 49 and 50, and that Theorems 48 and 49 are used to prove Theorem 51.

319. Corollary I. An angle inscribed in a semicircle is a right angle.



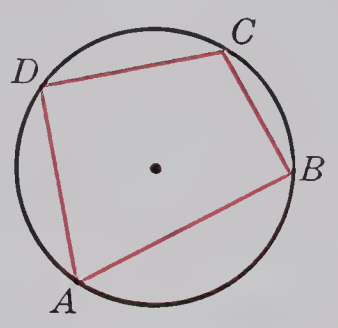
NOTE. This fact is said to have been discovered by Thales.

Given $\angle B$ inscribed in semicircle ABC.
To prove that $\angle B$ is a rt. $\angle$.

320. Corollary II. Inscribed angles which intercept the same arc are equal.

SUGGESTION. Draw two or more inscribed angles which intercept the same arc, and use § 318.

321. Corollary III. If a quadrilateral is inscribed in a circle, the opposite angles are supplementary.



SUGGESTIONS. $\angle A \doteq \frac{1}{2}$ which arc? $\angle C \doteq \frac{1}{2}$ which arc? $\angle A + \angle C \doteq \frac{1}{2}$ which arcs?

EXERCISES

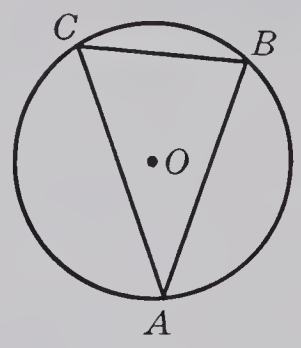
A

1. Find the number of degrees in each angle of $\triangle ABC$ if $\overset{\frown}{AC} = 140°$ and $\overset{\frown}{AB} = 100°$.

2. Find the number of degrees in each of the three arcs if $\angle A = 40°$ and $\angle C = 46°$.

3. Find the size of $\angle A$, $\angle B$, and $\angle C$ when $\overset{\frown}{AB} = 114°$, and $\overset{\frown}{AC} = 168°$.

4. Find the size of $\angle A$, $\angle B$, and $\angle C$ when $\overset{\frown}{AB} = \overset{\frown}{BC} = \frac{1}{2}\overset{\frown}{AC}$.



Ex. 1-4

5. *Prove:* A parallelogram inscribed in a circle is a rectangle.

6. Two sides of an inscribed triangle have arcs of 100° and 132°. How many degrees are there in each angle of the triangle?

B

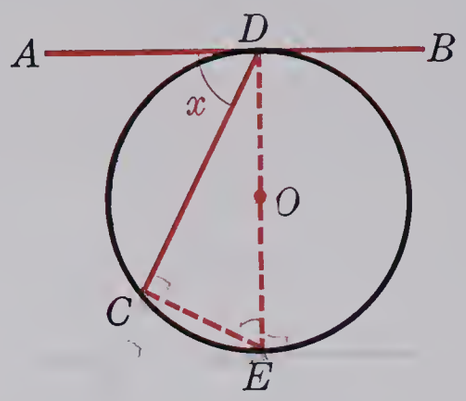
7. The diameter AB of $\odot O$ is 18 inches. AC and BC are chords of the circle. Find AC and BC if $\angle BOC$ is 60°.

8. Find the area of a parallelogram inscribed in a 34-inch circle if one of its sides is $\frac{8}{15}$ of another.

9. Prove that a right angle can be inscribed in a semicircle.

★Theorem 49

322. An angle formed by a tangent and a chord is equal in degrees to one half its intercepted arc.



Given $\angle ADC$ formed by tangent AB to $\odot O$ at D and chord CD.

To prove that $\angle x \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CD}$.

Planning the Proof: 1. We can compare angles with their arcs by §§ 315, 318.
2. We shall use § 318.

Proof

STATEMENTS	REASONS
1. Draw diameter DE and line CE.	1. Why possible?
2. $AB \perp DE$.	2. Why?
3. $\angle C$ is a rt. $\angle$.	3. Why?
4. $\therefore CE \perp CD$.	4. Why?
5. $\angle E = \angle x$.	5. § 127.
6. $\angle E \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CD}$.	6. Why?
7. $\therefore \angle x \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CD}$.	7. Why?

Use Statement 7 to prove that $\angle CDB \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{CED}$.

EXERCISES

A

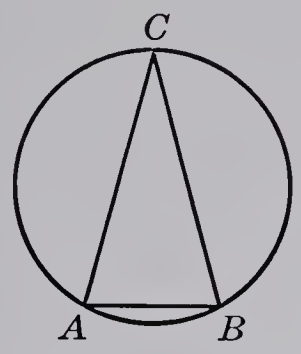
1. Using the figure above, find the number of degrees in $\angle x$ when $\overset{\frown}{CD} = 84°$; when $\overset{\frown}{CE} = 68°$.

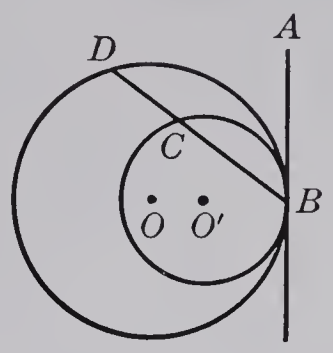
2. Using the figure above, find the number of degrees in $\overset{\frown}{CD}$ when $\angle x = 25°$; when $\angle E = 47°$.

3. The angle formed by a tangent and a chord contains 43°. How many degrees are in each arc of the circle?

4. If two angles of a triangle inscribed in a circle are 37° and 75° respectively, find the number of degrees in the three arcs of the circle.

5. A tangent to a circle at the midpoint of an arc is parallel to the chord of the arc.





6. *Given* $\triangle ABC$ inscribed in a circle with $AC = BC$ and $\overset{\frown}{BC} = 144°$.

Find $\measuredangle A$, B, and C.

7. *Given* Ⓢ O and O' tangent internally at B, BD a chord of $\odot O$.

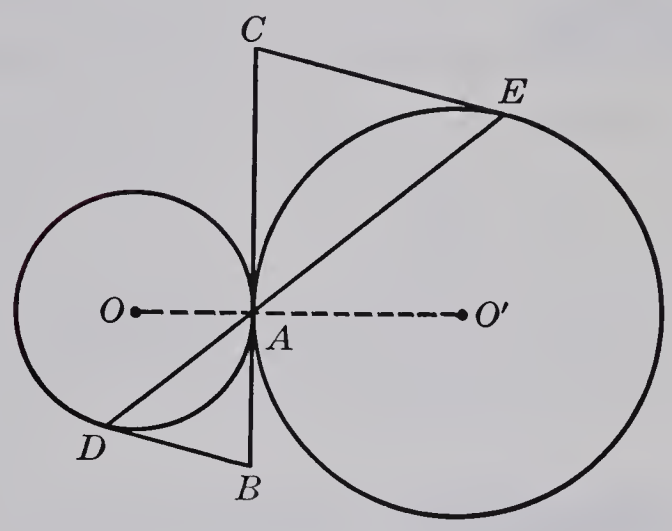
Prove that $\overset{\frown}{BD} \doteq \overset{\frown}{BC}$.

B

8. If two tangents drawn from a point to a circle form an angle of 60°, the tangents and the chord joining the points of contact form an equilateral triangle.

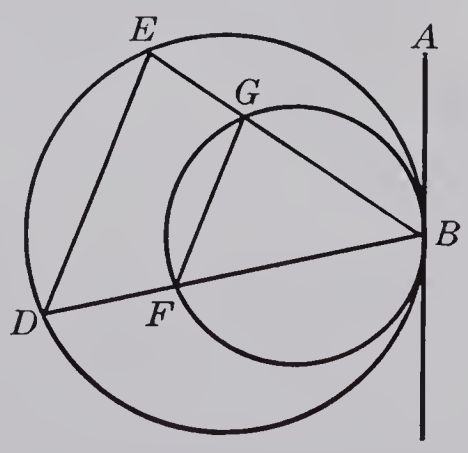
9. *Given* BC a common internal tangent to Ⓢ O and O', line DE passing through A the point of tangency of the two circles, and tangents DB and CE.

Prove $DB \parallel CE$.



C

10. If two circles are tangent externally and two lines are drawn through the point of contact and terminated by the circles, the chords joining the end points of these lines are parallel.



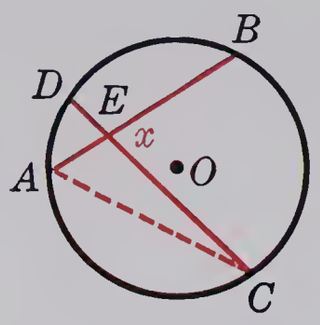
11. *Given* two circles tangent internally at B, and BE and BD chords of the larger circle.

Prove that $DE \parallel FG$.

12. In the inscribed quadrilateral $ABCD$, $AB = AD$ and $CB = CD$. Prove that AC is a diameter of the circle.

★Theorem 50

323. An angle formed by two chords intersecting within a circle is equal in degrees to one half the sum of the arcs intercepted by it and its vertical angle.



Given $\angle x$ formed by the chords AB and CD of $\odot O$.
To prove that $\angle x \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AD})$.

Planning the Proof: 1. We can compare angles with their arcs by §§ 315, 318, 322.
2. We shall use § 318.

Proof

STATEMENTS	REASONS
1. Draw AC.	1. Why possible?
2. $\angle x = \angle A + \angle C$.	2. § 123.
3. $\angle A \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{BC}$.	3. Why?
4. $\angle C \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{AD}$.	4. Why?
5. $\angle A + \angle C \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AD})$.	5. Why?
6. $\therefore \angle x \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AD})$.	6. Why?

EXERCISES — Angles and Arcs

A

Example. In the above figure, if $\angle x = 75°$ and $\overset{\frown}{AD} = 20°$, find $\overset{\frown}{BC}$.

Solution. From § 323, $\angle x \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AD})$.
Substituting, $75° = \frac{1}{2}(\overset{\frown}{BC} + 20°)$;
whence $150° = \overset{\frown}{BC} + 20°$,
and $\overset{\frown}{BC} = 130°$.

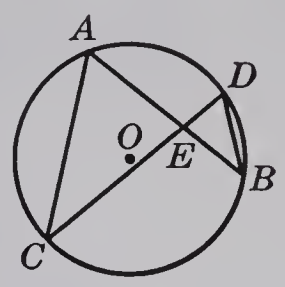
1. In the figure for Theorem 50, how many degrees are there in $\angle x$ if $\overset{\frown}{BC} = 150°$ and $\overset{\frown}{AD} = 64°$? if $\overset{\frown}{AD} = 79°$ and $\overset{\frown}{BC} = 141°$?

2. In the figure for Theorem 50, if $\angle x = 110°$ and $\overset{\frown}{BC} = 145°$, how many degrees are there in $\overset{\frown}{AD}$?

3. In the figure for Theorem 50, if $\overset{\frown}{AC} = 62°$ and $\overset{\frown}{DB} = 55°$, how many degrees are there in $\angle x$?

4. *Given* AB and CD chords of $\odot O$ intersecting at E.

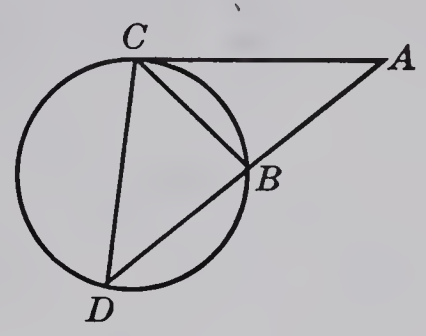
Prove that $\angle C = \angle B$ and that $\angle A = \angle D$.



5. If two chords of a circle are perpendicular to each other and two adjacent arcs contain 12° and 62° respectively, find the number of degrees in the other arcs.

6. *Given* $\odot CBD$, tangent AC, secant AD, $\overset{\frown}{BD} = 102°$, and $\angle D = 44°$.

Find $\angle A$; $\angle BCD$; $\angle ACB$.



7. The two diagonals drawn from one vertex of an inscribed regular pentagon trisect the angle from whose vertex they are drawn.

B

8. If a point is inside a circle, the lines joining it to the ends of a diameter form an obtuse angle.

9. A central angle and an inscribed angle intercept the same arc of the circle. Find the angles if their sum is 135°.

10. A rhombus inscribed in a circle is a square.

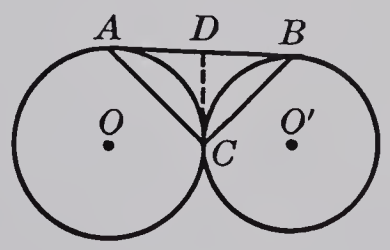
11. If one angle of an inscribed triangle formed by two chords and a diameter contains 30°, prove that the side opposite this angle is equal to a radius.

C

12. An exterior angle of an inscribed quadrilateral is equal to its opposite interior angle.

13. *Given* $\circledS O$ and O' tangent externally at C, AB a common external tangent, and chords AC and BC.

Prove that $\angle ACB$ is a rt. $\angle$.



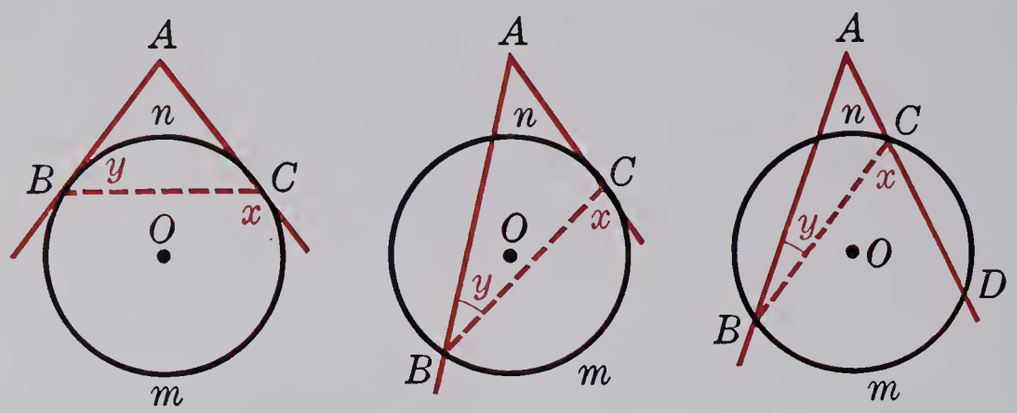
SUGGESTION. Draw DC, the common internal tangent.

14. *Given* $\triangle ABC$ inscribed in $\odot O$, chord $AD \perp BC$, and chord $BE \perp AC$.

Prove $\overset{\frown}{DC} = \overset{\frown}{CE}$.

★ Theorem 51

324. An angle formed by two tangents, a tangent and a secant, or two secants, intersecting outside a circle, is equal in degrees to one half the difference of the intercepted arcs.



Given the two lines AB and AC intercepting the two arcs m and n on $\odot O$.

To prove that $\angle A \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{m} - \overset{\frown}{n})$.

Planning the Proof: 1. We can compare angles with their arcs by §§ 315, 318, 322, 323.
2. We shall use §§ 318, 322.

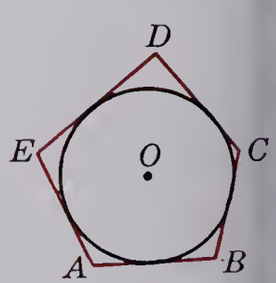
Proof

STATEMENTS	REASONS
1. Draw BC.	1. Why possible?
2. $\angle A + \angle y = \angle x$.	2. Why?
3. $\therefore \angle A = \angle x - \angle y$.	3. Why?
4. $\angle x \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{m}$.	4. Why?
5. $\angle y \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{n}$.	5. Why?
6. $\therefore \angle x - \angle y \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{m} - \overset{\frown}{n})$.	6. Why?
7. $\therefore \angle A \stackrel{\circ}{=} \frac{1}{2}(\overset{\frown}{m} - \overset{\frown}{n})$.	7. Why?

NOTE. Be careful in giving the reasons 4 and 5. They will differ according to the figure under consideration.

325. Circumscribed Polygon

A polygon is *circumscribed about a circle* if its sides are tangents to the circle. The circle is then said to be *inscribed in the polygon.* $ABCDE$ is a circumscribed polygon.



EXERCISES — Angles and Arcs

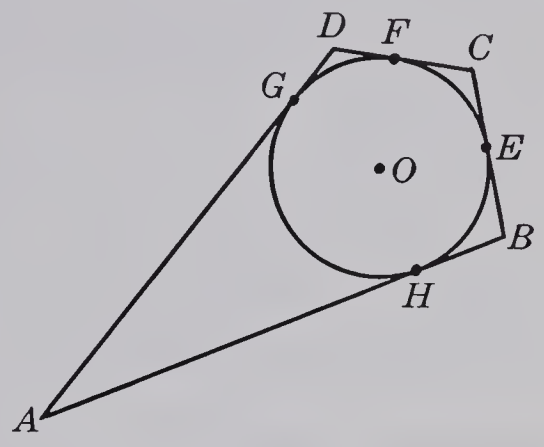
A

Example. In the figure for Theorem 51, if $\angle A = 30°$ and $\widehat{n} = 25°$, find the number of degrees in $\widehat{m}$.

Solution. From § 324, $\angle A \doteq \frac{1}{2}(\widehat{m} - \widehat{n})$.
Substituting, $30° = \frac{1}{2}(\widehat{m} - 25°)$;
whence $60° = \widehat{m} - 25°$, and $\widehat{m} = 85°$.

1. Two secants are drawn to a circle from an external point and intercept on the circle arcs of 112° and 54° respectively. How many degrees are there in the angle?

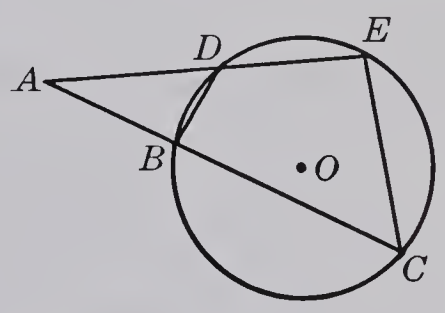
2. If a tangent and a secant to a circle form an angle of 63° and intercept one arc of 230°, find the number of degrees in the other intercepted arc.



3. The points of tangency of the circumscribed polygon $ABCD$ are E, F, G, and H. Find the number of degrees in $\angle A$, $\angle B$, $\angle C$, and $\angle D$ when $\widehat{EF} = 70°$, $\widehat{FG} = 60°$, and $\widehat{HE} = 80°$.

4. $BCED$ is an inscribed polygon whose sides CB and ED meet, when produced, in point A. If $\widehat{EC} = 100°$, $\angle A = 30°$, $\widehat{DE} = 70°$, find the number of degrees in:

a. $\widehat{BD}$ **b.** $\angle C$ **c.** $\widehat{BC}$ **d.** $\angle CBD$



B

5. Find the number of degrees in the arcs intercepted by two perpendicular tangents.

6. *Prove:* The bisector of the angle formed by a tangent and a chord drawn to the point of contact bisects the intercepted arc.

7. *Given* circles O and O' intersecting in points A and B, P a point free to move on major arc AB of $\odot O$, the common chord AB of both circles, secants PAC and PBD of circle O', and chords BC and DC.

Prove (informally) that as P moves on the major arc AB of $\odot O$ the sum of $\angle P$ and $\angle PCB$ is constant; that $\angle PBC$ is constant; that $\angle CBD$ is constant; and that CD is constant.

8. If AB is the common chord of two intersecting circles and BC and BD are two diameters, prove that CAD is a straight line.

C

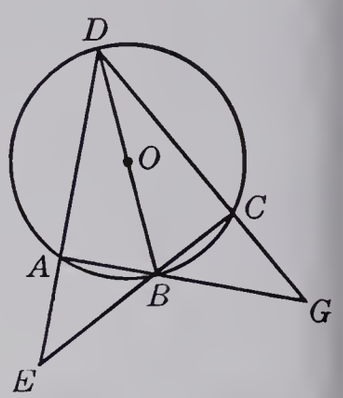
9. If two circles are tangent internally and the smaller circle passes through the center of the larger circle, any chord of the larger circle drawn to the point of tangency is bisected by the smaller circle.

10. If one leg of a right triangle is a diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other leg.

11. The vertex of an angle of 25° is without a circle and its sides intercept on the circle arcs whose sum is 152°. How many degrees are there in each arc?

12. BD is a diameter of $\odot O$; and ED, EC, GA, and GD are secants.

Find a relation between $\angle E$ and $\angle G$. Prove the relation you discover.



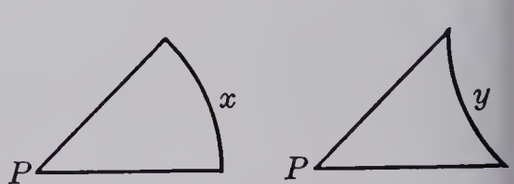
326. Principle of Continuity

In this chapter we have had one assumption and four theorems which compare angles with the arcs associated with them. Let us see how Assumption 46 and Theorems 48, 49, 50, and 51 can be combined in one general theorem, stated as follows:

General Theorem. The angle formed by two intersecting lines, each of which intersects or is tangent to a circle, is equal in degrees to one half the sum of the intercepted arcs.

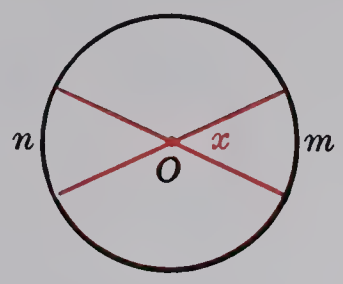
There are different ways of showing that this general theorem applies to all five cases.

Let us define positive and negative arcs in a sense suitable for our discussion. If from the vertex of an angle the intercepted arc appears convex, we call the arc positive; and if from the vertex of an angle the intercepted arc appears concave, we call the arc negative. Thus with respect to $\angle P$, arc x is positive and arc y is negative.



Now see how this general theorem applies to all five cases.

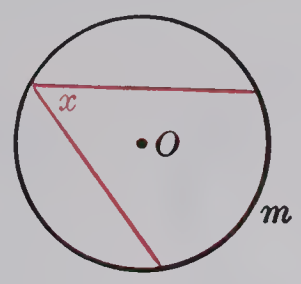
To § 315



$\overset{\frown}{m}$ positive
$\overset{\frown}{n}$ positive

$\overset{\frown}{m} = \overset{\frown}{n}$
$\angle x \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} + \overset{\frown}{n})$
$\overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} + \overset{\frown}{m}) \overset{\circ}{=} \overset{\frown}{m}$

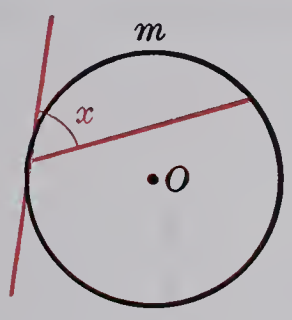
To § 318



$\overset{\frown}{m}$ positive
$\overset{\frown}{n} = 0$

$\angle x \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} + \overset{\frown}{n})$
$\overset{\circ}{=} \frac{1}{2}\overset{\frown}{m}$

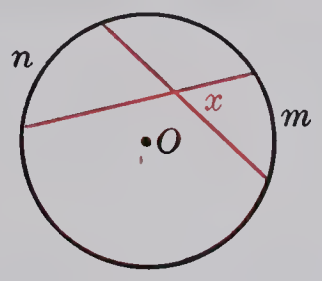
To § 322



$\overset{\frown}{m}$ positive
$\overset{\frown}{n} = 0$

$\angle x \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} + \overset{\frown}{n})$
$\overset{\circ}{=} \frac{1}{2}\overset{\frown}{m}$

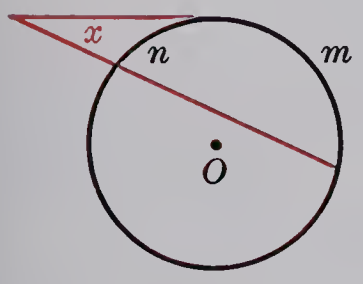
To § 323



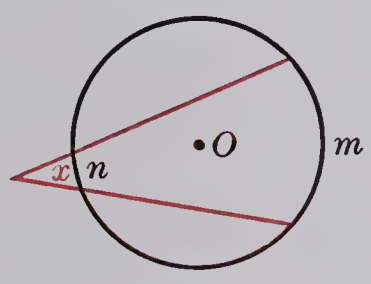
$\overset{\frown}{m}$ positive
$\overset{\frown}{n}$ positive

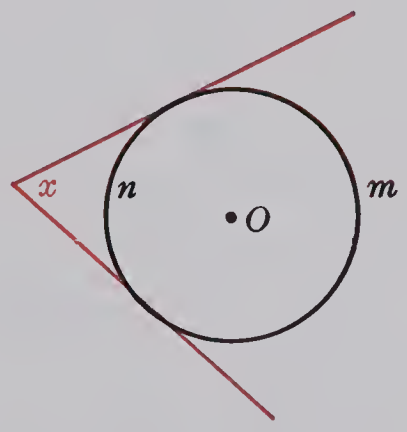
$\angle x \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} + \overset{\frown}{n})$

To § 324



$\overset{\frown}{m}$ positive
$\overset{\frown}{n}$ negative

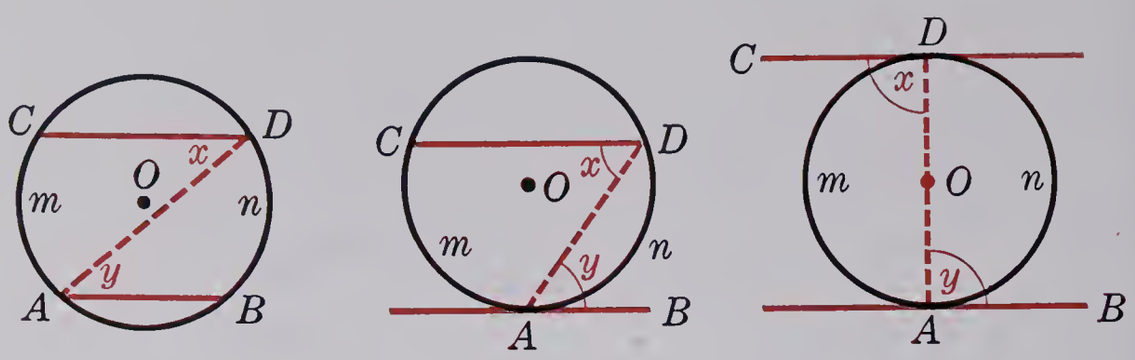




$\angle x \overset{\circ}{=} \frac{1}{2}[\overset{\frown}{m} + (-\overset{\frown}{n})]$
$\overset{\circ}{=} \frac{1}{2}(\overset{\frown}{m} - \overset{\frown}{n})$

Theorem 52

327. Parallel lines intercept equal arcs on a circle.



Given $\odot O$ and arcs m and n intercepted on the circle by the parallel lines AB and CD.

To prove $\overset{\frown}{m} = \overset{\frown}{n}$.

Planning the Proof: 1. We can prove arcs = by §§ 312, 313, 317.
2. We shall use § 317.

Proof

STATEMENTS	REASONS
1. $AB \parallel CD$.	1. Why?
2. Draw AD.	2. Why possible?
3. $\angle x \doteq \frac{1}{2}\overset{\frown}{m}$ and $\angle y \doteq \frac{1}{2}\overset{\frown}{n}$.	3. Why?
4. $\angle x = \angle y$.	4. Why?
5. $\therefore \frac{1}{2}\overset{\frown}{m} \doteq \frac{1}{2}\overset{\frown}{n}$.	5. Why?
6. $\overset{\frown}{m} = \overset{\frown}{n}$.	6. Asmt. 3, § 317.

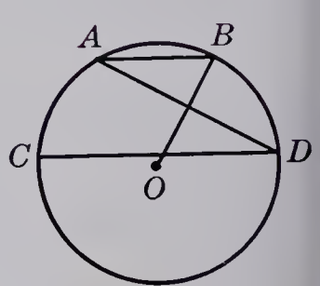
EXERCISES

A

1. *Prove:* A trapezoid inscribed in a circle is isosceles.

2. *Prove:* The base angles of an inscribed trapezoid are equal.

3. A rectangle is inscribed in a circle. One of the small arcs contains 82°. Draw the diagonals of the rectangle and find the number of degrees in each angle of the figure.

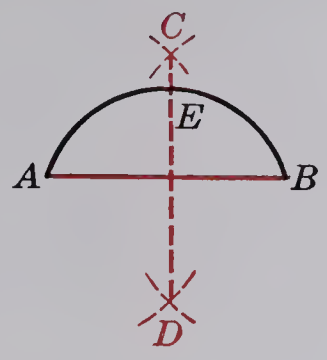


4. *Given* $\odot O$, chord $AB \parallel$ chord CD, and $OB \perp AD$.
Prove $\overset{\frown}{AC} = \overset{\frown}{AB}$.

5. State the contrapositive of Theorem 52.

Problem 13

328. To bisect a given arc of a circle.



Given $\overset{\frown}{AB}$ of a $\odot$.
To bisect $\overset{\frown}{AB}$.

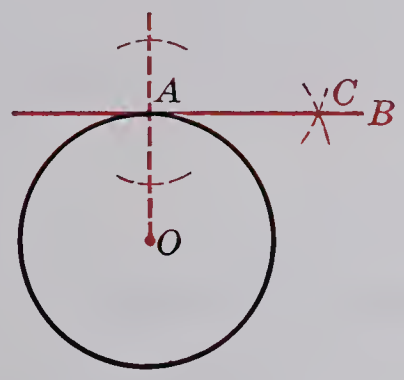
Construction

STATEMENTS	REASONS
1. Draw AB.	1. Why possible?
2. Construct CD, the $\perp$ bisector of AB, intersecting $\overset{\frown}{AB}$ at E. *Then* $\overset{\frown}{AE} = \overset{\frown}{EB}$.	2. Why possible?

Proof: (The proof is left to the student. Use §§ 292 and 290.)

★Problem 14

329. To construct the tangent to a given circle at a given point on the circle.



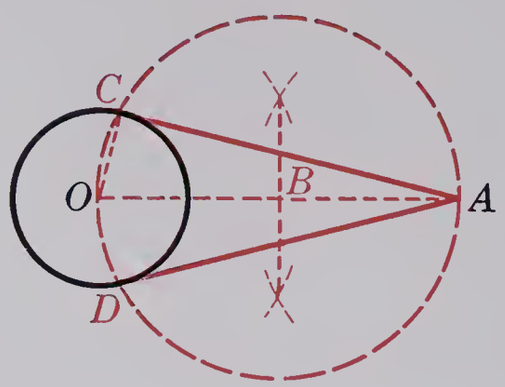
Given $\odot O$ with point A on the circle.
To construct a tangent to $\odot O$ at point A.
(The construction and proof are left to the student.)

EXERCISE

Construct a tangent parallel to a given chord.

★★ Problem 15

330. To construct the tangents to a given circle from a given external point.



Given $\odot O$ and external point A.

To construct the tangents to $\odot O$ from point A.

Construction

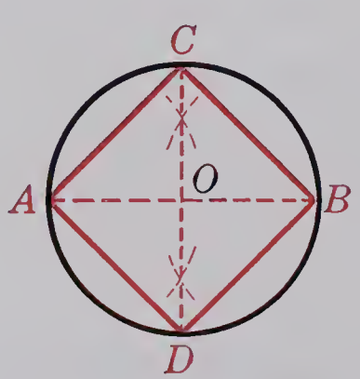
STATEMENTS	REASONS
1. Draw OA.	1. Why possible?
2. Bisect line segment OA, calling the point of intersection B.	2. Why possible?
3. With B as a center and BO as a radius draw a $\odot$ intersecting $\odot O$ at C and D.	3. Why possible?
4. Draw AC and AD.	4. Why possible?
Then AC and AD are the required tangents.	

Proof: (The proof is left to the student. See §§ 319 and 298.)

Discussion. How many tangents can be drawn to $\odot O$ from point A?

★★ Problem 16

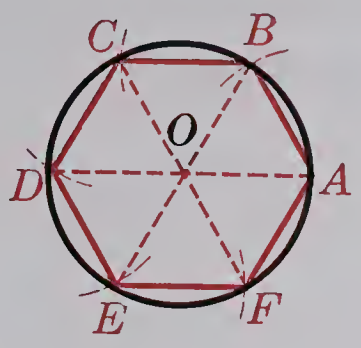
331. To inscribe a square in a given circle.



(The solution is left to the student.)

★★ Problem 17

332. To inscribe a regular hexagon in a given circle.



Given $\odot O$.

To inscribe a regular hexagon in $\odot O$.

Construction

STATEMENTS	REASONS
1. Draw any radius OA.	1. Why possible?
2. With A as a center and OA as a radius draw an arc intersecting $\odot O$ at B.	2. Why possible?
3. With B as a center and OA as a radius draw an arc intersecting $\odot O$ at C.	3. Why possible?
4. In like manner draw arcs intersecting $\odot O$ at D, E, and F.	4. Why possible?
5. Draw AB, BC, CD, DE, EF, and FA. *Then $ABCDEF$ is the required hexagon.*	5. Why possible?

Proof

1. Connect O to each vertex of the hexagon.	1. Why possible?
2. $\triangle AOB$ is equilateral.	2. Why?
3. $\therefore \triangle AOB$ is equiangular.	3. Why?
4. $\angle AOB = 60°$.	4. Why?
5. $\measuredangle BOC$, COD, DOE, and $EOF = 60°$.	5. Reasons 2–4.
6. Then $\angle FOA = 60°$.	6. Give full proof.
7. $\therefore \overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EF} = \overset{\frown}{FA}$.	7. Why?
8. $\therefore AB = BC = CD = DE = EF = FA$.	8. Why?
9. Each $\angle$ of the hexagon $= 120°$.	9. Give full proof.
10. $\therefore ABCDEF$ is a regular hexagon.	10. Why?

If desired, pages 513–517 may be studied now.

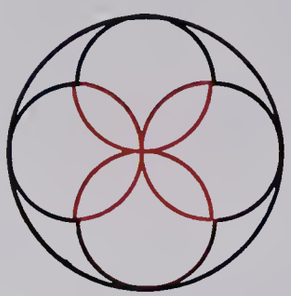
This problem gives us a foundation for inscribing in a circle an equilateral triangle and regular polygons having 12, 24, 48, . . . sides.

EXERCISES — Miscellaneous

A

1. Construct two parallel tangents to a given circle.

2. Construct the following designs:



3. Circumscribe a square about a given circle.

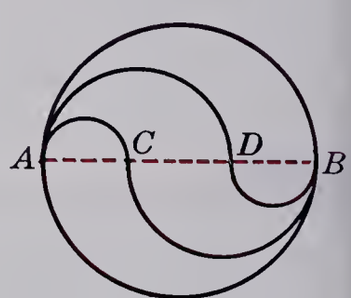
4. Inscribe an equilateral triangle in a given circle.

5. Construct four circles within a given square, so that each circle is tangent to two sides of the square and to two of the other circles.

B

6. Inscribe an equilateral octagon in a given circle.

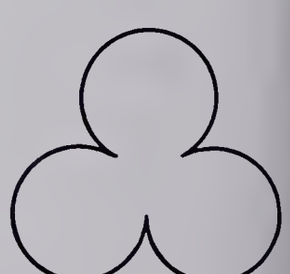
7. Inscribe an equilateral polygon of twelve sides in a given circle.



8. The space enclosed by a circle may be divided into three equal parts by the method shown here. The diameter AB is trisected at C and D. Make this construction for a circle with a radius of 2 inches.

9. Construct an original design for a linoleum pattern.

10. A *trefoil* and its basis of construction are shown at the right. Construct a similar trefoil in which the radii of the circles are 1 inch.



11. *Prove:* If the bisector of an inscribed angle is extended until it intersects the circle, and through this point of intersection a chord is drawn parallel to one side of the angle, it is equal to the chord which is the other side of the angle.

EXERCISES — Applications

1. Show how it is possible to find the center and the radius of a circle by using a carpenter's square. (See Fig. 1 at the right.)

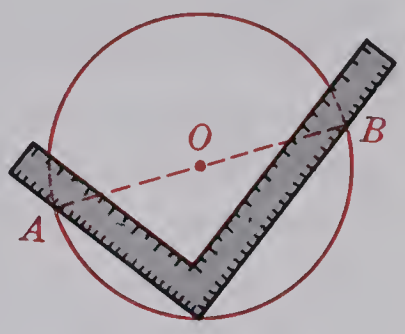


Fig. 1

2. Given a circle and the midpoint of a chord, construct the chord.

3. Prove that an inscribed angle is greater than an angle formed by two secants which intercept the same arc. (Fig. 2)

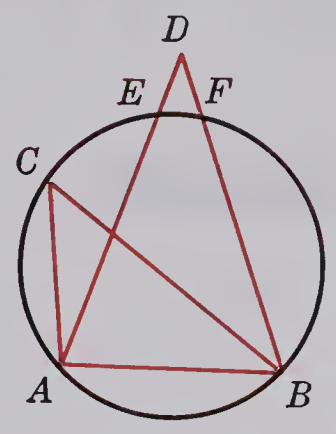


Fig. 2

4. When sailing near rocky coast lines, captains often steer their ships by the aid of two lighthouses and the "horizontal danger angle." In the figure, A and B represent two lighthouses, and $\angle ACB$ is the "horizontal danger angle." Show that if the angle formed by the lines drawn from the ship to the two lighthouses is less than $\angle C$, the ship is out of danger. (Fig. 3 below)

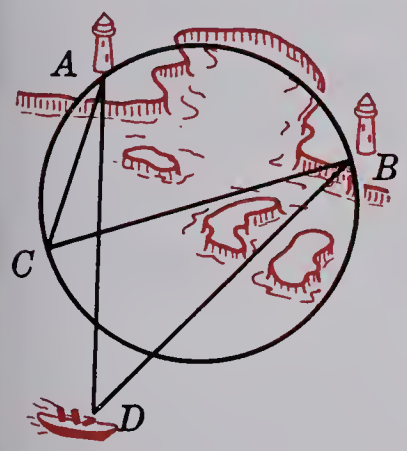


Fig. 3

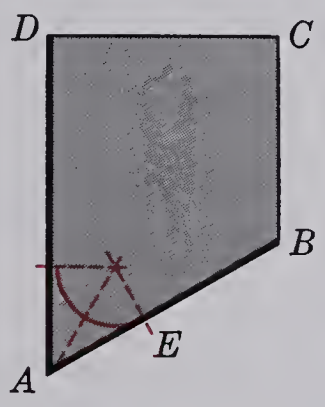


Fig 4.

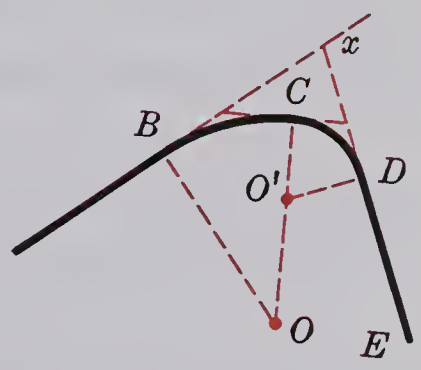
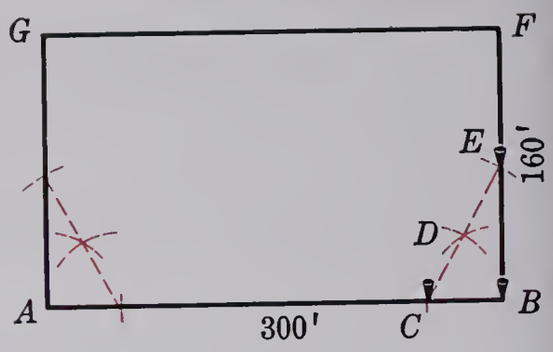


Fig. 5

5. Mr. Woods has a corner lot $ABCD$ with sidewalks along AD and AB. He wishes to round the corner at A by the use of a circular arc. Show how he can construct the arc, which is to be tangent to AB at E and tangent to AD. (Fig. 4 above)

6. $ABCDE$ is a portion of a street-car track. It consists of two straight pieces joined by a compound curve through B, C, and D. Explain how the compound curve BCD is constructed. Prove that $\angle x = \angle O + \angle O'$. (Fig. 5)

7. A coach laid off a football field, using only a steel tape and some stakes. To construct $BF \perp AB$ at B he set stakes at B and C 50 feet apart. Then he found the point D, 50 feet from B and C. He then set a stake at E in line with stakes at C and D and 50 feet from D. He said that EB was $\perp AB$. Can you prove it?



EXERCISES Reasoning

Optional

Which of the following conclusions are valid?

1. The Pyramid Theater always has good pictures. I know that *The Man Gone Wild* is good because the Pyramid showed it last week.

2. Olive has a good voice and she should have a leading role in *Naughty Marietta*.

3. Our basketball team won from Walton 52 to 36 and Walton beat Rochelle 49 to 44. From the results of these two games, I believe that we can win from Rochelle.

4. When Betty is home on Tuesday evenings, she always listens to the Quiz Show. Since she did not hear the Quiz Show last Tuesday evening, she was not at home.

5. Frank never misses school unless he is sick or there is sickness in his family. No member of his family is sick so Frank is in school.

6. The morning paper stated that Dick Best was arrested yesterday for driving a truck while intoxicated. He will have to pay a big fine.

7. Our next-door neighbor has an automobile that uses a lot of oil. The car is only one year old, too. I would not want that make of car.

8. Mrs. Franklin is very particular about her appearance and is always immaculate. I am sure that she keeps her home clean and tidy.

9. Mary went to college and Priscilla became a stenographer. So Mary is better educated than Priscilla.

10. Mr. Wertz is not an educated man. He never went to high school.

11. Our tax rate is lower this year than it was last year. Because of the decreased tax rate, our taxes will be lowered.

12. Most colleges require plane geometry for entrance. I cannot go to college because I am not studying geometry.

The conclusions in exercises 13–17 are based on the assumption, "If the farmer makes hay, it is not raining." Which are valid?

13. If it is not raining, the farmer makes hay.

14. If the farmer is not making hay, it is not raining.

15. It is raining if the farmer is not making hay.

16. If it is raining, the farmer is not making hay.

17. It is not raining so the farmer is not making hay.

REVIEW QUESTIONS — Chapter 10

1. What is the difference between an arc degree and an angle degree?

2. $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ contain 13° each. If the radius of $\overset{\frown}{AB}$ is 10 inches and the radius of $\overset{\frown}{CD}$ is 8 inches, which arc is the longer?

3. How many degrees are there in an angle that is inscribed in a semicircle?

4. What is the name of the point where a tangent touches the circle?

5. Name the order of sequence of the following theorems:

a. An angle formed by a tangent and a chord · · ·.
b. An angle formed by two tangents, a tangent and a secant · · ·.
c. An inscribed angle is equal in degrees · · ·.
d. An angle formed by two chords · · ·.

6. An angle is inscribed in a minor arc. Is the angle acute?

7. Can a square be inscribed in any circle?

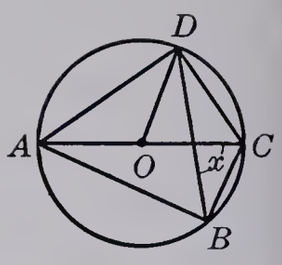
8. Define: arc; chord; tangent; secant; inscribed angle.

9. How many arcs does a chord have?

10. If one arc of a chord contains 80°, what kind of arc is it?

11. How many chords may be drawn from a point on a circle?

In the figure at the right, AC is a diameter, $\overset{\frown}{BC} = 50°$, and $\angle AOD = 110°$.

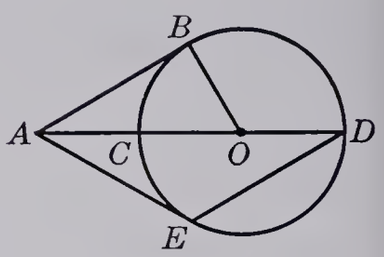


Ex. 12–13

12. Find the number of degrees in $\overset{\frown}{CD}$; in $\angle CAD$; in $\angle ADB$; in $\angle x$.

13. If $AC = 29$ inches and $CD = 20$ inches, how long is AD?

In this figure AB is tangent to $\odot O$ at B, and AE is tangent to the circle at E.



Ex. 14–15

14. If $\angle OAB = 30°$, find $\overset{\frown}{CB}$; $\overset{\frown}{BD}$; $\overset{\frown}{CE}$; $\angle COB$; $\angle ADE$.

15. If $\angle BOA = 60°$ and $BO = 8$ inches, find CO; AO; AC; AB; AE.

16. What do you know about the angles of a quadrilateral if the quadrilateral can be inscribed in a circle?

17. Is it true that the line which bisects two chords of a circle passes through the center of the circle?

18. AB is a diameter of circle O, and BC and AC are chords. Diameter AB is 10 inches and $\overset{\frown}{BC}$ is 60°. How long is chord BC?

19. The side DE of $\triangle DEF$ is a diameter of a circle, and F is a point on the circle. DF is 36 inches and EF is 15 inches. How long is DE?

20. ABC is a triangle inscribed in a circle, and tangents to the circle at B and C meet in a point D. If the angle at A is 50°, what is the angle at D?

SUMMARY of Principal Bases for Proof

333. Comparison of angles and their arcs

a. A central angle is equal in degrees to its intercepted arc.

b. An inscribed angle is equal in degrees to one half its intercepted arc.

c. An angle formed by a tangent and a chord drawn from the point of contact is equal in degrees to one half its intercepted arc.

d. An angle formed by two chords intersecting within a circle is equal in degrees to one half the sum of the arcs intercepted by it and its vertical angle.

e. An angle formed by two tangents, a tangent and a secant, or two secants, intersecting outside a circle, is equal in degrees to one half the difference of the intercepted arcs.

334. Two angles are equal

If they are inscribed angles intercepting the same arc.

335. Two lines are perpendicular

If they are sides of an angle inscribed in a semicircle.

336. Constructions

a. To bisect a given arc of a circle.
b. To construct the tangent to a given circle at a given point on the circle.
c. To construct the tangents to a given circle from a given external point.
d. To inscribe a square in a given circle.
e. To inscribe a regular hexagon in a given circle.

337. Two arcs are equal

a. If they are arcs of the same circle or of equal circles and have the same number of degrees.
b. If they are arcs of a circle intercepted by parallel lines.

TEST 25 — Multiple-Choice Statements

[8 Minutes]

On your paper write the word or group of words in boldface type which makes each statement true:

1. An angle inscribed in a semicircle is
an acute angle **a right angle** **an obtuse angle.**

2. A central angle is equal in degrees to
its intercepted arc **one half its intercepted arc**
twice its intercepted arc.

3. If a central angle is doubled, its arc
is doubled **remains the same** **is tripled.**

4. An inscribed angle is equal in degrees to **its intercepted arc**
one half its intercepted arc **twice its intercepted arc.**

5. An inscribed angle intercepting an arc of 90° is
acute **right** **obtuse.**

6. If a quadrilateral is inscribed in a circle, the opposite angles are
equal **complementary** **supplementary.**

7. Inscribed angles which intercept the same arc are
equal **complementary** **supplementary.**

8. An angle formed by two secants is equal in degrees to
one half the sum of its intercepted arcs
one half the difference of its intercepted arcs
the difference of its intercepted arcs.

9. An angle formed by two intersecting chords is equal in degrees to **the sum** **the difference** **one half the sum** **one half the difference** of the arcs intercepted by it and its vertical angle.

10. A polygon is inscribed in a circle if its sides are
tangents **secants** **chords**
of the circle.

TEST 26 — Constructions

[17 Minutes]

Make the following geometric constructions:

1. In a given circle inscribe an angle of 60°.

2. Given the arc of a circle, locate its center.

3. Construct a tangent to a given circle at a given point on the circle.

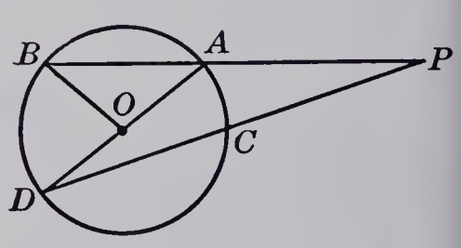
4. Inscribe a square in a given circle.

TEST 27 — Applications

[30 Minutes]

In this figure, DA is a diameter of $\odot O$, $\angle D = 24°$, and $\overset{\frown}{BD} = 86°$. Find the number of degrees in:

1. $\angle BOD$. 2. $\overset{\frown}{AC}$. 3. $\angle P$. 4. $\overset{\frown}{BA}$.



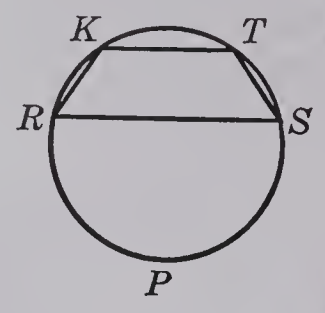
Ex. 1-4

5. Two tangents to a circle are perpendicular to each other. How many degrees are there in the minor arc?

6. Two tangents to a circle form an angle of 60°. How many degrees are there in the major arc of the circle?

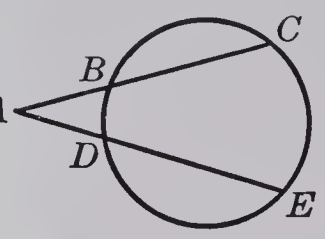
7. How many degrees are there in each of the arcs formed by the sides of a regular pentagon inscribed in a circle?

8. A circle is divided into six equal arcs. If the diameter of the circle is 14 inches, how many inches are there in a chord joining any two successive points of division?



9. In this figure, $RSTK$ is an inscribed trapezoid with bases RS and KT, $\overset{\frown}{ST} = 40°$, and $\overset{\frown}{KT} = 50°$. Find the number of degrees in $\angle R$; in $\angle T$.

10. AB is a diameter and AC is a chord of a circle. If AC is equal to a radius of the circle, how many degrees are there there in $\overset{\frown}{BC}$?



11. In the figure, if $\angle A = 32°$ and $\overset{\frown}{BD} = 24°$, how many degrees are there in $\overset{\frown}{CE}$?

12. If the hypotenuse of a right triangle is 10 inches, what is the length of the median on the hypotenuse?

13. $\triangle ABC$ is inscribed in a circle. Chord CD intersects AB in E and bisects $\angle ACB$. Find $\angle AEC$ if $\overset{\frown}{BC} = 60°$ and $\angle DBC = 100°$.

CHAPTER 11

Loci

In this chapter you will make a more thorough study of locus and its application to theorems and construction problems.

338. Locus of Points

A *locus of points* is a geometric figure containing *all the points, and only those points,* that satisfy a given condition or set of conditions. A locus may consist of one or more points, lines, surfaces, or any combination of them. The word *locus,* the plural of which is *loci* (pronounced lō′sī), is the Latin word meaning "place" or "location."

"Locus" may also be defined as the path of a moving point satisfying a given condition. As the point moves according to given conditions, it traces a path all points of which satisfy the given conditions. Thus the phrases "locus of points" and "locus of a point" have the same general meaning. Locus lines will be indicated by long-dash lines to distinguish them from given lines and construction lines.

There are four important steps in determining a locus, the first three of which will be illustrated in the following examples:

Example 1. What is the locus of points $\frac{3}{8}$ inch from a given point O?

The given condition is that the points shall be $\frac{3}{8}$ inch from O.

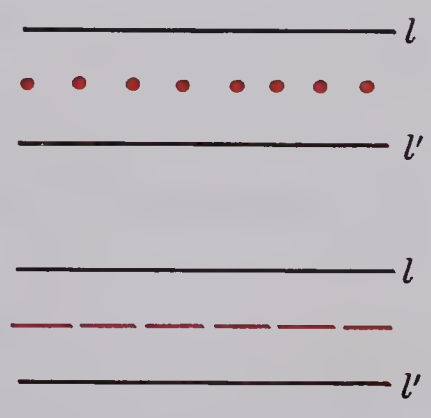
STEP 1. We first locate several points that are $\frac{3}{8}$ inch from O. We know that there is an infinite number of them.

STEP 2. We draw a smooth curve through these points, using a dash line.

STEP 3. We describe the geometric figure which the locus appears to be. The locus of points $\frac{3}{8}$ inch from the given point O is a circle with center O and a radius of $\frac{3}{8}$ inch.

Example 2. What is the locus of points equidistant from two given parallel lines l and l'?

The given condition is that the points shall be the same distance from l and l'.



STEP 1. We locate several points that are equidistant from l and l'. We know that there is an infinite number of such points.

STEP 2. The points appear to lie on a straight line; so we draw a straight dash line through the points.

STEP 3. Now we describe the geometric locus. The locus of points equidistant from two given parallel lines l and l' is the line parallel to l and l' and midway between them.

Example 3. Find the locus of points $\frac{1}{2}$ inch from a given line l.

The given condition is that the points shall be $\frac{1}{2}$ inch from l.

STEP 1. Draw a given line l. Locate several points on both sides of and $\frac{1}{2}$ inch from l.

STEP 2. Draw a dash line through the points on one side of l. Draw another dash line through the points on the other side of l.

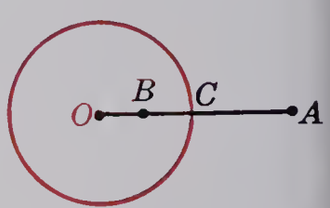
STEP 3. The locus of points $\frac{1}{2}$ inch from a given line l is two lines parallel to l and $\frac{1}{2}$ inch from it.

339. Directions for Determining a Locus:

1. Locate several points which satisfy the given conditions.
2. Draw a smooth line* or lines through these points.
3. Describe accurately the geometric figure you think is the locus.
4. Prove that this figure is the locus.

340. Distance from a Point to a Circle

The distance from a point to a circle is measured on the line joining the point to the center of the circle and is the length of the line segment between the point and the circle. AC is the distance of point A from $\odot O$, and BC is the distance of point B from $\odot O$. Show that AC is less than any other line segment drawn from A to the circle.



EXERCISES

A

Making use of the directions given in § 339 for determining a locus, draw a figure and describe the locus for each of the following exercises. No proof is required.

1. The locus of points which are 3 inches from a given point.

2. The locus of the center of a ball as it rolls on a straight line over a level floor.

3. The locus of the center of a circle that rolls around the outside of another circle.

4. The locus of points on this page 3 inches from the top.

*This line may be a straight line or a curved line.

5. The locus of all cities that are 5 miles from a given railroad.

6. The locus of points 5 inches from a circle whose radius is 5 inches.

7. The locus of points equidistant from the top and bottom of this page.

8. The locus of points less than 5 inches from a given point.

9. The locus of a point equidistant from two given points.

10. The locus of points within an angle equidistant from the sides of the angle.

11. The locus of the vertex of a right triangle with a given hypotenuse as a base.

12. The locus of the centers of all circles tangent to a given line at a given point.

13. The locus of points equidistant from the points of intersection of two given intersecting circles.

14. The locus of the midpoints of parallel chords of a given circle.

15. The locus of the midpoints of all equal chords of a given circle.

16. The locus of the midpoints of the radii of a given circle.

17. What is the locus of a point that is inside a 4-inch square and 1 inch from a side?

18. What is the locus of points inside a circle whose radius is 3 inches?

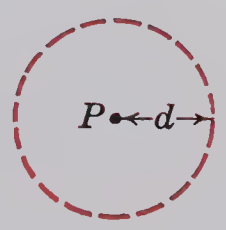
19. What is the locus inside a circle of a point whose distance from the circle is equal to a radius?

20. What is the locus of the center of a circle that rolls around the inside of a larger circle and is always tangent to the larger circle?

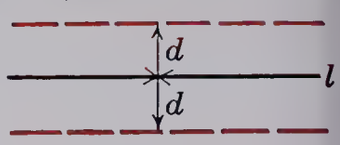
341. Assumptions on Loci

From the examples of § 338, and previous definitions, we may state the following assumptions on loci:

Assumption 47. **The locus of points at a given distance from a given point is a circle with the given point as the center and the given distance as the radius.**



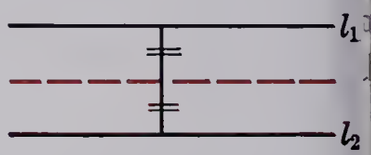
Assumption 48. The locus of points at a given distance from a given line is a pair of lines parallel to the given line and at the given distance from it.



(In the figure, l is the given line and d the given distance.)

This assumption means that if the two broken lines are parallel to the given line l and each at a distance d from l, then: (a) any point which is on either of the parallels is at a distance d from l, and (b) any point which is at a distance d from l is on one of the parallels.

Assumption 49. The locus of points equidistant from two given parallel lines is the line parallel to each of the given lines and midway between them.



(In the figure, l_1 and l_2 are the given lines.)

342. Proving a Locus Theorem

To prove that a geometric figure is a locus, we must prove not only that the figure contains all the points that satisfy the given conditions, but also that it contains no other points. In other words, we must prove that the locus points and the points of the figure coincide.

To prove that two geometric figures coincide, it is only necessary to show that all points of the first are points of the second and that all points of the second are points of the first.

For example, consider the portion of planes bounded by the circles C and C'.

In Fig. 1, no point in either circle lies in the other circle.

In Fig. 2, there are points (but not all) in either circle that lie in the other circle.

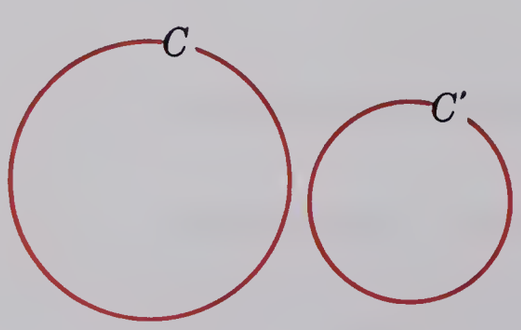


Fig. 1

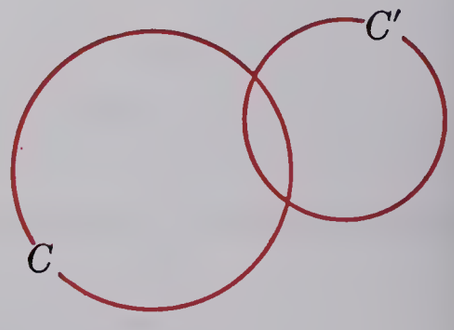


Fig. 2

In Fig. 3, all the points in circle C' lie in circle C but not all the points in circle C lie in circle C'.

In Fig. 4 the converse is true.

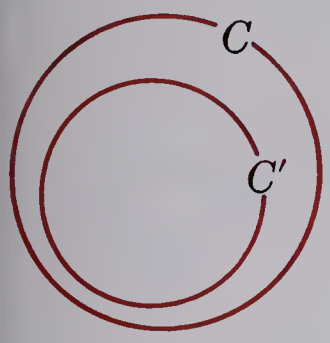


Fig. 3

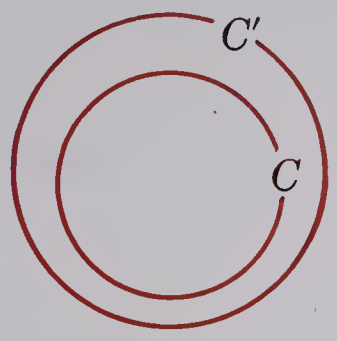


Fig. 4

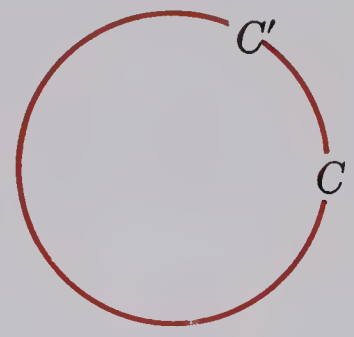


Fig. 5

In Fig. 5 all the points in circle C lie in circle C' and all the points in circle C' lie in circle C. In Fig. 5 the portions of the planes bounded by the circles coincide. The circles in Fig. 5 fulfill the conditions:

If a point is in C, it is in C'; and if a point is in C', it is in C.

When we say that a locus of points is a circle, we mean that the locus of points and the circle coincide. When we say that a line is a locus of points, we mean that the line and the locus coincide. In general, when we say that a locus is a certain geometric figure, we mean that all points of the locus are points of the geometric figure and that all points of the geometric figure are points of the locus.

From this discussion the following directions can be stated.

To prove that a geometric figure is a locus:

1. Prove that every point of the figure is a point of the locus.
2. Prove that
   - **a.** every point of the locus is a point of the figure, or
   - **b.** every point outside the figure is not a point of the locus.

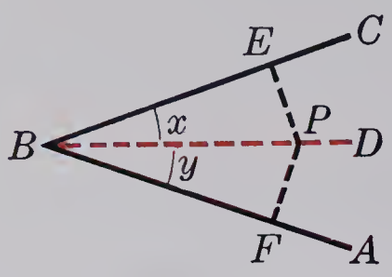
Step 1 (or its equivalent) is needed to show that all the geometric figure is a part of the locus. Either step 2 (*a*) or 2 (*b*) is needed to show that all the locus is part of the figure. If either step 2 (*a*) or step 2 (*b*) is true, we know by § 132 that the other is true.

When studying Theorem 53 on the next page, note carefully the two parts of the demonstration. Part I proves that every point of the figure is a point of the locus and Part II proves that every point of the locus is a point of the figure.

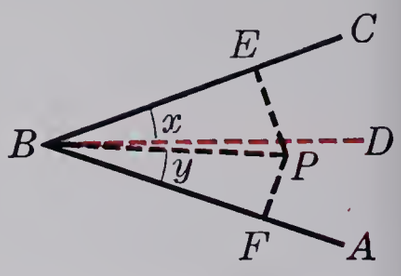
★Theorem 53

343. The locus of points within an angle equidistant from the sides is the bisector of the angle.

Part I



Part II



Part I. Any point on the bisector of the angle is equidistant from the sides of the angle.

Given BD the bisector of $\angle ABC$, and P any point in BD.

We proved Part I in § 126.

Part II. Any point equidistant from the sides of an angle is on the bisector of the angle.

Given BD the bisector of $\angle ABC$, and any point P which is equidistant from BC and BA.

To prove that P is on BD. (We proved Part II in § 141.)

Then the bisector BD of $\angle ABC$ is the required locus (§ 338).

★**344. Corollary.** The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.

EXERCISES — Loci

A

To decide on these loci use steps 1 and 2 of § 339. Then construct each locus with straightedge and compasses. Do not prove.

1. Determine the locus of the midpoints of the radii of a given circle.

2. Determine the locus of the midpoints of all chords of a given length in a given circle.

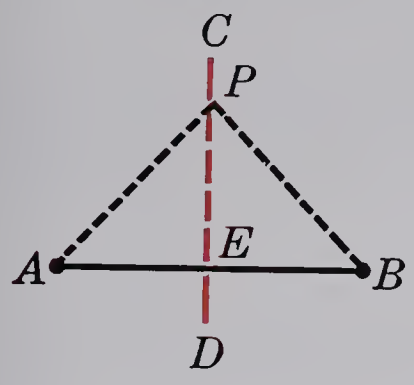
3. Determine the locus of the vertices of all triangles having a given base and a given altitude.

4. Determine the locus of points equidistant from two circles having the same center.

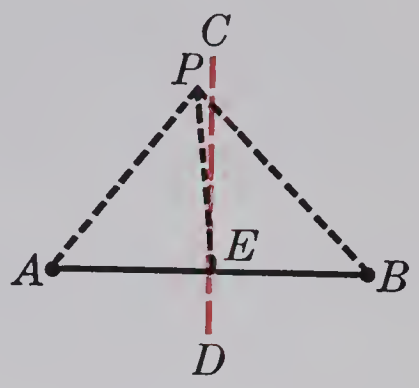
★★ Theorem 54

345. The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.

Part I



Part II



Part I. Any point on the perpendicular bisector of a line segment is equidistant from the end points of the segment.

We proved Part I in § 79.

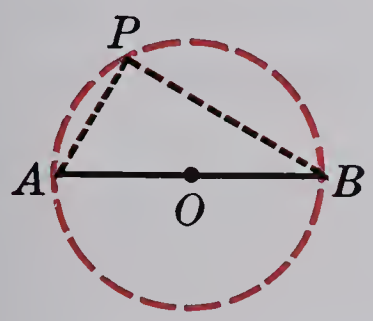
Part II. A point equidistant from the end points of a line segment lies on the perpendicular bisector of the segment.

We proved Part II in § 139.

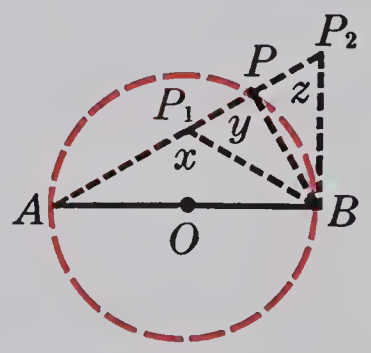
Theorem 55

346. The locus of the vertex of a right triangle with a given hypotenuse as the base is a circle upon the hypotenuse as a diameter.

Part I



Part II



Part I. If from any point on a circle lines are drawn to the end points of a diameter, a right triangle is formed and the point is the vertex of the triangle.

Part II. If from any point not on a circle lines are drawn to the end points of a diameter of the circle, the triangle so formed is not a right triangle.

The proofs of Part I and Part II are left to the student.

SUGGESTION. For proof of Part II use §§ 319, 93.

Theorem 56

347. **The locus of the centers of all circles tangent to a given line at a given point is the perpendicular to the line at that point.**

(The demonstration is left to the student.)

NOTE. The loci developed in Assumptions 47–49 and Theorems 53–56 should be thoroughly understood, as most loci problems are applications of them.

348. Concentric Circles

Concentric circles are circles having the same center.

EXERCISES*

A

1. Determine the locus of the centers of all circles passing through two given points.

2. Determine the locus of the center of a circle tangent to the sides of a given angle.

3. Determine the locus of points within a given circle which are equidistant from the ends of a given chord.

4. Determine the locus of the center of a circle which has a given radius and is tangent to a given line.

5. Determine the locus of the centers of all circles (a) tangent to two given parallel lines; (b) tangent to two given intersecting lines.

B

6. Determine the locus of the centers of all circles tangent to a given circle at a given point on the circle.

7. Determine the locus of points at a given distance from a given circle (three different loci, depending on the relative length of the radius and the given distance).

8. Determine the locus of the midpoints of all chords which can be drawn from a given point on a given circle.

9. Determine the locus of the centers of all circles with a given radius and passing through a given point.

*You may omit step 4, § 339, unless your teacher requires it.

C

10. Determine the locus of the midpoints of all chords formed by secants drawn to a circle from a given external point.

11. In a rectangle $ABCD$, AB equals 2 inches and BC equals 1 inch. A line EB is drawn from E, the midpoint of CD, to B. Draw the locus of the midpoints of lines parallel to EB and terminated by the sides of the rectangle.

12. If the range of a field artillery gun is between 1000 yards and 5000 yards, and the gun can be turned through an angle of 135°, draw the locus of points where the projectiles may fall.

13. See if you can plot and give the names of the following two loci, which are curves but not circles: (**a**) the locus of points equidistant from a given line and a given point; (**b**) the locus of points the sum of whose distances from two given points is a given length.

349. Ways of Expressing a Theorem and Its Converse [Optional]

You have learned that the proof of a locus theorem consists of proving a theorem and its converse (or of proving a theorem and its inverse).

There are two ways of describing a situation when both a theorem and its converse are true. One way is to use the words "if and only if" and the other is to use the words "necessary and sufficient."

For example, let us use two assumptions, each of which is the converse of the other:

(**a**) If two central angles of a circle are equal, the intercepted arcs are equal.

(**b**) If two arcs of a circle are equal, the central angles that intercept them are equal.

Assumption (**a**) states that the intercepted arcs are equal if the central angles are equal; and assumption (**b**) states that the arcs are equal only if the central arcs are equal. We may express both assumptions by "Two intercepted arcs of a circle are equal if, and only if, the angles that intercept them are equal."

Assumption (**a**) gives a sufficient condition for the arcs to be equal and assumption (**b**) gives a necessary condition for the arcs to be equal. Both assumptions can be expressed by "The necessary and sufficient condition for two arcs of a circle to be equal is that the central angles that intercept them be equal."

EXERCISES

Optional

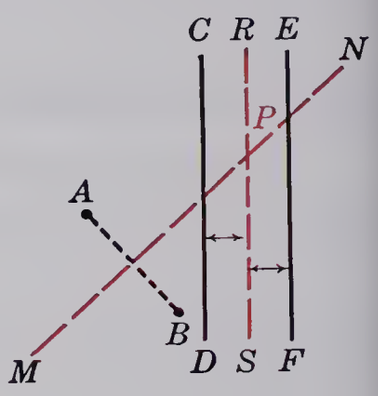
1. Use the words "if and only if" to express Theorems 3 and 12.

2. Use the words "necessary and sufficient" to express Theorems 3 and 12.

350. Compound Locus

Sometimes points are required to satisfy two separate conditions. In such cases the points will be the intersections of the separate loci determined by each condition. A *compound locus* is determined by the intersection of two or more loci.

In solving a locus problem place the given parts in the most general positions in order to determine the most general locus, and then in a discussion state the locus for all special conditions.



Example. Find all the points equidistant from two given points and also equidistant from two given parallel lines.

Given the points A and B and the two $\parallel$s CD and EF.

To find all points equidistant from A and B and also equidistant from CD and EF.

Solution

STATEMENTS	REASONS
1. MN, the $\perp$ bisector of AB, the line segment joining A and B, is the locus of points equidistant from A and B.	1. § 345.
2. RS, the line $\parallel CD$ and EF and midway between them, is the locus of points equidistant from CD and EF.	2. Asmt. 49.
3. The required point is P, the intersection of MN and RS.	3. § 350.

Discussion.

1. If $RS \parallel MN$, there are no points.
2. If RS coincides with MN, every point of MN is a required point.
3. In all other positions there is only one point.

EXERCISES

A

1. Find all the points equidistant from three given points. (Find the locus for two points at a time.)

2. Find all the points $1\frac{1}{2}$ inches from a given point and equidistant from two given parallels which are 1 inch apart.

3. Find all the points equidistant from two given intersecting lines and at a distance of 1 inch from the point of intersection of the two lines.

4. Find all the points at a distance of 2 inches from each of two given points which are 3 inches apart.

5. Find all the points equidistant from two given parallel lines and at a given distance from a third given line.

6. Find all the points equidistant from two given points and at a given distance from a third given point.

7. Find all the points equidistant from two given points and at a given distance from a given line.

8. Find all the points at a given distance from a given line and at a given distance from a given point.

B

9. Find all the points equidistant from two given parallels and also equidistant from two given intersecting lines.

10. Find all the points equidistant from two given intersecting lines and at a given distance from a third given line.

11. Find all the points at a given distance from a given circle and equidistant from two given points.

12. Find all the points at a distance of the radius of a given circle from the given circle and at the same distance from a given tangent to the given circle.

C

13. Find all the points equidistant from two given concentric circles and also equidistant from two given parallels intersecting the circles.

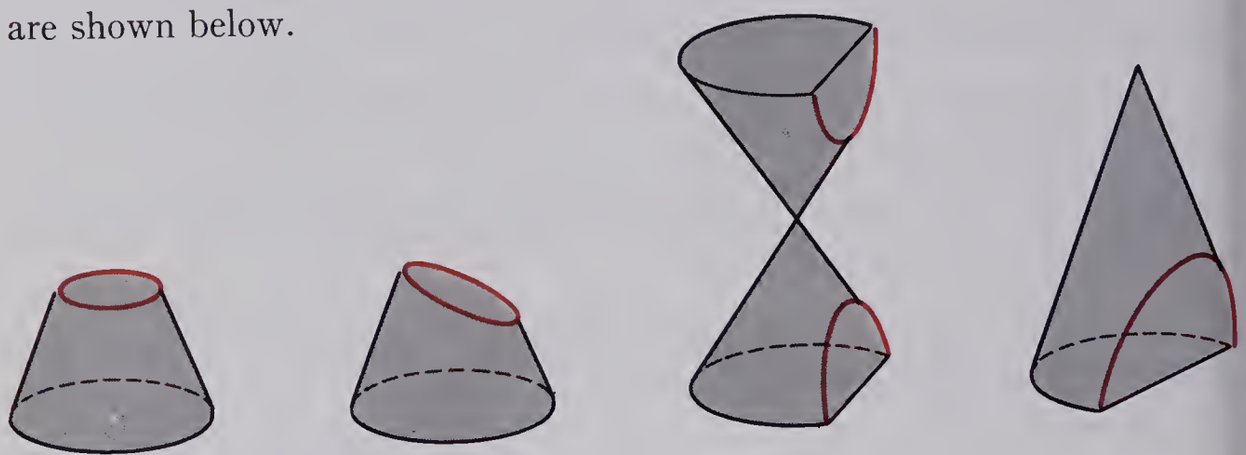
14. Find all the points at a given distance from a given circle and also equidistant from two given lines that intersect at the center of the given circle.

15. Find all the points equidistant from two given equal intersecting circles and at the distance of their common chord from their line of centers.

16. AB is a line segment 3 inches long. Locate all the points which are 1 inch from AB and are vertices of right angles whose sides pass through the end points A and B. What would be the distance of the vertices of the right angles from AB if the locus consisted of only two points?

351. Loci Which Are Conic Sections

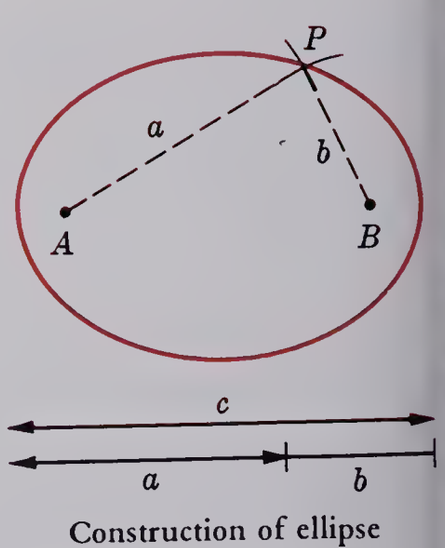
There are four important loci which are conic sections, that is, figures formed by cutting a right circular cone with a plane. The four are shown below.



Circle Ellipse Hyperbola Parabola

You have already studied the circle. It is the only conic section which can be constructed when we limit ourselves to compasses and a straightedge. Of course, in constructing a circle we use only compasses.

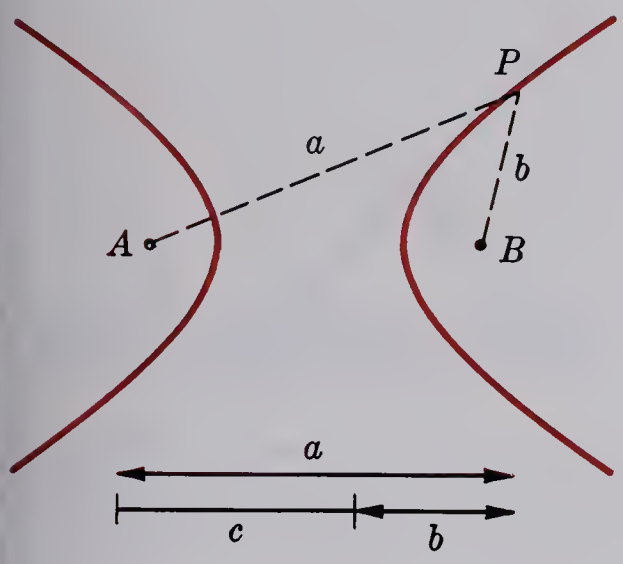
An *ellipse* is the locus of points the sum of whose distances from two fixed points is constant. Even though we cannot construct an ellipse with compasses and a straightedge, we can construct as many points of the ellipse as we desire. In the figure at the right, A and B are fixed points and c a fixed length. Study the figure to see that point P was located by drawing two intersecting arcs, one having center A and radius a, the other having center B and radius b, and such that $a + b = c$. By using the same centers and other radii a' and b', such that $a' + b' = c$, another point P' can be



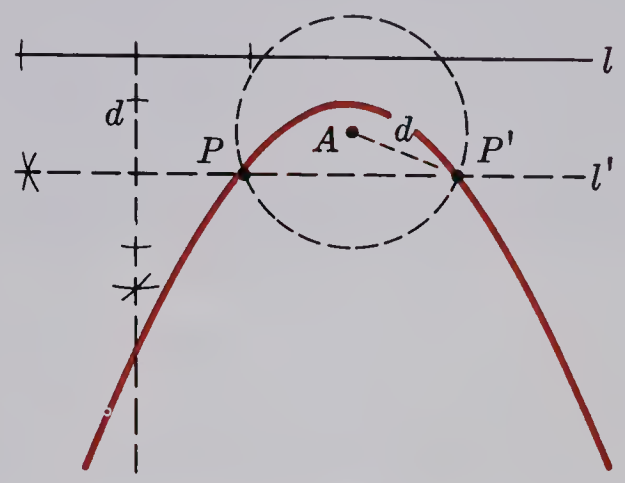
Construction of ellipse

located on the ellipse. By continuing the process indefinitely the ellipse takes shape. Some automobile springs, gears in machines, and arches of bridges are applications of the ellipse. The orbits of planets as they move around the sun are ellipses. An ellipse has the property that light or sound waves emanating (i.e. issuing) from one of the fixed points will come together at the other.

A *hyperbola* is the locus of points the difference of whose distances from two fixed points is constant. The hyperbola has the property that a ray of light emanating from one of the fixed points is reflected along a line whose extension passes through the other fixed point.



Construction of hyperbola



Construction of parabola

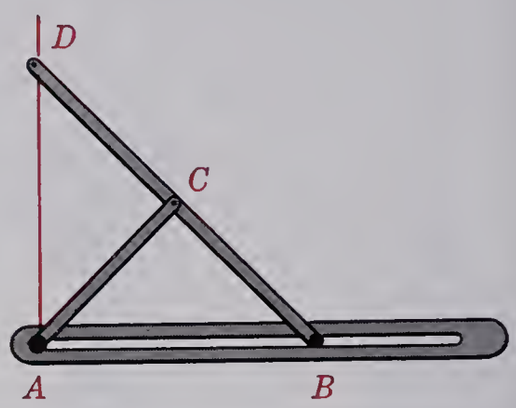
A *parabola* is the locus of points equidistant from a fixed line and a fixed point. In the drawing at the right above, l is a fixed line and A a fixed point. Study the drawing to see that two points P and P' of the parabola were located by drawing l' parallel to l and at a distance d from it, after which l' was intersected by a circle having center A and radius d. By repeating the process with other distances d', d'', $\cdots$, other points of the parabola can be found. A ball thrown into the air follows approximately the path of a parabola. A parabola has the property that light or sound waves emanating from the fixed point are reflected in parallel lines each of which is perpendicular to the fixed line. This property makes the parabola a good shape to use for automobile headlights. A parabolic surface is an ideal surface to use when light, sound, or radio waves from a distant source must be brought to a focus. This is because rays which strike the surface perpendicular to the fixed line are all reflected to meet at the fixed point. A picture of a parabolic antenna is shown on page 361.

352. Algebraic Loci

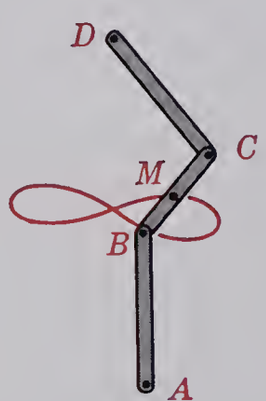
Since all of the conic sections can be represented by algebraic equations, these loci can be studied from the algebraic point of view. The same can be said for a straight line (the locus of a point moving in a fixed direction). In Chapter 17 you will learn more about algebraic loci.

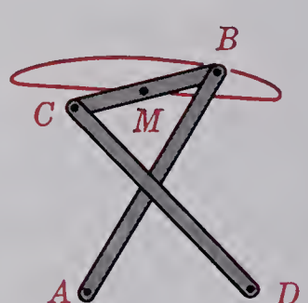
353. Linkages

Sometimes by linking together bars and other geometric figures, so that the points of joining are movable, we can obtain interesting and useful loci. In the device shown at the right, AB is a bar so slotted that B of bar BD can move back and forth along it. Point A is stationary. Bar AC is connected to bars AB and BD in pivoted joints A and C, respectively. $AC = \frac{1}{2} BD$ and C is the midpoint of BD. Show that for any position of B, DA is perpendicular to AB. How many lines can be drawn perpendicular to AB at A? Do you see that as B moves back and forth along AB, a pencil inserted at D will trace straight line AD?



In the linkage at the left the equal bars AB and DC are connected to bar BC with movable joints B and C and are pivoted to a plane surface in fixed points A and D. As the bars move back and forth M, the midpoint of BC, traces out the figure-eight curve shown.





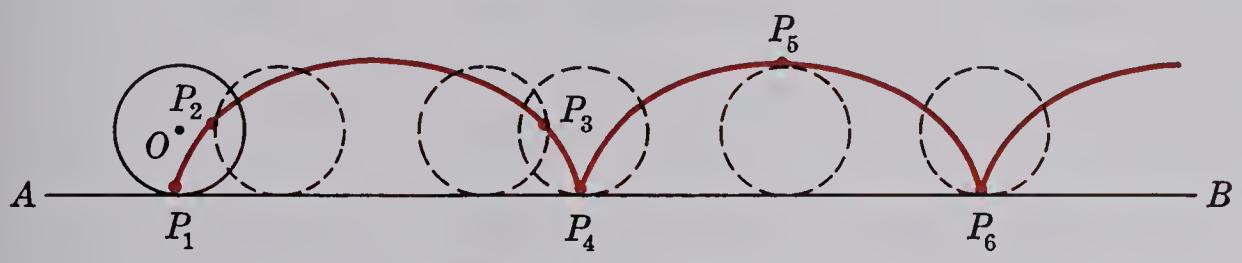
If the bars described above are crossed and attached to a flat surface with fixed but pivoted joints at A and D, M traces the curve shown at the right above.

This parabolic antenna reflects incoming parallel radio or TV waves at one point, the focus, at which the receiver is located

American Telephone & Telegraph Company

354. Other Loci

There are many other interesting loci. For example, the locus of a given point on a circle as it rolls along a straight line is a cycloid.



The curve $P_1P_2P_3P_4$ is one arch of a cycloid made by point P on circle O as the circle rolls along AB.

In the figure at the right, CE is half an arch of a cycloid. If an object is released at C it will travel to E in less time than if it followed any other path from C to E. For this reason the cycloid is called "the curve of quickest descent."

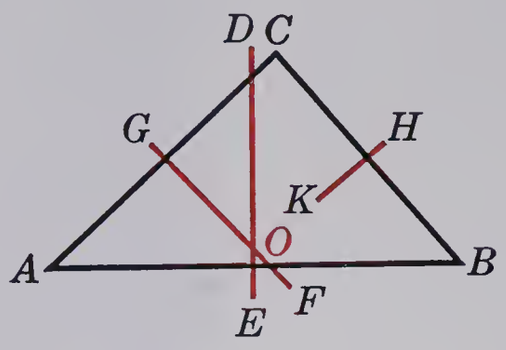
NOTE. If desired, §§ 548, 549 may be studied now.

355. Concurrent Lines

Concurrent lines are three or more lines passing through the same point.

★ Theorem 57

356. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.



Given $\triangle ABC$ with DE, FG, and HK the $\perp$ bisectors of the sides AB, AC, and BC respectively.

To prove that DE, FG, and HK are concurrent in a point equidistant from A, B, and C.

Planning the Proof: 1. We can prove a point equidistant from other points by §§ 276, 345.
2. We shall use § 345.

Proof

STATEMENTS	REASONS
1. DE and FG will intersect in some point, as O.	1. § 115.
2. DE is the locus of points equidistant from A and B.	2. § 345.
3. FG is the locus of points equidistant from A and C.	3. Why?
4. $\therefore O$ is equidistant from A, B, and C.	4. Asmt. 6.
5. $\therefore O$ is on HK, the $\perp$ bisector of BC.	5. Theorem 54, Part II.
6. $\therefore DE$, GF, and HK are concurrent in a point equidistant from A, B, and C.	6. Statements 1, 4, 5.

357. Circumcenter

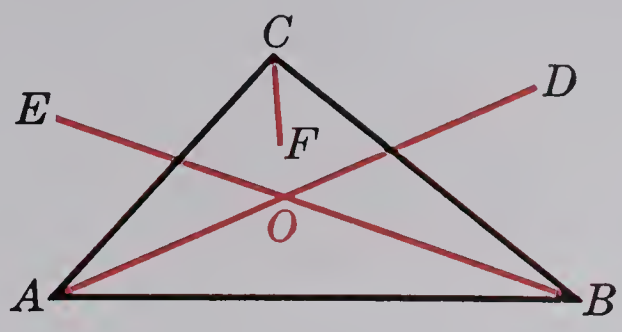
The point where the perpendicular bisectors of the sides of a triangle are concurrent is called the *circumcenter* of the triangle.

EXERCISE

In the figure above, find OB and OC if OA equals 5.

★ Theorem 58

358. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.



Given $\triangle ABC$ with AD, BE, and CF bisecting $\measuredangle A$, B, and C respectively.

To prove that AD, BE, and CF are concurrent in a point equidistant from AB, BC, and AC.

Planning the Proof: 1. We can prove a point equidistant from lines by Asmt. 49, §§ 343, 344.
2. We shall use § 343.

Proof

STATEMENTS	REASONS
1. AD and BE will intersect in some point O.	1. Give full proof.*
2. AD is the locus of points equidistant from AB and AC.	2. Why?
3. BE is the locus of points equidistant from AB and BC.	3. Why?
4. $\therefore O$ is equidistant from AB, AC, and BC.	4. Why?
5. $\therefore O$ is on CF, the bisector of $\angle C$.	5. § 343.
6. $\therefore AD$, BE, and CF are concurrent in a point equidistant from the sides AB, BC, and AC.	6. Statements 1, 4, and 5.

359. Incenter

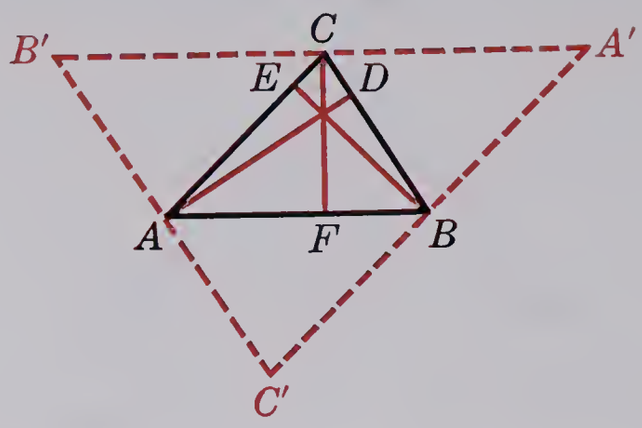
The point where the bisectors of the angles of a triangle are concurrent is called the *incenter* of the triangle.

It will be proved in § 364 that the *circumcenter* of a triangle is the center of the circle circumscribed about the triangle (the circle passing through the three vertices). Why do you think the incenter of a circle is so called?

*If AD and BE do not intersect, $AD \parallel BE$. Why? Then $\angle DAB + \angle EBA = 1$ st. $\angle$ (§ 162). This is impossible (§ 120.)

Theorem 59

360. The altitudes of a triangle are concurrent.



Given $\triangle ABC$ with the altitudes AD, BE, and CF.
To prove that AD, BE, and CF are concurrent.

Planning the Proof: 1. We can prove lines concurrent by §§ 356, 358.
2. We shall use § 356.

Proof

STATEMENTS	REASONS
1. Draw $B'A'$ through $C \parallel AB$; $C'A'$ through $B \parallel AC$; and $C'B'$ through $A \parallel BC$.	1. § 174.
2. $ABCB'$ and $ABA'C$ are Ⓢ.	2. Why?
3. $\therefore B'C = AB$ and $CA' = AB$.	3. Why?
4. $\therefore B'C = CA'$.	4. Why?
5. $CF \perp AB$.	5. Why?
6. $CF \perp B'A'$.	6. Why?
7. $\therefore CF$ is the $\perp$ bisector of $B'A'$.	7. Statements 4 and 6.
8. In like manner, BE and AD are the perpendicular bisectors of $C'A'$ and $B'C'$ respectively.	8. Reasons 2–7.
9. $\therefore AD$, BE, and CF are concurrent.	9. § 356.

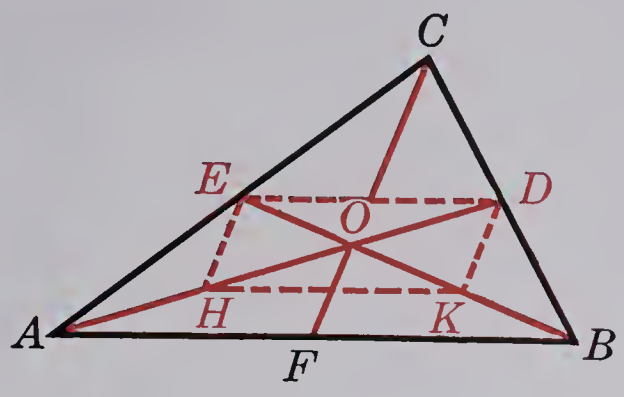
361. Orthocenter

The point where the altitudes of a triangle are concurrent is called the *orthocenter* of the triangle. "Ortho" is from the Greek word *orthos*, meaning "straight," or "right."

In mathematics, words which begin with "ortho" refer to right angles. For example, orthogonal circles are circles which intersect so that tangents to the circles at a point of intersection are perpendicular to each other.

★ Theorem 60

362. The medians of a triangle are concurrent in a point which lies two thirds of the distance from each vertex to the midpoint of the opposite side.



Given $\triangle ABC$ with the medians AD, BE, and CF.

To prove that AD, BE, CF are concurrent in a point which is two thirds of the distance from each vertex to the midpoint of the opposite side.

Planning the Proof: 1. We can prove line segments = by §§ 97, 157, 217, 308, 309, 343, 345.
2. We shall use § 217 c

Proof

STATEMENTS	REASONS
1. AD and BE will meet in some point, such as O.	1. Prove indirectly.
2. Draw HK joining the midpoints, H and K, of the line segments AO and BO respectively.	2. Why possible?
3. Draw EH, ED, and DK.	3. Why possible?
4. In $\triangle AOB$, $HK \parallel AB$ and $HK = \frac{1}{2} AB$.	4. §§ 221 b, 218 a.
5. In $\triangle ABC$, $ED \parallel AB$ and $ED = \frac{1}{2} AB$.	5. Why?
6. $\therefore HK = ED$ and $HK \parallel ED$.	6. Why?
7. $\therefore HKDE$ is a $\square$.	7. Why?
8. $\therefore HO = OD$ and $EO = OK$.	8. § 217 c.
9. $\therefore AH = HO = OD$ and $EO = OK = KB$.	9. Why?
10. $\therefore AD$ and BE intersect in a point $\frac{2}{3}$ the distance from A to D and from B to E respectively.	10. Why?
11. Likewise, AD and CF intersect in a point which lies $\frac{2}{3}$ the distance from A to D and C to F respectively.	11. Reasons 2–10.
12. $\therefore CF$ passes through O.	12. Why?
13. $\therefore AD$, BE, and CF are concurrent in a point which lies two thirds of the distance from each vertex to the midpoint of the opposite side.	13. Why?

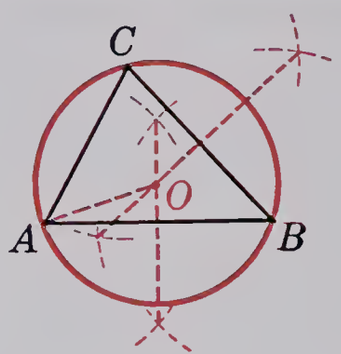
363. Centroid

The point where the medians of a triangle are concurrent is called the *centroid* of the triangle. It is the center of gravity of the triangle. In the drawing shown here the teacher is balancing a triangular shaped board on his index finger. Before class time he drew two medians of the triangle to find the centroid. He is balancing the board by supporting it at the centroid.

If a triangular piece of board of uniform thickness and material is suspended from any vertex, the centroid of the triangle will lie on a plumb line through the vertex. How would you use a plumb line to find the centroid of the triangular board?

★★ Problem 18

364. To circumscribe a circle about a given triangle.



(The solution is left to the student. See § 356.)

365. Circumcircle

The radius of the circle circumscribed about a triangle is called the *circumradius* of the triangle and the circle the *circumcircle* of the triangle. In the figure for § 364, OA is the circumradius, and the circle with center at O the circumcircle, of triangle ABC. O is the circumcenter of triangle ABC. (See § 357.)

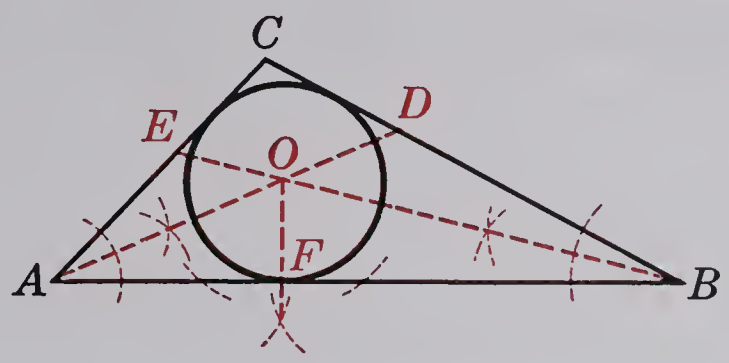
★ **366. Corollary I.** Through three points not in a straight line one circle, and only one, can be drawn. (The three points can be considered as the vertices of a triangle.)

367. Corollary II. Two circles cannot intersect in more than two points.

368. Corollary III. A straight line cannot intersect a circle in more than two points. (This corollary is the same as Assumption 27.)

★★ Problem 19

369. To inscribe a circle in a given triangle.



(The solution is left to the student. See § 358.)

370. Incircle

The radius of the circle inscribed in a triangle is called the *inradius* of the triangle, and the circle the *incircle* of the triangle. In the figure for § 369, OF is the inradius, and the circle with its center at O the incircle, of triangle ABC. O is the incenter of triangle ABC. (See § 359.)

371. Segment of a Circle

A *segment of a circle* is the figure formed by an arc of a circle and its chord. Notice that the definitions for the segment of a circle and the segment of a line do not correspond. A part of a line is called a line segment; a part of a circle is called an arc. A segment of a circle is a figure composed of both a part of a circle and a part of a line.

EXERCISES — Constructions

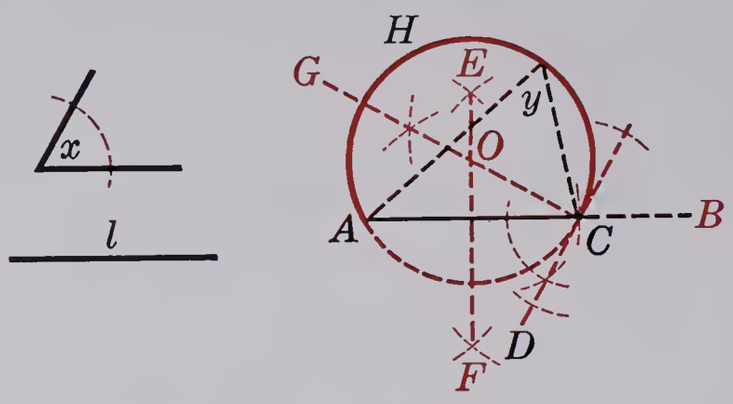
A

1. Given a side, construct an equilateral triangle and circumscribe a circle about it.

2. Construct a 30°-60° right triangle and inscribe a circle in it. Circumscribe a circle about the triangle.

Problem 20

372. Upon a given line segment as a chord to construct an arc of a circle in which a given angle can be inscribed.



Given line segment l and $\angle x$.

To construct on l as a chord an arc of a circle in which $\angle x$ can be inscribed.

Construction

STATEMENTS	REASONS
1. On any line AB construct $AC = l$.	1. Why possible?
2. At C construct $\angle ACD = \angle x$.	2. Why possible?
3. Construct EF, the $\perp$ bisector of AC.	3. Why possible?
4. Construct $CG \perp CD$ at C.	4. Why possible?
5. CG will intersect EF at some point, such as O.	5. § 115.
6. With O as center and OC as radius draw a $\odot$. *Then AHC is the required arc of a $\odot$.*	6. Why?

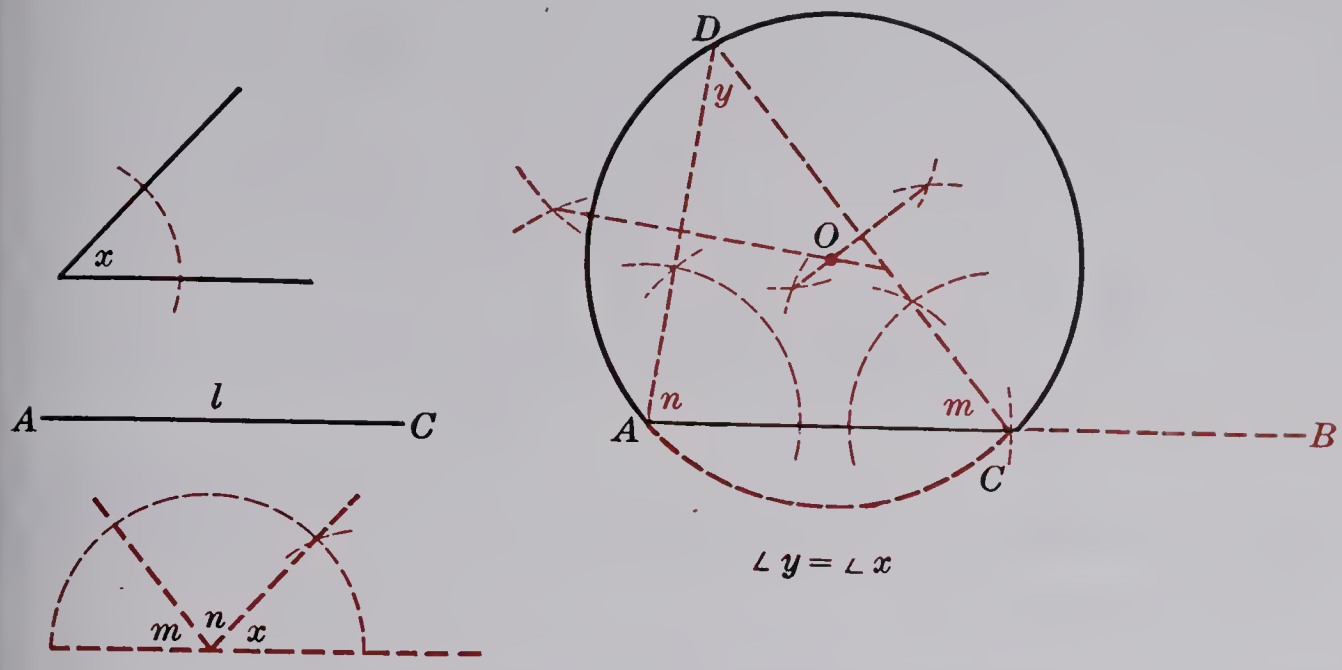
Proof

1. Inscribe any $\angle$ as $\angle y$ in the arc AHC.	1. Why possible?
2. $\angle ACD \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{AC}$.	2. Why?
3. $\angle y \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{AC}$.	3. Why?
4. $\therefore \angle y \stackrel{\circ}{=} \angle ACD$.	4. Why?
5. But $\angle x \stackrel{\circ}{=} \angle ACD$.	5. Why?
6. $\therefore \angle x = \angle y$.	6. Why?

EXERCISES

1. Why will all angles that are inscribed in $\overset{\frown}{AHC}$ equal $\angle x$?
2. What do you know about $\overset{\frown}{AHC}$ when $\angle x$ is acute?

Construction of Problem 20 by a high-school pupil.



Construction. 1. On any line AB construct $AC = l$. 2. Construct an angle $= \angle x$. 3. Separate the supplement of $\angle x$ into $\angle m$ and $\angle n$. 4. Construct $\triangle ACD$ (§ 170). 5. Circumscribe a circle about $\triangle ACD$. Then $\overset{\frown}{ADC}$ is the required arc.

373. Notation in a Triangle

It is convenient to denote certain points and lines connected with the triangle by definite letters. In a $\triangle ABC$, the sides opposite $\measuredangle A$, B, and C are denoted by a, b, and c respectively; the altitudes by h_a, h_b, h_c; the medians by m_a, m_b, m_c; the bisectors of the angles by t_a, t_b, t_c; and the radii of the circumcircle and incircle by R and r respectively. In a right triangle, $\angle C$ denotes the right angle and c the hypotenuse.

374. Analysis of a Problem

The *analysis* of a problem consists in an assumption that the problem is solved, the drawing of a figure approximately satisfying the given conditions, and an investigation of the relation among the given and the unknown parts.

375. Loci in Construction

In many cases the solution of a construction problem depends on the finding of a point which satisfies certain conditions. A thorough knowledge of the theorems on loci often makes it possible to discover immediately where the required point is to be located.

Example. Construct a right triangle, given the hypotenuse and the altitude on the hypotenuse.

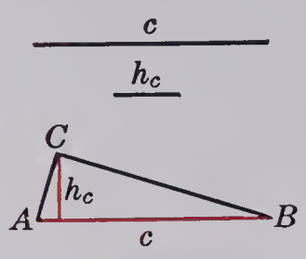


Fig. 1

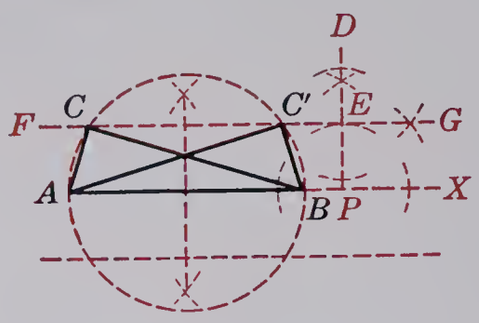


Fig. 2

Given the hypotenuse c and the altitude h_c.

To construct the rt. $\triangle$.

Analysis. 1. The drawing when completed will appear as in Fig. 1.

2. Since h_c is given and we know it is $\perp AB$, the vertex of the rt. $\angle$ must lie on a line $\parallel AB$ at the distance h_c from it. (§ 341, Asmt. 48.)

3. We know that the locus of the vertex of a rt. $\triangle$ with a given hypotenuse as the base is a $\odot$ upon the hypotenuse as a diameter. (§ 346.)

4. $\therefore C$ will lie on the intersection of two loci.

Construction

STATEMENTS	REASONS
1. On any line AX construct $AB = c$.	1. Why possible?
2. Construct a $\odot$ on AB as a diameter.	2. Why possible?
3. At any point on AX, as P, construct $PD \perp AX$.	3. Why possible?
4. On PD construct $PE = h_c$.	4. Why possible?
5. Through E construct $FG \parallel AX$.	5. Why possible?
6. Let C and C' be the points where the $\odot$ and FG intersect.	6. § 368.
7. Join C to A and B, and C' to A and B. *Then either* $\triangle ABC$ or $\triangle ABC'$ *is the required* $\triangle$.	7. Why possible?

Proof

In $\triangle ABC$ or $\triangle ABC'$,	
1. AB is the required hypotenuse.	1. Const.
2. PE is the required altitude.	2. Const.
3. $\angle ACB$ or $\angle AC'B$ is a rt. $\angle$.	3. § 319.

Discussion.

1. The construction shows two right triangles having $h_c < \frac{1}{2}c$.

2. If the complete locus of step 2 in the analysis had been drawn in the construction, two other right triangles fulfilling the given conditions could have been drawn.

3. If $h_c = \frac{1}{2}c$, there would be two right triangles, one above AB and one below AB.

4. If $h_c > \frac{1}{2}c$, there would be no solution. Why?

EXERCISES — Construction

A

Construct a right triangle, given

1. The altitude on the hypotenuse and one acute angle.
2. An acute angle and the radius of the circumcircle.
3. A leg and the altitude on the hypotenuse.
4. A leg and the radius of the circumcircle.

Construct an isosceles triangle, given

5. The altitude on the base and a base angle.
6. One of the equal sides and the altitude on it.

Construct an isosceles triangle, given

7. The base and the radius of the circumcircle.
8. The base and the radius of the incircle.

Construct a triangle, given

9. The base, altitude, and an angle at the base.
10. The base, altitude, and one of the other sides.
11. One side and the altitude and median on it.
12. Two sides and the altitude on the third side.
13. The base, the median on the base, and one of the other sides.

Construct a parallelogram, given

14. One angle, one side, and the altitude on that side.

Construct an isosceles trapezoid, given

15. One base, the diagonal, and the angle included by them.

Construct a circle that

16. Has a given radius and is tangent to a given circle at a given point.

17. Is tangent to a given line at a given point and has its center on another given line.

18. Is tangent to two given intersecting lines, to one of them at a given point.

B

Construct a right triangle, given

19. Both the altitude and the median on the hypotenuse.

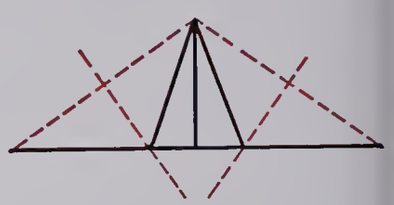
20. A leg and the radius of the incircle.

Construct an isosceles triangle, given

21. The vertex angle and the radius of the circumcircle.

22. The base and the altitude on one of the equal sides.

23. The perimeter and the altitude on the base. (See the figure at the right.)



24. The vertex angle and the radius of the incircle.

Construct an equilateral triangle, given

25. The radius of the incircle.

26. The radius of the circumcircle.

Construct a triangle, given

27. Two angles and the altitude on the side included by the angles.

28. One side, an adjacent angle, and the radius of the circumcircle.

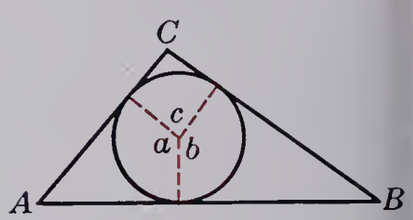
29. An angle, the bisector of the angle, and the altitude on a side adjacent to the angle.

30. The midpoints of the three sides.

SUGGESTION. Join the three points and see § 204.

31. Two angles and the radius of the incircle.

SUGGESTION. $\angle A + \angle a = 180°$.



Ex. 31

32. The three medians.

33. One side, an adjacent angle, and the radius of the incircle.

34. Construct an isosceles trapezoid, given the bases and diagonal.

35. Construct a circle that is tangent to two given intersecting lines and has a given radius.

C

Construct a right triangle, given

36. The segments of the hypotenuse made by the bisector of the right angle.

37. An acute angle and the radius of the incircle.

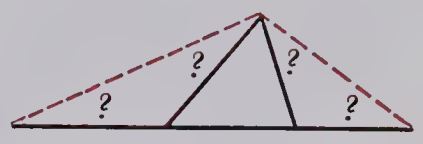
38. The sum of the legs and an acute angle.

Construct a triangle, given

39. The base, the median on the base, and the vertex angle.

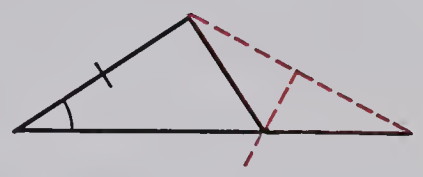
40. The base, the angle at the vertex, and one of the other sides.

41. Two angles and the perimeter. (See the figure at the right.)



42. The base, altitude, and vertex angle.

43. One side and the two medians on the other two sides.



44. One side, an adjacent angle, and the sum of the other two sides. (See the figure at the right.)

Construct a triangle, given

45. One side, an adjacent angle, and the difference of the other two sides

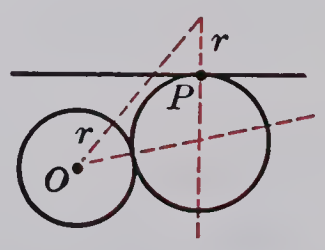
46. Two sides and the median on the third side.

SUGGESTION. The diagonals of a $\square$ bisect each other.

Construct a circle that

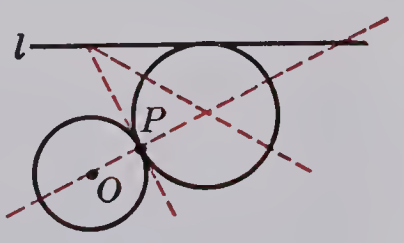
47. Has a given radius and is tangent to two given unequal circles.

48. Is tangent to two given parallel lines and passes through a given point between the parallels.



49. Is tangent to a given circle and also is tangent to a given line at a given point. (In the figure, O is the given circle.)

50. Is tangent to a given circle at a given point and passes through a given external point.



51. Is tangent to a given line and is also tangent to a given circle at a given point. (In the figure, O is the given circle.)

52. Is tangent to two equal circles and has a given radius.

53. Construct a line tangent to a given circle and perpendicular to a given line.

54. Construct a line tangent to a given circle and parallel to a given line.

55. Construct a line passing through a given point and making equal angles with two given lines.

56. Given the three radii, construct three circles, each tangent externally to the other two.

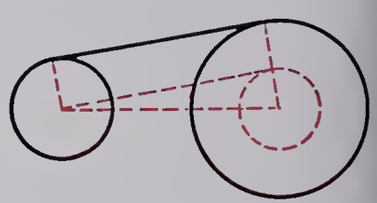
57. Construct four equal circles inside a square, each tangent to one side of the square and to two of the circles.

58. Through one vertex of a triangle draw two lines, each of which is equidistant from the other two vertices.

59. Construct the bisector of the angle between two given nonparallel lines without extending them to their point of intersection.

60. Construct a right triangle, given the radii of the incircle and the circumcircle.

61. Construct a common external tangent to two given circles. (See the figure at the right.)



62. Construct a common internal tangent to two given circles.

63. Three lines are concurrent. Construct a line segment terminating in the outer two and bisected by the inner one.

64. Inscribe in a given circle a rectangle with one side equal to a given line segment.

65. Inscribe a circle in a given square.

66. Inscribe a circle in a given rhombus.

67. In a given sector of a circle inscribe another circle.

68. In a given circle inscribe three equal circles each tangent to two others and to the given circle.

69. Construct a tangent to a given circle without using any points or lines within the circle.

70. Determine the locus of the orthocenters of acute triangles with a given base and a given vertex angle.

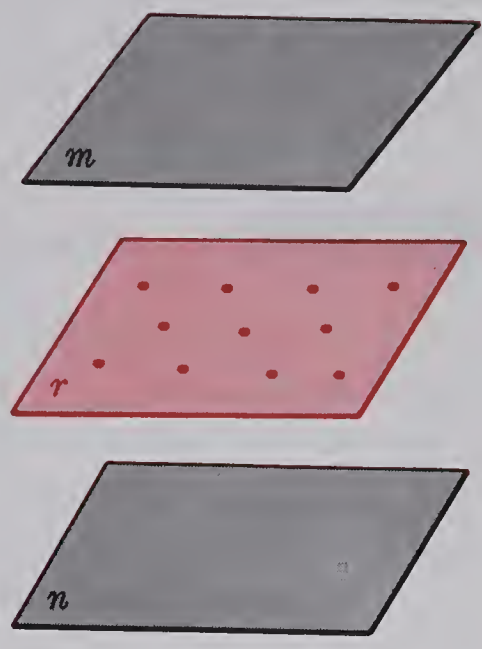
71. Determine the locus of the centroids of triangles with a given base and a given vertex angle.

SPACE GEOMETRY

[Optional]

376. Loci in Space

The loci we have studied have consisted of points and lines in a plane. In space a locus may also include surfaces. The more common surfaces are planes, spheres, and cylinders. In space geometry we cannot actually construct points fulfilling a given condition but must visualize where they lie.



Example. What is the locus of points in space equidistant from two parallel planes m and n?

Solution. If we think of the top of a table and the ceiling, and then visualize several points midway between the two, we conclude that the locus is a plane r parallel to m and n and midway between them.

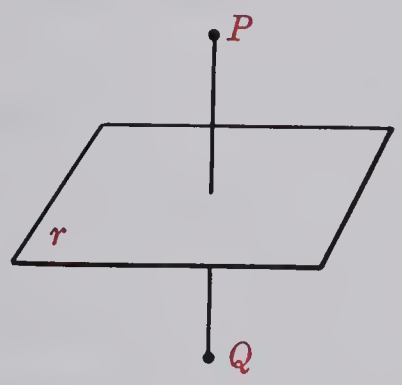
EXERCISES — Space Loci

State without proof the following loci:

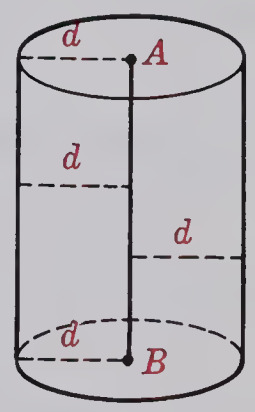
1. The locus of points in space equidistant from two given points P and Q.

2. The locus of points in space at a given distance from a given point.

3. The locus of points in space at a given distance d from a given line AB.



Ex. 1



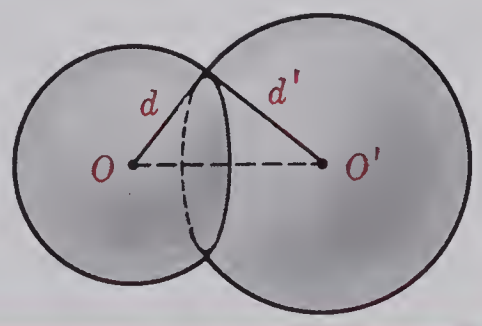
Ex. 3

4. The locus of points in space at a given distance from a given plane.

5. The locus of points in space equidistant from all points on a circle.

6. The locus of points in space equidistant from the vertices of a triangle.

7. The locus of points in space at distances d and d' from two given points O and O'.



377. Indirect Reasoning [Optional]

In solving problems by indirect reasoning consider all the possible conclusions, then one by one eliminate those that are impossible. The remaining conclusion will be the correct one.

EXERCISES Optional

1. One Sunday a farmer said that he would have to go to town on an afternoon of one of the remaining days of the week. He said that he would have to go on a day that he could deliver a load of hogs at the stockyards, go to his bank, and visit his dentist. The dentist's office was open only on Monday, Wednesday, and Friday. The stockyards were not open Monday and Saturday. The bank was not open Wednesday and Saturday afternoons. What day did the farmer go to town?

2. Four couples, Mr. and Mrs. Smith, Mr. and Mrs. Jones, Mr. and Mrs. Brown, and Mr. and Mrs. White, went into a restaurant for their dinner. They were seated at two square tables, four at each table, so that no husband and wife were at the same table. Mr. Brown and Mr. White did not sit at the same table. Mrs. Jones sat opposite Mr. Brown, and Mrs. Smith sat next to Mr. White. Mr. White sat opposite his daughter. Show by a diagram how they were seated at the two tables.

3. On a certain passenger train three passengers and the conductor engaged in a conversation. The following facts were given:

a. The names of the passengers were Smith, Jones, and Robinson.

b. The conductor said that these names were the names of his crew, which consisted of engineer, fireman, and brakeman.

c. Mr. Robinson said that he lived in Detroit.

d. The conductor said that his engineer lived halfway between Chicago and Detroit.

e. Mr. Jones said that he earned $8000 a year.

f. The conductor said that his fireman always lost at billiards to Smith, one of the crew members.

g. One of the three passengers lived next door to the engineer and received exactly three times as much salary as the brakeman.

h. The engineer had the same name as the passenger living in Chicago.

What is the brakeman's name? Give proof.

REVIEW QUESTIONS — Chapter 11

1. What is the definition of a locus of points?

2. Name the steps in determining a locus.

3. State the three assumptions on loci.

4. What is the locus of points equidistant from the sides of an angle?

5. What is the locus of points equidistant from two given points?

6. What is the locus of the vertex of a right triangle having a fixed hypotenuse as a base?

7. Complete: If the incenter and circumcenter of a triangle coincide, the triangle is __?__.

8. Complete: Concentric circles have the same __?__.

9. What is the locus of the centers of all circles tangent to a given line at a given point?

10. What are the two steps necessary in proving that a geometric figure is a locus?

11. What is a compound locus?

12. In this chapter you have studied four centers of a triangle. Name and define each.

13. How many circles can be drawn through three points not in a straight line?

14. In how many points may two circles intersect?

15. Define incircle and circumcircle.

16. What figure is formed by the arc of a circle and its chord?

17. Into what ratio does the centroid of a triangle divide a median?

18. On what line is the distance from a point to a circle measured?

19. What is the locus of points equidistant from two equal circles that are tangent externally?

20. What is the locus of the midpoint of the leg BC of a right triangle ABC whose hypotenuse is fixed?

21. Find the inradius of equilateral $\triangle ABC$ if $AB = 8$ inches.

SUMMARY of Principal Bases for Proof

378. Loci

a. The locus of points within an angle equidistant from the sides is the bisector of the angle.

b. The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.

c. The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.

d. The locus of the vertex of a right triangle with a given hypotenuse as the base is the circle upon the hypotenuse as a diameter.

e. The locus of the centers of all circles tangent to a given line at a given point is the perpendicular to the line at that point.

379. Concurrent line theorems

a. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

b. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.

c. The altitudes of a triangle are concurrent.

d. The medians of a triangle are concurrent in a point which lies two thirds of the distance from each vertex to the midpoint of the opposite side.

380. Constructions

a. To circumscribe a circle about a given triangle.

b. To inscribe a circle in a given triangle.

c. Upon a given line segment as a chord to construct an arc of a circle in which a given angle can be inscribed.

WORD LIST

Can you spell and use each of the following words?

centroid	circumradius	incenter	loci
circumcenter	concentric	incircle	locus
circumcircle	concurrent	inradius	orthocenter

TEST 28 True-False Statements

[10 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number.

1. The locus of points 3 inches from a given point P is a circle with the given point as center and a radius of 3 inches.

2. The locus of points 5 inches from a given line is two lines each parallel to the given line and 5 inches from the given line.

3. The locus of points equidistant from two given parallel lines is a line perpendicular to each of the given lines.

4. The locus of points equidistant from two given points is two circles whose centers are the given points and whose radii are equal to one half the line segment joining the given points.

5. The locus of the centers of all circles tangent to both sides of an angle is the bisector of the angle.

6. The circumcenter of a right triangle is a point on the hypotenuse.

7. The locus of the midpoints of all chords of a given circle O passing through a given point P within the circle is a circle whose diameter is PO.

8. The locus of the centers of all circles tangent to a given line at a given point is the perpendicular to the line at that point.

9. The medians of a triangle are concurrent in a point that is equidistant from the three vertices of the triangle.

10. Only one circle may be drawn through 3 points not in a straight line.

TEST 29 Constructions

[25 Minutes]

Make the following geometric constructions as accurately as you can:

1. Construct the medians of a given triangle.

2. Construct a circle through three given points not in a straight line.

3. Construct the locus of points equidistant from two given points.

4. Construct a circle tangent to the sides of a given triangle.

5. With a given radius, construct a circle tangent to two given unequal circles which are tangent externally.

CHAPTER 12

Proportion; Proportional Line Segments

In this chapter
you will learn more about proportion
and you will see
how line segments can be compared
by means of ratios and proportions.

381. Ratios

You know that the ratio of one number to another is the quotient obtained by dividing the first by the second.

For example, the ratio of 3 to 4 is $\frac{3}{4}$, or 3 : 4. In this book a ratio is usually written in the form of a fraction.

382. Proportion

A *proportion* is an equation whose two members are ratios. Thus $\frac{x}{3} = \frac{5}{9}$ is a proportion. The proportion $\frac{x}{3} = \frac{5}{9}$ may be read "x divided by 3 equals 5 divided by 9" or "x is to 3 as 5 is to 9." The word "as" in this case means "equals."

383. Terms of a Proportion

The four quantities which form a proportion are called its terms. Since the proportion $\frac{a}{b} = \frac{c}{d}$ may be written $a : b = c : d$, a is called the *first term*, b the *second term*, c the *third term*, and d the *fourth term*. The second and third terms, being in the middle, are called the *means*, and the first and fourth terms are called the *extremes*.

384. Fourth Proportional

The *fourth proportional* to three given quantities is the fourth term of the proportion whose first three terms are the three quantities taken in order. Thus in the proportion $\frac{a}{b} = \frac{c}{d}$, d is the fourth proportional to a, b, and c.

385. Mean Proportional and Third Proportional

When the means of a proportion are equal, either mean is said to be the *mean proportional* between the first and fourth terms of the proportion. Also, the fourth term is said to be the *third proportional* to the first and second terms.

Thus in the proportion $\frac{a}{b} = \frac{b}{c}$, b is the mean proportional between a and c, and c is the third proportional to a and b.

EXERCISES

A

Express each of the following ratios in lowest terms:

1. $\frac{8}{12}$.
2. $\frac{10}{25}$.
3. $\frac{m^2}{2m}$.
4. $\frac{3xy}{12x^2y}$.
5. $15abc : 18b^2$.
6. $\frac{x^2-7x+12}{4x^2-12x}$.

Which of the following are proportions?

7. $\frac{4}{5}=\frac{12}{15}$.
8. $\frac{5}{7}=\frac{25}{35}$.
9. $\frac{3}{4}=\frac{9}{16}$.
10. $\frac{11}{12}=\frac{10}{11}$.
11. $\frac{ab}{b^2}=\frac{a}{b}$.
12. $\frac{x}{1}=\frac{6x}{6}$.

13. Is 6 a ratio? Is $\frac{6}{1}$ a ratio?

Find the value of x in each of the following proportions:

14. $\frac{x}{12}=\frac{4}{3}$.
15. $\frac{8}{x}=\frac{24}{9}$.
16. $\frac{x+3}{4}=\frac{7}{5}$.
17. $\frac{x}{5-x}=\frac{7}{8}$.
18. $\frac{2x-1}{3}=\frac{3}{1}$.
19. $\frac{3x+2}{5}=\frac{x-4}{4}$.
20. $\frac{x}{a}=\frac{b}{c}$.
21. $\frac{x}{4}=\frac{4}{x}$.
22. $\frac{bx}{3}=\frac{b}{6}$.

23. Tell what are the first, second, third, and fourth terms of the proportion $\frac{3}{4}=\frac{6}{8}$; of the proportion $(a+b):(c+d)=m:n$; of the proportion $(a^2+1):a=(a-1):5$.

24. Tell what terms are the means and what terms are the extremes in $\frac{5}{6}=\frac{15}{18}$; in $\frac{4x}{5}=\frac{7}{8}$; in $\frac{a^2}{b}=\frac{c}{d}$.

25. Write a proportion stating that x is the mean proportional between 4 and 9.

26. Write the proportion stating that 32 is the third proportional to 2 and 8.

27. Solve for x:

a. $1:7=4:x$.
b. $7:21=x:6$.
c. $5:(x-3)=2:(2x+1)$.
d. $12.5:x=x:2$.

Fundamental theorems on proportion

386.* In a proportion the product of the extremes is equal to the product of the means.

Given $\frac{a}{b} = \frac{c}{d}$.

To prove that $ad = bc$.

Proof

STATEMENTS	REASONS
1. $\frac{a}{b} = \frac{c}{d}$.	1. Given.
2. $bd \times \frac{a}{b} = bd \times \frac{c}{d}$, or $ad = bc$.	2. Asmt. 3.

387. If the product of two quantities is equal to the product of two other quantities, either pair may be made the means of a proportion and the other pair the extremes.

Given $ad = bc$.

To prove that $\frac{a}{b} = \frac{c}{d}$.

SUGGESTION. By what must you divide both members of $ad = bc$ to obtain $\frac{a}{b} = \frac{c}{d}$?

388. If the numerators of a proportion are equal, the denominators are equal; and conversely.

Given $\frac{a}{x} = \frac{b}{y}$ and $a = b$.

To prove that $x = y$.

SUGGESTION. Use § 386. The converse is left to the student.

389. If three terms of one proportion are equal respectively to the corresponding terms of another proportion, the remaining terms are equal.

Given $\frac{a}{b} = \frac{c}{d}$, $\frac{w}{x} = \frac{y}{z}$, $a = w$, $b = x$, and $c = y$.

To prove that $d = z$.

Proof: (The proof is left to the student.)

*If desired, the proofs of these theorems on proportion may be omitted.

390. In a series of equal ratios, the sum of the numerators is to the sum of the denominators as any numerator is to its denominator.

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

To prove that $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Proof

STATEMENTS	REASONS
1. Let $\frac{a}{b}=r$. Then $\frac{c}{d}=r$, and $\frac{e}{f}=r$.	1. Why?
2. Then $a=br$, $c=dr$, and $e=fr$.	2. Why?
3. $a+c+e=(b+d+f)r$.	3. Why?
4. $\therefore \frac{a+c+e}{b+d+f}=r$.	4. Why?
5. Then $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.	5. Why?

A proportion such as $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ is said to be a continued proportion.

391.* In a proportion the means may be interchanged; that is, in a proportion the terms are in proportion by alternation.

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{a}{c}=\frac{b}{d}$.

SUGGESTION. First use § 386, and second, § 387.

392. In a proportion the ratios may be inverted.

That is, the second term is to the first term as the fourth term is to the third term.

Given $\frac{a}{b}=\frac{c}{d}$.

To prove that $\frac{b}{a}=\frac{d}{c}$.

Proof: (The proof is left to the student.)

*Also, the extremes of a proportion can be interchanged.

393. In a proportion the terms are in proportion by addition or subtraction.

That is, the sum (or difference) of the first and second terms is to the second term as the sum (or difference) of the third and fourth terms is to the fourth term.

Given $\frac{a}{b}=\frac{c}{d}$.

To prove that $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$.

SUGGESTIONS. $\frac{a}{b}+1=\frac{c}{d}+1$. Why? Then $\frac{a+b}{b}=\frac{c+d}{d}$. Why?

EXERCISES — Proportions

A

1. If the second and third terms of a proportion are alike, what is each of these terms called?

2. Name the extremes and the means in the proportion $m : n = s : t$.

3. Can you suggest any way of remembering which terms of a proportion form the means? the extremes?

4. Solve the following proportions for x:

a. $\frac{4}{x}=\frac{5}{7}$. **c.** $\frac{x-5}{x+3}=\frac{1}{5}$. **e.** $\frac{x^2-6}{x^2}=\frac{1}{x}$.

b. $\frac{2x}{3}=\frac{7}{8}$. **d.** $\frac{4}{x}=\frac{x}{16}$. **f.** $\frac{a+b}{b}=\frac{a+x}{x}$.

5. If $\frac{m}{n}=\frac{r}{s}$ and $\frac{m}{n}=\frac{r}{t}$, why does $s=t$?

6. Find the mean proportional between 9 and 16; between 7 and 63.

7. Find the fourth proportional to

a. 2, 3, and 4. **b.** 4, 6, and 10. **c.** $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. **d.** m, n, and p.

8. From each of the following proportions, form another proportion by inversion:

a. $\frac{2}{5}=\frac{14}{35}$. **c.** $\frac{14}{9}=\frac{35}{22.5}$. **e.** $\frac{a+b}{b}=\frac{c+d}{d}$.

b. $\frac{8}{3}=\frac{24}{9}$. **d.** $\frac{m}{n}=\frac{s}{t}$. **f.** $x : 3 = y : 5$.

9. From each of the proportions in Ex. 8, form another proportion (a) by interchanging the means; (b) by addition; (c) by subtraction.

EXERCISES Ratios

A

1. Using § 387, find the ratio of x to y in each of the following:

a. $2x = 3y$ c. $4x = y$ e. $\frac{3}{5}x = \frac{7}{8}y$ g. $x(a+b) = y(c+d)$

b. $5x = 7y$ d. $x = \frac{1}{5}y$ f. $ax = by$ h. $mx + ny = rx + sy$

2. Find the ratio of x to y if

a. $3x = 4y$. b. $mx = 5y$. c. $x = 4y$. d. $y = 7x$.

394. Commensurable and Incommensurable Quantities

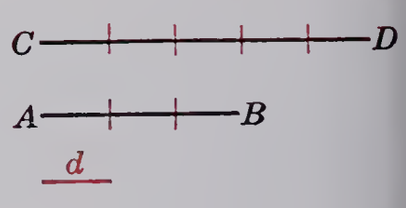
Two quantities of the same kind are *commensurable* if there is a common unit of measure that is contained in each of them an integral number of times. Thus 2 feet and 3 feet 7 inches are commensurable, since 1 inch is contained in each of them an integral number of times; 5 inches and $2\frac{7}{8}$ inches are commensurable, since $\frac{1}{8}$ inch is contained 40 times in 5 inches and 23 times in $2\frac{7}{8}$ inches.

Two quantities of the same kind are *incommensurable* when there is no common unit of measure which is contained in each of them an integral number of times. Thus 4 and $\sqrt{3}$ are incommensurable, since there is no number which is contained in each of them an integral number of times.

NOTE. In this text we prove theorems for the commensurable case only, and assume that the theorems are true for the incommensurable case. You will find them proved in more advanced mathematics courses.

395. Ratio of Geometric Quantities

The *ratio of two geometric quantities* of the same kind is equal to the ratio of their numerical measures. In the figure the unit of measure d is contained exactly 3 times in AB and 5 times in CD. Therefore the ratio $\frac{AB}{CD} = \frac{3}{5}$.



396. Line Segments in Proportion

Two line segments are *divided proportionally* when the segments of one have the same ratio as the corresponding segments of the other. Thus in the figure of Ex. 6 on page 387, CA and CB are divided proportionally if $\frac{CD}{DA} = \frac{CE}{EB}$, or $\frac{DA}{DC} = \frac{EB}{EC}$.

EXERCISES — Ratios

A

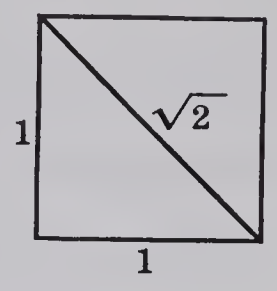
1. Name three common units of measure which are contained an integral number of times in each of the following:

a. 4 feet and 12 feet.
b. 8 centimeters and 6 centimeters.
c. $\frac{1}{2}$ foot and $\frac{1}{3}$ foot.
d. 4.3 inches and 5.4 inches.

2. Find a common unit of measure of

a. 1 inch and 1.4 inches.
b. 1 inch and 1.41 inches.
c. 1 inch and 1.414 inches.
d. 1 inch and 1.4142 inches.

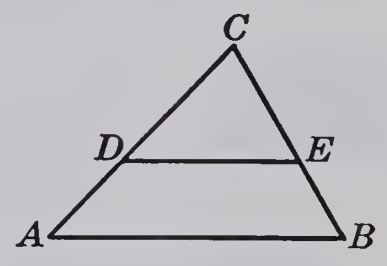
3. The diagonal of a square 1 inch on a side is $\sqrt{2}$ inches. Are the side and diagonal of a square commensurable?



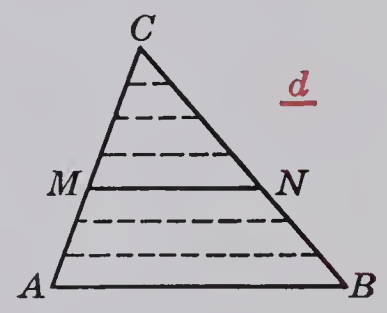
4. Are $\sqrt{2}$ and $\sqrt{3}$ commensurable?

5. Are $\sqrt{4}$ and 3 commensurable?

6. In $\triangle ABC$ at the right, $\frac{CD}{DA} = \frac{CE}{EB}$. Transform this proportion by alternation; by inversion. Transform it by addition and then substitute single line segments in the numerators.



7. In $\triangle ABC$, MN is drawn parallel to AB; d is contained four times in MC and three times in AM. To what numerical ratio is $\frac{MC}{AM}$ equal? The dash lines are parallel to AB. Are they parallel to each other? Why? Are the seven segments of BC equal to each other? Why? To what numerical ratio is $\frac{NC}{BN}$ equal? Is $\frac{MC}{AM} = \frac{NC}{BN}$? Why? Are the segments of BC equal to the segments of AC?



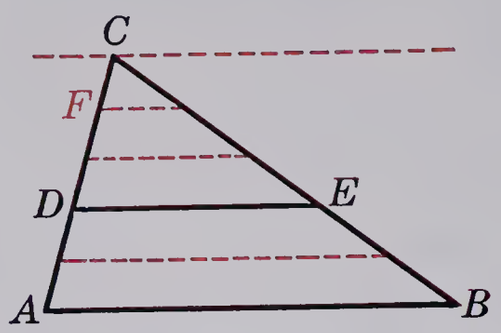
8. In the figure of Ex. 7, what is the ratio of $\frac{CM}{CA}$? of $\frac{CN}{CB}$? of $\frac{AM}{AC}$? of $\frac{BN}{BC}$?

9. In the figure of Ex. 7, show that $\frac{CM}{CA} = \frac{CN}{CB}$.

10. If $\frac{x-3}{4} = \frac{c}{2}$, then $\frac{x+1}{4} = \frac{c+2}{2}$. Why?

★★ Theorem 61

397. If a line is parallel to one side of a triangle and intersects the other two sides, it divides these sides proportionally.



Given $\triangle ABC$ with $DE \parallel AB$.

To prove that $\frac{CD}{DA} = \frac{CE}{EB}$.

Planning the Proof: 1. We can prove line segments proportional by § 396.
2. We shall use § 396.

Proof

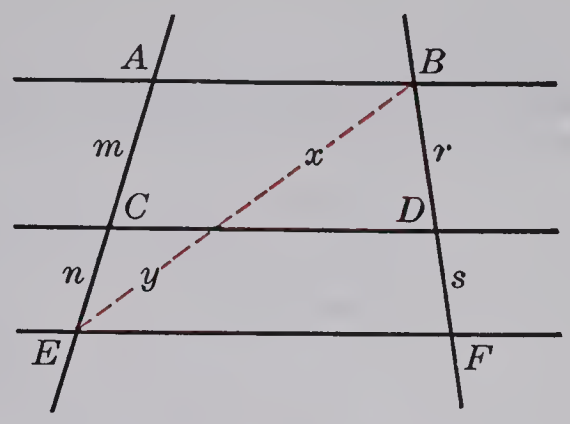
STATEMENTS	REASONS
1. Let CF be a common unit of measure that is contained 3 times in CD and 2 times in DA.	1. § 394.
2. Then $\frac{CD}{DA} = \frac{3}{2}$.	2. § 395.
3. Through C and the points of division on CA draw lines $\parallel AB$.	3. Why possible?
4. These lines divide CE into 3 parts and EB into 2 parts, all $=$.	4. Why?
5. Then $\frac{CE}{EB} = \frac{3}{2}$.	5. Why?
6. $\therefore \frac{CD}{DA} = \frac{CE}{EB}$.	6. Asmt. 7.
7. $\therefore AC$ and BC are divided proportionally.	7. § 396.

NOTE. The proof is the same if any numbers other than 3 and 2 are used.

398. Corollary I. If a line is parallel to one side of a triangle and intersects the other two sides, it divides these sides so that either side is to one of its segments as the other side is to its corresponding segment.

SUGGESTION. Use §§ 397 and 393.

399. Corollary II. Parallel lines intercept proportional segments on two transversals.



SUGGESTIONS. Draw BE.

Does $\frac{m}{n}=\frac{x}{y}$? Does $\frac{r}{s}=\frac{x}{y}$? Does $\frac{m}{n}=\frac{r}{s}$?

EXERCISES

A

1. In § 397, if CD is twice DA, what is true of CE and EB?

2. In § 397, if CD is two thirds of CA, what is true of CE and CB?

3. In § 397, if $CD = 6$ inches, $DA = 3$ inches, and $CE = 8$ inches, find EB.

4. In § 397, if $CA = 10$ feet, $CD = 6$ feet, and $CE = 7\frac{1}{2}$ feet, find CB.

5. In § 397, if $CE = 12$, $EB = 8$, and $CA = 15$, find CD and DA.

6. In § 399, if $AC = 5$, $CE = 8$, and $BD = 12$, find DF.

7. The three sides DF, EF, and DE of $\triangle DEF$ are 18 inches, 24 inches, and 30 inches respectively. The line parallel to the side DE intersects DF in the point S and the side EF in the point T. Find ET if $DS = 8$.

B

8. Determine the locus of the midpoints of line segments drawn from a given point to a given line.

9. The nonparallel sides of a trapezoid are 10 inches and 15 inches respectively. A line parallel to the bases divides the 10-inch side in the ratio of 1 to 4. Find the segments of the 15-inch side.

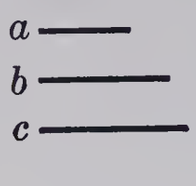
C

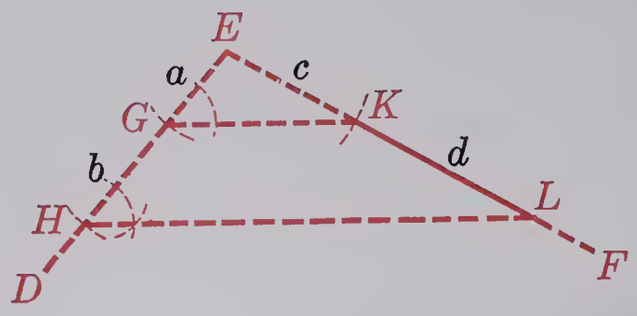
10. A line drawn through the centroid of a triangle and parallel to one side divides each of the other two sides into segments having the ratio of 2 to 1.

11. The nonparallel sides of a trapezoid are 6 inches and 9 inches respectively. A line parallel to the bases divides the longer diagonal in the ratio of 3 to 4. Find the segments of the nonparallel sides.

★★ Problem 21

400. To construct the fourth proportional to three given line segments.





Given the line segments a, b, and c.

To construct a line segment d so that $\frac{a}{b} = \frac{c}{d}$.

Construction

STATEMENTS	REASONS
1. Draw any $\angle DEF$; on ED construct $EG = a$ and $GH = b$; and on EF construct $EK = c$.	1. Why possible?
2. Draw GK.	2. Why possible?
3. Through H construct $HL \parallel GK$.	3. Why possible?

Then $\frac{a}{b} = \frac{c}{d}$.

Proof

1. $GK \parallel HL$.	1. Why?
2. $\frac{EG}{GH} = \frac{EK}{KL}$.	2. Why?
3. $\therefore \frac{a}{b} = \frac{c}{d}$.	3. Why?

EXERCISES Construction

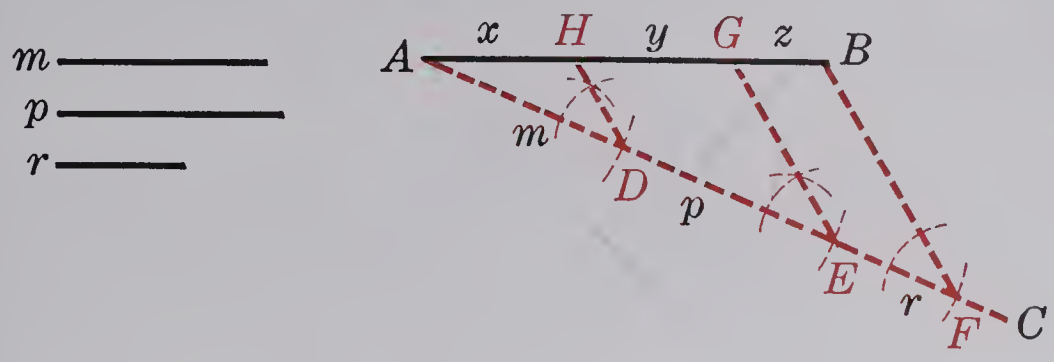
A

1. Construct the fourth proportional to three given line segments which are 2, 5, and 3 inches in length respectively. Measure the resulting segment and check by solving the proportion $\frac{2}{5} = \frac{3}{x}$ by algebra.

2. Given line segments a, b, and c, construct a line segment x so that $bx = ac$.

★ Problem 22

401. To divide a given line segment into n parts which are proportional to n given line segments.



Given the line segment AB and line segments m, p, and r.

To divide AB into segments x, y, and z so that $\frac{x}{m} = \frac{y}{p} = \frac{z}{r}$.

Construction

STATEMENTS	REASONS
1. Through A draw any line AC.	1. Why possible?
2. On AC construct $AD = m$, $DE = p$, and $EF = r$.	2. Why possible?
3. Draw FB.	3. Why possible?
4. Through E construct $EG \parallel FB$.	4. Why possible?
5. Through D construct $DH \parallel FB$.	5. Why possible?

Then $\frac{x}{m} = \frac{y}{p} = \frac{z}{r}$.

(The proof is left to the student. See § 399.)

NOTE. In the construction above, $n = 3$. The same method is used when n has other values.

EXERCISES

B

1. Divide a given line segment into two parts proportional to two given line segments m and n.

2. Divide a given line segment into parts proportional to 1, 2, and 3.

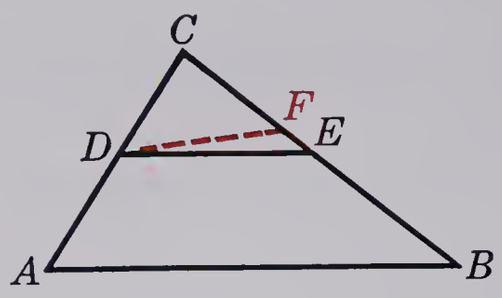
3. Divide the base of a triangle into segments proportional to the other two sides.

4. Trisect a given line segment.

5. Divide a line segment into segments having the ratio 2 to 5.

★★ Theorem 62

402. If a line divides two sides of a triangle proportionally, it is parallel to the third side.



Given $\triangle ABC$ with DE drawn so that $\frac{CD}{DA} = \frac{CE}{EB}$.

To prove that $DE \parallel AB$.

Planning the Proof: 1. We can prove lines $\parallel$ by §§ 160, 221.
2. We shall prove that DE coincides with a line parallel to AB.

Proof

STATEMENTS	REASONS
1. Through D draw $DF \parallel AB$, intersecting CB at F.	1. Why possible?
2. Then $\frac{CA}{DA} = \frac{CB}{FB}$.	2. § 398.
3. But $\frac{CD}{DA} = \frac{CE}{EB}$.	3. Given.
4. Then $\frac{CA}{DA} = \frac{CB}{EB}$.	4. § 393.
5. From (2) and (4), $FB = EB$.	5. § 389.
6. $\therefore F$ falls on E and DF coincides with DE.	6. Why?
7. $\therefore DE \parallel AB$.	7. Why?

★ **403. Corollary.** If a line divides two sides of a triangle so that either side is to one of its segments as the other side is to its corresponding segment, the line is parallel to the third side. (Use subtraction, § 393, and then § 402.)

EXERCISE

1. State the contrapositive of Theorem 62. Why is it true?

404. Internal and External Division of a Line Segment

A line segment is *divided internally* into two segments if the point of division lies on the line segment. In the figure below, P divides the line segment AB internally into two segments, AP and PB, whose ratio is 2 to 3.

A line segment is *divided externally* into two segments if the point of division lies on the line segment extended. In this figure, P divides the line segment AB externally into segments AP and PB, whose ratio is 3 to 7.

If P is between A and B, it divides AB internally; if P is not between A and B, it divides AB externally. In either case, AP is one segment of AB and PB is the other. When a line segment is divided internally, the sum of the segments is equal to the line segment. When the line segment is divided externally, the difference of the segments is equal to the line segment.

EXERCISES

A

1. A line segment AB is divided internally into segments AP and PB. If $AP : PB = 2 : 3$, AP is what part of AB? PB is what part of AB? If $2x$ represents AP, what will represent PB?

2. With compasses and straightedge divide a given line segment internally into two segments whose ratio is 3 to 4. (See § 212.)

3. A line segment 16 inches long is divided internally into two segments which have the ratio 3 : 5. Compute the lengths of the two segments. (Represent the segments by $3x$ and $5x$.)

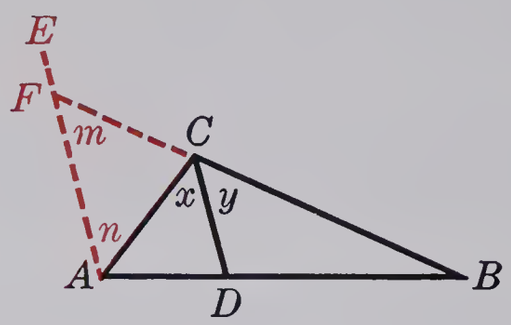
B

4. A line segment AB is divided externally at P into segments AP and PB so that $AP : PB = 2 : 5$. AP is what part of AB? PB is what part of AB? If $2x$ represents AP, what will represent PB? What will represent AB?

5. A 14-inch line segment is divided externally into two segments whose ratio is $\frac{7}{9}$. Compute the lengths of the segments. (What does $9x - 7x$ equal?)

Theorem 63

405. The bisector of an interior angle of a triangle divides the opposite side internally into segments which are proportional to the adjacent sides.



Given $\triangle ABC$ with CD bisecting $\angle ACB$.

To prove that $\dfrac{AD}{DB}=\dfrac{AC}{CB}$.

Planning the Proof: 1. We can prove line segments proportional by §§ 396, 397, 398, 399.
2. We shall use § 397.

Proof

STATEMENTS	REASONS
1. Through A draw $AE \parallel DC$.	1. Why possible?
2. Extend BC to meet AE at F.	2. Asmts. 13, 35, 37.
3. In $\triangle FAB$, $\dfrac{AD}{DB}=\dfrac{FC}{CB}$.	3. Why?
4. $\angle x = \angle y$.	4. Why?
5. But $\angle x = \angle n$ and $\angle y = \angle m$.	5. Two reasons.
6. $\therefore \angle m = \angle n$.	6. Why?
7. $\therefore FC = AC$.	7. Why?
8. From (3) and (7), $\dfrac{AD}{DB}=\dfrac{AC}{CB}$.	8. Why?

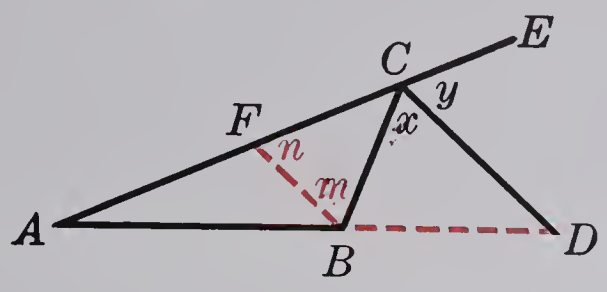
EXERCISES

A

1. In § 405, find AD if $DB = 6$, $AC = 8$, and $CB = 10$.
2. In § 405, find AC if $CB = 12$, $AD = 4$, and $DB = 6$.
3. State and prove the converse of Theorem 63.

Theorem 64

406. The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides.



Given $\triangle ABC$, AC extended to E, and CD bisecting $\angle BCE$.

To prove that $\dfrac{AD}{DB} = \dfrac{AC}{CB}$.

Planning the Proof: 1. We can prove line segments proportional by §§ 396, 397, 398, 399, 405.
2. We shall use § 398.

Proof

STATEMENTS	REASONS
1. Through B draw $BF \parallel DC$.	1. Why possible?
2. In $\triangle ADC$, $\dfrac{AD}{DB} = \dfrac{AC}{CF}$.	2. § 398.
3. $\angle x = \angle y$.	3. Why?
4. But $\angle m = \angle x$ and $\angle n = \angle y$.	4. Why?
5. $\therefore \angle m = \angle n$.	5. Why?
6. $\therefore CF = CB$.	6. Why?
7. From (2) and (6), $\dfrac{AD}{DB} = \dfrac{AC}{CB}$.	7. Why?

407. Harmonic Division of a Line Segment

A line segment is divided *harmonically* when it is divided internally and externally in the same ratio.

Then P and P' divide AB harmonically since $\dfrac{AP}{PB} = \dfrac{2}{1} = \dfrac{AP'}{P'B}$. The division is said

to be harmonic because musical strings whose lengths are proportional to AP, AB, and AP' produce harmonic tones. This fact was discovered by Pythagoras.

EXERCISES

B

1. In what kind of triangle is the bisector of an exterior angle parallel to one side of the triangle? Prove your answer.

2. In § 406, find AD if $DB = 12$, $AC = 14$, and $CB = 7$.

3. The sides of a triangle are 9, 12, and 14 inches respectively. Find the lengths of the segments into which the longest side is divided by the bisector of the opposite angle.

4. The sides of a triangle are 8, 12, and 15 inches respectively. Find the segments into which each side is divided by the bisector of the opposite angle. (Draw a figure for each computation.)

5. In $\triangle ABC$, $AB = 12$, $BC = 7$, and $AC = 10$. If the exterior angle at C is bisected, will the bisector meet the opposite side extended through B or extended through A? Find the segments of AB made by the bisector of the exterior angle at C.

6. The sides of a triangle are 4, 6, and 8. The smallest exterior angle is bisected. Find the segments into which the bisector divides the opposite side.

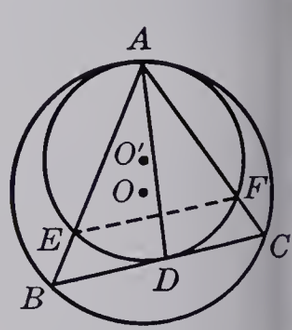
7. By the use of Th. 63 divide a given line segment into segments having the ratio 2 : 3.

8. Construct a 30°-60° right triangle. Show that the bisector of the 60° angle divides the opposite side into segments having the ratio 2 : 1.

9. The perimeter of a triangle is 80″. The bisector of one angle divides the opposite side into segments of 8″ and 12″. Find the lengths of the other two sides.

C

10. If two circles are tangent internally at A and a chord BC of the larger circle is tangent to the smaller circle at D, then AD bisects $\angle CAB$.

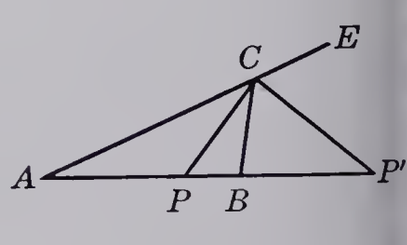


SUGGESTIONS. Prove that $EF \parallel BC$. (See Ex. 11, p. 327.) Then show that $\overset{\frown}{ED} = \overset{\frown}{DF}$.

11. State and prove the converse of Th. 64.

12. *Given* $\triangle ABC$ with CP bisecting $\angle ACB$ and CP' bisecting $\angle BCE$.

Prove that AB is divided harmonically.



408. The Use of Analysis in Algebraic Proofs

The theorems on proportion, which were given at the beginning of this chapter, deal with principles of algebra. Although the proofs of some of them are easily forgotten, they may be easily discovered by analysis. Since the converses of these theorems are usually true, the proofs of the theorems can usually be made by first proving the converses and then reversing the order of the statements. This method is useful when the statement to be proved is more complex than the given statement.

Example. In a proportion the terms are in proportion by addition.

Given $\frac{a}{b}=\frac{c}{d}$.

To prove that $\frac{a+b}{b}=\frac{c+d}{d}$.

Proof of Converse

STATEMENTS	REASONS
1. $\frac{a+b}{b}=\frac{c+d}{d}$.	1. Given
2. $b(c+d)=d(a+b)$.	2. § 386.
3. $bc+bd=ad+bd$.	3. Asmt. 6.
4. $bc=ad$.	4. Asmt. 2.
5. $\frac{a}{b}=\frac{c}{d}$.	5. § 387.

Proof of Theorem

STATEMENTS	REASONS
1. $\frac{a}{b}=\frac{c}{d}$.	1. Given.
2. $bc=ad$.	2. § 386.
3. $bc+bd=ad+bd$.	3. Asmt. 1.
4. $b(c+d)=d(a+b)$.	4. Asmt. 6.
5. $\frac{a+b}{b}=\frac{c+d}{d}$.	5. § 387.

In some of the algebraic proofs on page 398 you may wish to use the plan of first proving the converse and then reversing the statements. In others you may prefer to build the proof directly from the given fact.

EXERCISES — Algebraic Proofs

B

Given $\frac{a}{b}=\frac{c}{d}$, prove:

1. $\frac{a-1}{b}=\frac{bc-d}{bd}$.

2. $\frac{a+1}{1}=\frac{bc+d}{d}$.

3. $\frac{a}{b}=\frac{c-a}{d-b}$.

4. $\frac{a-b}{b}=\frac{c-d}{d}$.

5. $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

6. $\frac{a^2-b}{ac-d}=\frac{a}{c}$.

REVIEW QUESTIONS — Chapter 12

1. What is a ratio? a proportion?

2. Can two quantities be in proportion?

3. What are commensurable quantities? Give an example.

4. What are incommensurable quantities? Give an example.

5. Which of the following are true?

a. $\frac{3}{7}=\frac{9}{21}$. b. $\frac{3}{5}=\frac{12}{20}$. c. $\frac{10}{3}=\frac{15}{6}$. d. $\frac{7x}{5}=\frac{35x}{25}$.

6. State six fundamental theorems on proportion.

7. Transform $\frac{x}{y}=\frac{m}{n}$ by

a. Alternation. b. Inversion. c. Addition. d. Subtraction.

8. Write two proportions from $ab=cd$.

9. State the methods you have had in this chapter of proving

a. Lines parallel. b. Line segments in proportion.

10. State the numbered construction problems you have had in this chapter.

11. In the proportion $\frac{x}{y}=\frac{y}{z}$, what is y called? what is z called?

12. What is the mean proportional between 4 and 9?

13. Find the fourth proportional to 10, 15, and 18.

14. Find the third proportional to 10 and 12.

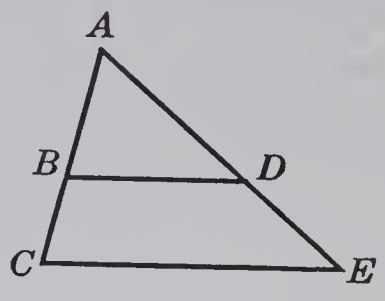
15. The sides of a triangle are 24, 36, and 40. Find the segments of the longest side made by the bisector of the opposite angle.

In $\triangle ACE$, $BD \parallel CE$.

16. Find DE if $AB = 6$, $BC = 4$, and $AD = 10$.

17. Find AB if $AD = 16$, $DE = 12$, and $BC = 8$.

18. Find AE if $AC = 21$, $BC = 6$, and $AD = 18$.



19. A line segment 27 inches long is divided internally into two segments having the ratio 4 : 5. How long is each segment?

20. Given line segments a and b, construct a line segment x so that $x = b^2 \div a$.

SUMMARY of Principal Bases for Proof

409. Line segments in proportion

a. If a line is parallel to one side of a triangle and intersects the other two sides, it divides these sides proportionally.

b. If a line is parallel to one side of a triangle and intersects the other two sides, it divides these sides so that either side is to one of its segments as the other side is to its corresponding segment.

c. Parallel lines intercept proportional segments on two transversals.

d. The bisector of an interior (or exterior) angle of a triangle divides the opposite side internally (or externally) into segments which are proportional to the adjacent sides.

410. Lines parallel

a. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

b. If a line divides two sides of a triangle so that either side is to one of its segments as the other side is to its corresponding segment, the line is parallel to the third side.

411. Constructions

a. To construct the fourth proportional to three given line segments.

b. To divide a given line segment into n parts which are proportional to n given line segments.

WORD LIST

alternation	exterior	internal	proportion
commensurable	external	inversion	proportional
denominator	interior	numerator	ratio

TEST 30 — True-False Statements

[15 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, place a T after its number. If a statement is *not always* true, place an F after its number. Do not guess.

1. The fraction $\frac{2}{3}$ is a proportion.

2. A ratio is an equation.

3. If $\frac{x}{y} = \frac{m}{n}$, then $xm = yn$.

4. $\frac{2}{3} = \frac{10}{15}$ is a proportion.

5. If $\frac{x}{12} = \frac{4}{3}$, then $x = 9$.

6. If $ab = cd$, then $\frac{a}{d} = \frac{c}{b}$.

7. 6 is a mean proportional between 4 and 9.

8. In the proportion $\frac{x}{y} = \frac{z}{w}$, w is the fourth proportional to x, y, and z.

9. The extremes of the proportion $\frac{7}{8} = \frac{x}{6}$ are 8 and 6.

10. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$ by alternation.

11. If $\frac{x}{y} = \frac{z}{w}$, then $\frac{x}{w} = \frac{z}{y}$ by inversion.

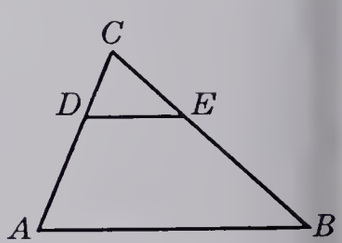
12. If $\frac{m}{n} = \frac{r}{s}$, then $\frac{m+n}{n} = \frac{r+s}{s}$ by addition.

13. If the second and third terms of a proportion are equal, either term is a mean proportional between the first and fourth terms.

14. If the first and fourth terms of a proportion are equal, the second and third terms are equal.

15. If the first and third terms of a proportion are equal, the second and fourth terms are equal.

16. A proportion always contains two equal ratios.



17. In $\triangle ABC$, if $AC = 75$, $DC = 30$, $CE = 36$, and $BC = 90$, then $DE \parallel AB$.

18. In $\triangle ABC$, if $CD = 12$, $AD = 40$, $BE = 70$, and $EC = 21$, then $DE \parallel AB$.

19. The line segment RS is divided internally into segments RP and PS such that $RP : PS = 2 : 1$. Then $RP = \frac{1}{3} RS$.

20. The third proportional to 4 and 8 is 16.

TEST 31 — Applications

[20 Minutes]

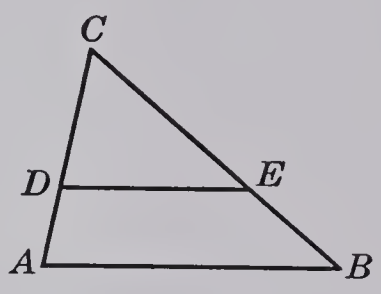
1. Find the fourth proportional to 2, 3, and 4.

2. Solve for x in the equation $\frac{2x}{5} = \frac{12}{10}$.

3. Transform $\frac{m}{6} = \frac{2n}{5}$ by inversion.

4. Name a measure common to 6 inches and 9 inches.

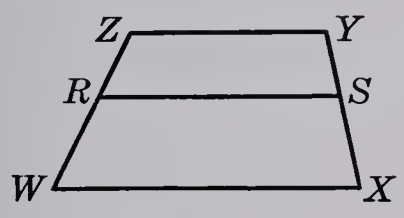
5. In the figure, $DE \parallel AB$. If $AD = 10$, $DC = 15$, and $EC = 18$, find EB.

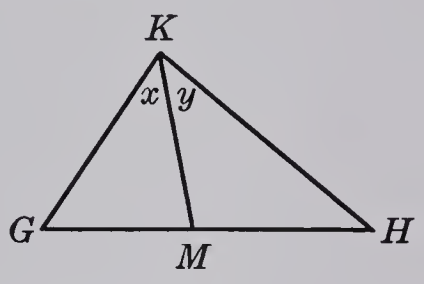


6. In the figure of Ex. 5, if $AC = 24$, $AD = 9$, and $BC = 40$, find BE.

7. In the figure of Ex. 5, if $CD = 14$, $AD = 7$, and $CB = 18$, find BE.

8. Find two complementary angles whose ratio is 2 to 3.





9. $WXYZ$ is a trapezoid with $RS \parallel$ base WX. If $ZR = 10$, $WR = 12$, and $YS = 8$, find SX.

10. In the figure of Ex. 9, if $YX = 27$, $ZR = 12$, and $RW = 18$, find YS.

11. In the figure, $\angle x = \angle y$. If $GK = 9$, $GM = 6$, and $KH = 12$, find MH.

12. In the figure of Ex. 11, if $GM = 10$, $GH = 25$, and $KH = 16$, find GK.

13. If $7c = 8m$, then $7 : m =$ __?__ : __?__.

14. If $\frac{a}{b-d} = \frac{b+d}{c}$, then $ac =$ __?__.

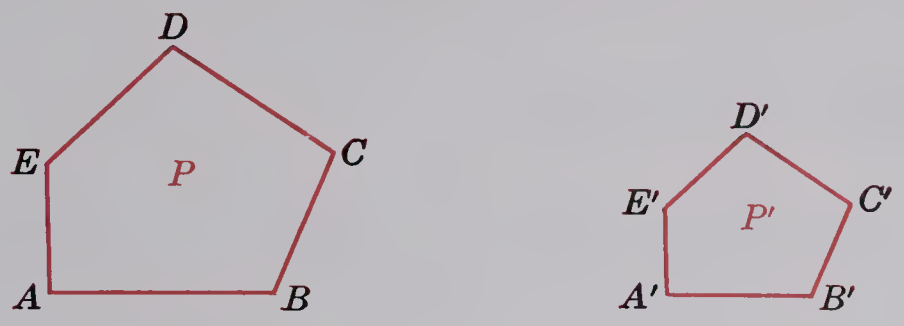
CHAPTER 13

Similar Polygons

IN THIS CHAPTER
you will study polygons
that have the same shape.

412. Similar Polygons

Similar polygons are polygons which have their corresponding angles equal and their corresponding sides proportional.



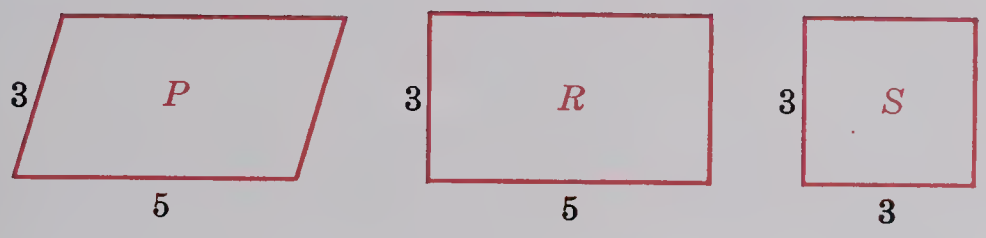
The polygons P and P' above are similar ($\sim$) if

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$;

(2) $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.

In connection with similar polygons, the word "corresponding" means "having the same relative position"; thus, in the figure above, $\angle A$ and $\angle A'$ occupy relatively the same position with respect to the other sides and vertices of their respective polygons, regardless of how the polygons are turned. Similarly, AB corresponds to $A'B'$.

Two polygons may have their corresponding sides proportional and not be similar, or they may have their corresponding angles equal and not be similar. Thus in the figures below, P and R have their sides pro-



portional but are not similar, since their corresponding angles are not equal; R and S have their corresponding angles equal but are not similar, since their corresponding sides are not proportional.

413. Conditions of Similarity

From the definition of similar polygons (§ 412), it follows that for two polygons to be similar, two distinct conditions must be fulfilled:

1. The corresponding angles must be equal.
2. The corresponding sides must be proportional.

Also, since a definition is reversible, if two polygons are known to be similar, their corresponding angles are equal and their corresponding sides are proportional.

414. Ratio of Similitude

If two polygons are similar, the ratio of any two corresponding sides is called the *ratio of similitude.*

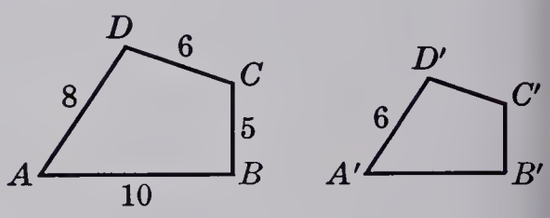
EXERCISES Similar Polygons

A

1. The legs of a right triangle are 5 inches and 7 inches respectively. If the shorter leg of a similar triangle is 15 inches, find the other leg.

2. The sides of a triangle are 3, 6, and 8 inches respectively. If the longest side of a similar triangle is 12 feet, find the other two sides.

3. Polygons $ABCD$ and $A'B'C'D'$ are similar. If $A'D' = 6$, find $A'B'$, $B'C'$, and $C'D'$.



4. The sides of a pentagon are 4, 5, 6, 8, and 12 respectively. If the longest side of a similar polygon is 18, find the lengths of the remaining sides. What is the ratio of similitude of the two polygons?

5. If each side of one square is 8 inches and a side of another square is 12 inches, show that their corresponding sides are proportional. Are their corresponding angles equal? Are the two squares similar? Are all squares similar?

6. Are two congruent polygons similar? Why? Are two similar polygons congruent? Why? What is the ratio of similitude of two congruent triangles?

7. If two polygons are similar, do they have the same shape? Illustrate your answer with figures.

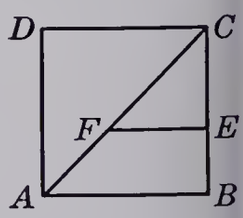
8. Can a pentagon be similar to a triangle? to a quadrilateral? to another pentagon?

B

9. Prove that any two equilateral triangles are similar.

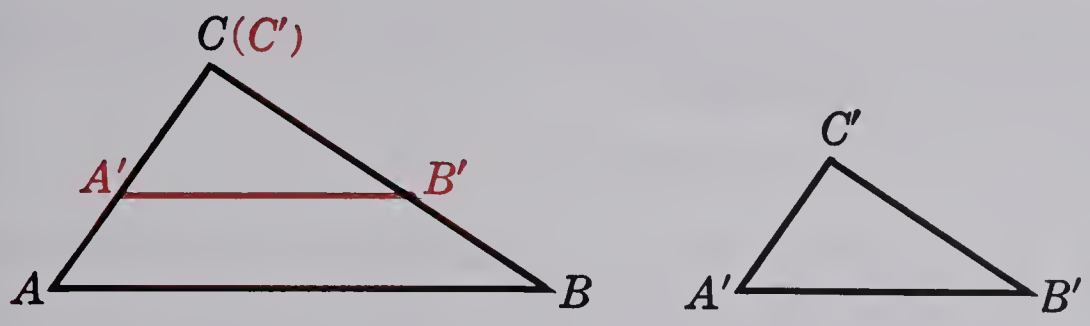
10. Prove that any two equiangular triangles are similar.

11. $ABCD$ is a square 15 inches on a side. The diagonal is 21.21 inches long. If $BE = 6$ inches and $FE \parallel AB$, what is the length of AF?



★Theorem 65

415. If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.



Given $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$ and $\angle B = \angle B'$.
To prove that $\triangle ABC \sim \triangle A'B'C'$.

Planning the Proof: 1. We can prove triangles $\sim$ by § 413.
2. We shall use § 413.

Proof

STATEMENTS	REASONS
1. $\angle A = \angle A'$ and $\angle B = \angle B'$.	1. Given.
2. Then $\angle C = \angle C'$.	2. § 158 c.
3. $\therefore \triangle ABC$ and $A'B'C'$ have their corresponding angles =.	3. Statements 1 and 2.
4. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle C'$ coincides with $\angle C$, $\triangle A'B'C'$ taking the position $A'B'C$.	4. Why?
5. $A'B' \parallel AB$.	5. § 160 b (2).
6. $\therefore \frac{AC}{A'C} = \frac{BC}{B'C}$, or $\frac{AC}{A'C'} = \frac{BC}{B'C'}$.	6. Why?
7. In the same manner, by placing $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle B'$ coincides with its equal $\angle B$, it may be proved that $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.	7. Reasons 4–6.
8. $\therefore \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.	8. Asmt. 6.
9. $\therefore \triangle ABC \sim \triangle A'B'C'$.	9. § 413.

416. Corollary I. If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.

417. Corollary II. The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.

SUGGESTION. Base proof on § 416.

418. Corollary III. If two triangles are similar to a third triangle, they are similar to each other.

SUGGESTION. Base proof on Theorem 65.

419. How to Determine the Corresponding Sides of Similar Triangles

We have said that corresponding parts of similar polygons occupy relatively the same positions with regard to the other parts of their respective polygons. Do you see that when the polygons are similar triangles, corresponding sides are opposite equal angles, and do you see that if the corresponding sides of two similar triangles are known, the equal angles are those which lie opposite the corresponding sides? Theorem 65 shows this clearly. Notice that in the proof of the theorem, statements 1 through 6 lead us from the given information, "$\angle A = \angle A'$ and $\angle B = \angle B'$," to the conclusion, "$\therefore \frac{AC}{A'C'} = \frac{BC}{B'C'}$." Do you see that AC and $A'C'$ are opposite the equal angles B and B', respectively, while BC and $B'C'$ are opposite the equal angles A and A', respectively? Of course, then, as statement 7 shows, AB and $A'B'$ are opposite the equal angles, C and C', respectively.

420. Ways of Proving Line Segments Proportional

In Chapter 12 you learned five ways of proving line segments proportional. You now have additional ways. To see three of these ways study the following paragraphs.

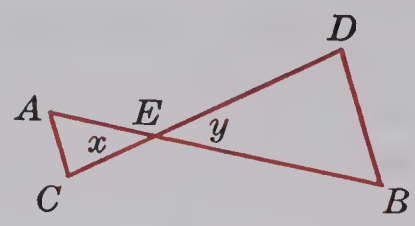
a. To prove that four line segments are proportional, show that the line segments are corresponding sides of similar triangles.

The steps in applying this procedure are:

1. Find two triangles each of which has two of the four segments as sides.
2. Prove that these triangles are similar.
3. Form a proportion using the fact that these line segments are corresponding sides of the similar triangles.
4. If necessary, transform the proportion.

Example 1. *Given* AB and CD intersecting at E, and $AC \parallel BD$.

To prove that $\dfrac{AE}{BE} = \dfrac{AC}{BD}$.



Proof

STATEMENTS	REASONS
1. $\angle C = \angle D$.	1. Why?
2. $\angle x = \angle y$.	2. Why?
3. $\therefore \triangle ACE \sim \triangle BDE$.	3. Why?
4. $\therefore \dfrac{AE \text{ (opp. } \angle C \text{ in } \triangle ACE)}{BE \text{ (opp. } \angle D \text{ in } \triangle BDE)} = \dfrac{AC \text{ (opp. } \angle x \text{ in } \triangle ACE)}{BD \text{ (opp. } \angle y \text{ in } \triangle BDE)}$.	4. Why?

NOTE. In statement 3 above, the vertices of the corresponding angles are paired, A with B, C with D, and E with E. This arrangement makes it easy to check for corresponding sides. For example, side AC corresponds to side BD. In statement 4 note that the numerators in the proportion are obtained from one triangle and the denominators from another triangle. (The expressions in parentheses are to help the student in selecting the corresponding sides. They may be omitted in written proofs.)

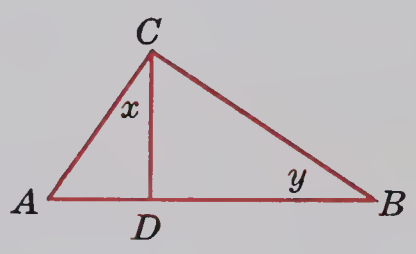
b. To prove that four line segments which are not sides of similar triangles are proportional, show that each of the two given ratios is equal to a third ratio.

c. To prove that a line segment is the mean proportional between two other line segments, prove that two triangles which have the first line segment as a common side are similar.

Example 2. *Given* $\angle ACB$ a rt. $\angle$ and $CD \perp AB$.

To prove that $\dfrac{AD}{AC} = \dfrac{AC}{AB}$.

Since AC is to be proved the mean proportional between AD and AB, we try to prove it a side common to two similar triangles.



Proof

STATEMENTS	REASONS
In $\triangle ADC$ and ACB,	
1. $\angle A = \angle A$.	1. Why?
2. $\angle ADC = \angle ACB$.	2. Why?
3. $\therefore \triangle ADC \sim \triangle ACB$.	3. Why?
4. $\therefore \dfrac{AD \text{ (opp. } \angle x \text{ in } \triangle ADC)}{AC \text{ (opp. } \angle y \text{ in } \triangle ACB)} = \dfrac{AC \text{ (opp. rt. } \angle \text{ in } \triangle ADC)}{AB \text{ (opp. rt. } \angle \text{ in } \triangle ACB)}$.	4. Why?

421. Ways of Proving Products of Line Segments Equal

a. To prove that the product of two line segments is equal to the product of two other line segments, show that the segments are proportional (§ 420 a) and apply § 386.

For example, from statement 4 in Example 1 on page 407, we know that $AE \times BD = BE \times AC$.

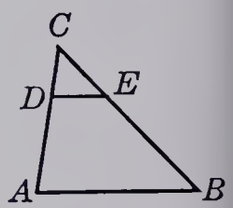
b. To prove that the square of one line segment is equal to the product of two others, prove that the first segment is the mean proportional between the two others (§ 420 c) and then use § 386.

For example, from statement 4 in Example 2 on page 407, $\overline{AC}^2 = AD \times AB$.

EXERCISES — Similar Triangles

A

1. In the figure, $AC = 18$, $DC = 6$, $BC = 24$, and $EC = 8$. Is $DE \parallel AB$? Does $\angle CDE = \angle A$? Does $\angle CED = \angle B$? Is $\triangle CDE \sim \triangle CAB$? CD is what part of CA? DE is what part of AB?

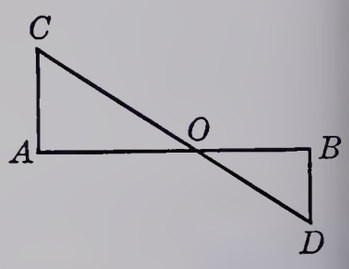


2. From §§ 409 and 420 state six ways of proving line segments in proportion.

3. A tower casts a shadow 120 feet long when a 20-foot telephone pole casts a shadow 15 feet long. How high is the tower?

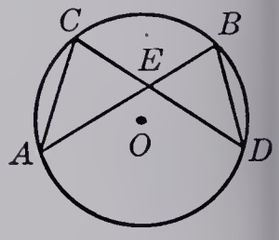
4. *Given* $AC \perp AB$ and $BD \perp AB$ in the adjoining figure.

Prove that $\dfrac{AC}{CO} = \dfrac{BD}{DO}$.



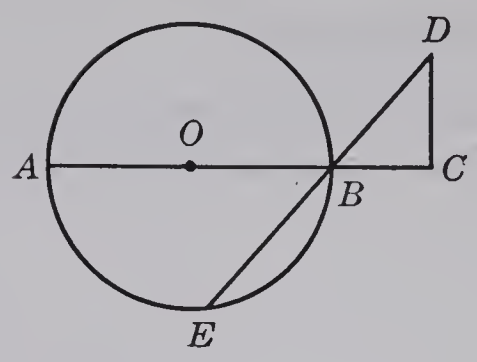
5. Triangle ABC is inscribed in a circle, AB being a diameter of the circle. ED is a perpendicular drawn from any point E on AC to AB. Prove that triangle ABC is similar to triangle ADE.

6. *Given* $\odot O$ with two chords, AB and CD, intersecting at E.



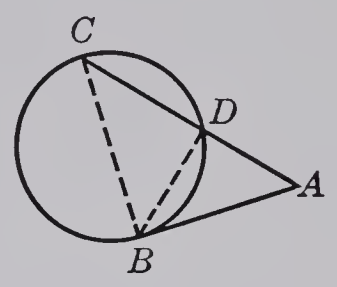
Find and prove as many relationships as you can involving the segments AE, EB, CE, and ED.

7. *Given* $\odot O$ having chord EB extended to D, and $DC \perp$ diameter AB at C.
Prove that $AB \times BC = EB \times BD$.



B

8. *Given* AB tangent to a $\odot$ at B and AC a secant to the $\odot$.



a. *Prove* that $\triangle ABD \sim \triangle ABC$.

b. *Prove* that $\dfrac{AC}{AB} = \dfrac{AB}{AD}$.

c. *Prove* that $\overline{AB}^2 = AC \times AD$.

9. *Given* rt. $\triangle ABC$, in which $\angle ACB$ is the rt. $\angle$, and $CD \perp AB$.

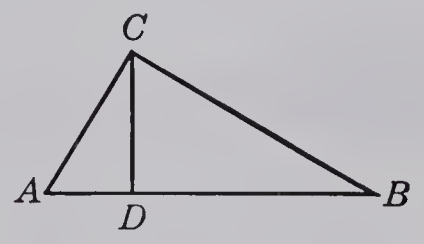
a. *Prove* that $\triangle ADC \sim \triangle ABC$.

b. *Prove* that $\triangle BCD \sim \triangle ABC$.

c. *Prove* that $\triangle ADC \sim \triangle BCD$.

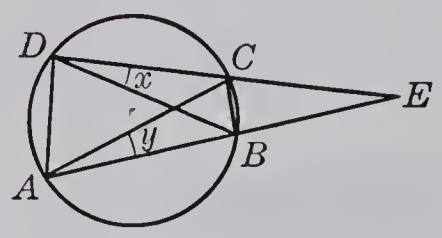
d. *Prove* that $\dfrac{AD}{DC} = \dfrac{DC}{DB}$.

e. *Prove* that $\overline{DC}^2 = AD \times DB$.



10. The lines joining the midpoints of the sides of a triangle form a triangle that is similar to the given triangle.

11. *Given* the inscribed quadrilateral $ABCD$ with the diagonals AC and BD, and the sides AB and DC produced to meet at E.



a. *Prove* that $\triangle BDE \sim \triangle ACE$.

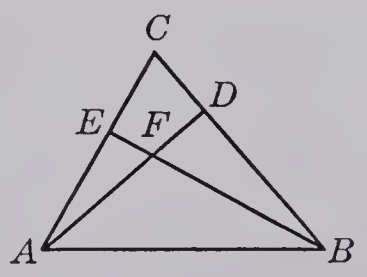
b. *Prove* that $\dfrac{BD}{CA} = \dfrac{DE}{AE}$.

C

12. The bisectors of two corresponding angles of two similar triangles have the same ratio as a pair of corresponding sides.

13. The diagonals of a trapezoid divide each other so that the corresponding segments are proportional.

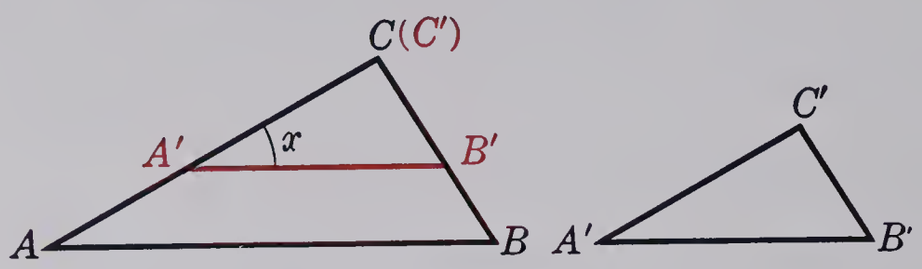
14. *Given* $\triangle ABC$ with two altitudes, AD and BE, intersecting at F.



Prove that a. $\dfrac{AF}{BF} = \dfrac{EF}{FD}$; b. $\dfrac{AD}{BE} = \dfrac{AC}{BC}$.

★★ Theorem 66

422. If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar.



Given $\triangle ABC$ and $A'B'C'$ having $\angle C = \angle C'$ and $\dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Planning the Proof: 1. We can prove $\triangle\sim$ by §§ 413, 415, 416, 418.
2. We shall use § 415.

Proof

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle C'$ coincides with its equal $\angle C$, $\triangle A'B'C'$ taking the position $A'B'C$.	1. Why?
2. $\dfrac{AC}{A'C} = \dfrac{BC}{B'C}$, or $\dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$.	2. Why?
3. $\therefore A'B' \parallel AB$.	3. § 410 b.
4. $\therefore \angle A = \angle x$, or $\angle A'$.	4. Why?
5. $\therefore \triangle ABC \sim \triangle A'B'C'$.	5. Why?

EXERCISES

A

1. In a $\triangle DEF$, $DF = 20$, $DE = 16$, $EF = 24$. Between H, a point in DF, and K, a point in EF, HK is drawn so that $FH = 15$ and $FK = 18$. Show that $\triangle HKF \sim \triangle DEF$. How long is HK?

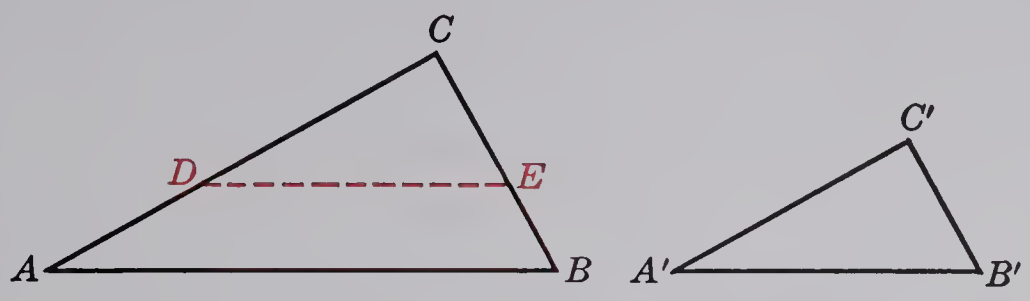
2. Two isosceles triangles are similar if a base angle of one equals a base angle of the other.

3. The legs of one right triangle are 3 in. and 4 in. and the legs of another are 9 in. and 12 in. Are the triangles similar?

4. In $\triangle ABC$ and $\triangle A'B'C'$, $\angle A = \angle A' = 70°$, $AB = 4$ in., $AC = 6$ in., $A'B' = 2$ in., and $A'C' = 5$ in. Are the triangles similar?

★ Theorem 67

423. If two triangles have their sides respectively proportional, they are similar.



Given $\triangle ABC$ and $A'B'C'$ having $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Planning the Proof: 1. We can prove $\triangle\sim$ by §§ 413, 415, 416, 418, 422.
2. We shall use § 418.

Proof

STATEMENTS	REASONS
1. On CA construct $CD = C'A'$ and on CB construct $CE = C'B'$.	1. Why possible?
2. Draw DE.	2. Why possible?
3. $\frac{AC}{A'C'} = \frac{BC}{B'C'}$.	3. Why?
4. $\therefore \frac{AC}{DC} = \frac{BC}{EC}$.	4. Why?
5. $\angle C = \angle C$.	5. Why?
6. $\triangle ABC \sim \triangle DEC$.	6. Why?
7. $\therefore \frac{AB}{DE} = \frac{AC}{DC}$, or $\frac{AB}{DE} = \frac{AC}{A'C'}$.	7. Why?
8. But $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.	8. Why?
9. $\therefore A'B' = DE$.	9. § 389.
10. $\therefore \triangle A'B'C' \cong \triangle DEC$.	10. Why?
11. $\therefore \triangle A'B'C' \sim \triangle DEC$.	11. § 415.
12. $\therefore \triangle ABC \sim \triangle A'B'C'$.	12. § 418.

EXERCISES

Is the converse of Theorem 67 a theorem? an assumption? a definition? a part of a definition?

EXERCISES — Similar Triangles

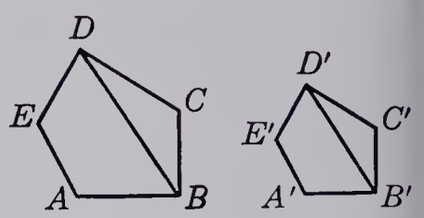
A

1. What ways have you had of proving two triangles similar? Find these ways on page 570. Are two polygons necessarily similar if their corresponding sides are proportional? What kind of polygons are similar if their corresponding sides are proportional? Why?

2. A line intersects two sides of a triangle so that the segments of one side are 8 and 22 and the corresponding segments of the other side are 12 and 33. Is the line parallel to the third side?

3. *Given* polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Prove that $\triangle BCD \sim \triangle B'C'D'$.



SUGGESTION. Use § 422.

4. Prove that two isosceles triangles are similar if the vertex angle of one equals the vertex angle of the other.

B

5. In $\triangle ABC$, $\angle C$ is acute, AD is perpendicular to BC, and BE is perpendicular to AC. Find a pair of similar triangles. Prove that the triangles are similar. Prove that $\frac{AD}{BE} = \frac{AC}{BC}$. State this proportion as a theorem.

6. The corresponding medians of two similar triangles have the same ratio as any two corresponding sides. (Use § 422.)

7. Two corresponding medians of two similar triangles have the same ratio as the bisectors of the angles from whose vertices the medians are drawn.

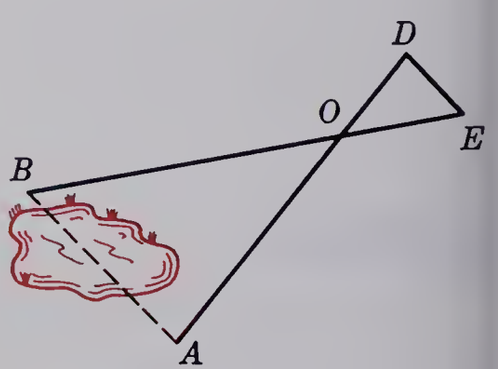
C

8. *Given* quadrilateral $ABCD$ inscribed in a $\odot$. BD bisects $\angle ABC$ and intersects AC at E.

a. *Prove* that $\triangle BEC \sim \triangle BDA$.

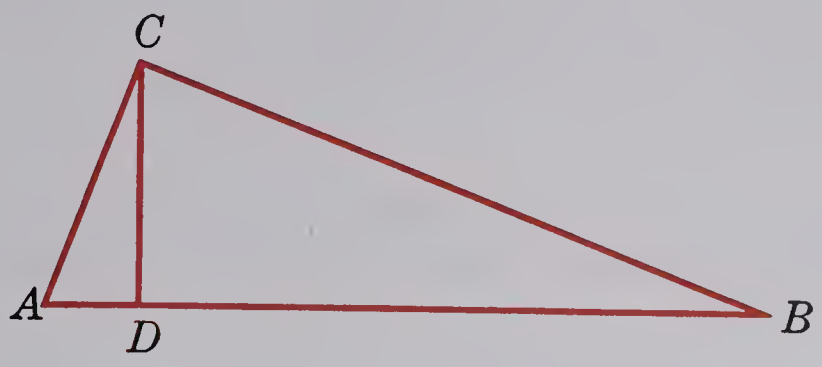
b. *Prove* that $\triangle BEA \sim \triangle BDC$.

9. A and B are two points on opposite sides of a pond. Show how to find the distance between them without using congruent triangles.



★ Theorem 68

424. The altitude on the hypotenuse of a right triangle forms two right triangles which are similar to the given triangle and to each other.



Given rt. $\triangle ABC$ having $CD \perp$ hypotenuse AB.

To prove that $\triangle ADC \sim \triangle ACB \sim \triangle CDB$.

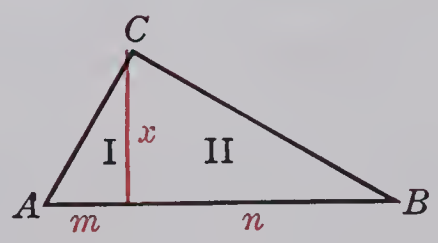
Planning the Proof: 1. We can prove $\mathbb{S} \sim$ by §§ 413, 415, 416, 418, 422, 423.

2. We shall use §§ 416, 418.

Proof

STATEMENTS	REASONS
1. $\mathbb{S}\, ADC$ and ACB are rt. $\mathbb{S}$.	1. Why?
2. $\angle A = \angle A$.	2. Why?
3. $\therefore \triangle ADC \sim \triangle ACB$.	3. Why?
4. $\mathbb{S}\, CDB$ and ACB are rt. $\mathbb{S}$.	4. Why?
5. $\angle B = \angle B$.	5. Why?
6. $\therefore \triangle CDB \sim \triangle ACB$.	6. Why?
7. $\therefore \triangle ADC \sim \triangle CDB$.	7. § 418.

★ **425. Corollary I.** The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

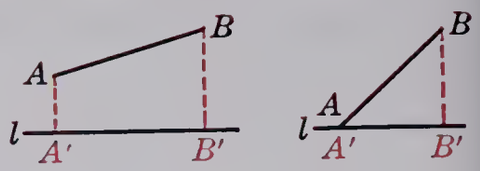


SUGGESTIONS. $\triangle$ I $\sim \triangle$ II by § 424. Then m of $\triangle$ I corresponds to x of $\triangle$ II and x of $\triangle$ I corresponds to n of $\triangle$ II. $\therefore \frac{m}{x} = \frac{x}{n}$, or $x^2 = mn$.

NOTE. This corollary was known by Archytas (400 B.C.), one of the greatest mathematicians of early times. He was one of the first men to write on mechanics, and was very skillful in making toys and models.

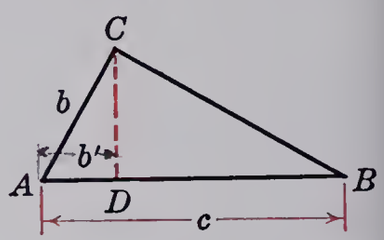
426. Projection of a Line Segment

The *projection of a line segment* on a line is the segment that is cut off on the line by perpendiculars to the line from the extremities of the line segment. Thus, in the figures, $A'B'$ is the projection of AB on the line l.

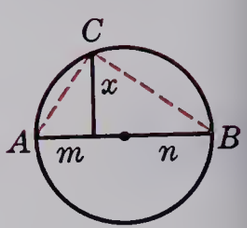


427. Corollary II. **Either leg of a right triangle is the mean proportional between the hypotenuse and its projection on the hypotenuse.**

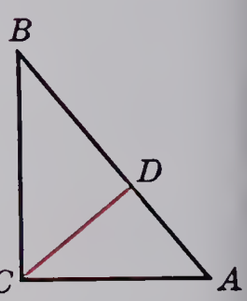
SUGGESTIONS. b' is the projection of b on c. $\triangle ADC \sim \triangle ACB$. Why? b' of $\triangle ADC$ corresponds to b of $\triangle ACB$ and b of $\triangle ADC$ corresponds to c of $\triangle ACB$. $\therefore \frac{b'}{b} = \frac{b}{c}$, or $b^2 = b'c$.



428. Corollary III. **The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the segments of the diameter.**



Example 1. Given rt. $\triangle ACB$ with $CD \perp$ hypotenuse AB, $AD = 11$, and $BD = 15$. Find CD.



Solution. From § 425, $\frac{AD}{CD} = \frac{CD}{BD}$.

Substituting, $\frac{11}{CD} = \frac{CD}{15}$.

$\therefore \overline{CD}^2 = 165$.

$CD = \pm\sqrt{165}$, or $\pm 12.845+$.

$CD = 12.845+$. (The negative value of CD is meaningless.)

Example 2. If, in the figure of Ex. 1, $AC = 6$ and $BD = 10$, what is the length of AD?

Solution. From § 427, $\frac{AB}{AC} = \frac{AC}{AD}$.

Let $x = AD$. Substituting, $\frac{x+10}{6} = \frac{6}{x}$.

$x^2 + 10x = 36$.

Adding $(\frac{1}{2}$ of $10)^2$, $x^2 + 10x + 25 = 61$.

Extracting the square roots, $x + 5 = \pm\sqrt{61}$

$x = -5 \pm \sqrt{61}$

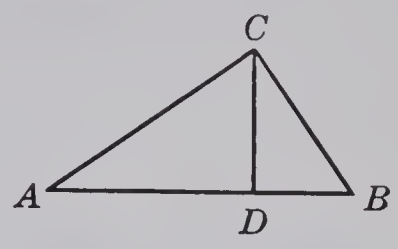
$x = 2.810+$. (The negative value of x is meaningless.)

EXERCISES

A

1. Name two ways of proving that a line segment is the mean proportional between two other line segments.

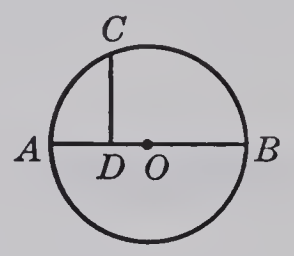
2. In $\triangle ABC$, $\angle ACB = 90°$, and $CD \perp AB$.



a. Find DB if $AD = 5$ and $CD = 10$.
b. Find AB if $CD = 8$ and $DB = 4$.
c. Find CD if $AB = 35$ and $DB = 7$.
d. Find BD if $AD = 20$ and $BC = 24$.
e. Find AC if $AD = 15$ and $DB = 25$.

3. In $\triangle MNP$, $\angle P$ is a right angle and PS is perpendicular to MN.
a. If $MS = 9$ and $SN = 4$, find PS.
b. If $MS = 21$ and $PS = 56$, find SN to the nearest tenth.
c. If $PS = 24$ and $SN = 18$, find MS.
d. If $MS = 18$ and $SN = 20$, find PS to the nearest tenth.
e. If $PS = 12.3$ and $SN = 5.4$, find MS to the nearest tenth.

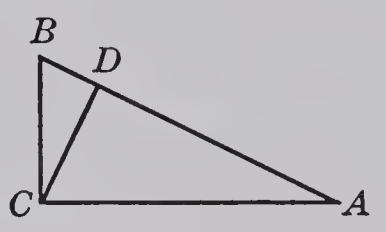
4. AB is a diameter of circle O, C is a point on the circle, and CD is perpendicular to AB.



a. If $AD = 3$ and $DB = 27$, find CD.
b. If $AD = 4$ and $AB = 20$, find CD.
c. If $CD = 9$ and $DB = 16.2$, find AD.
d. If $AD = 8.2$ and $DB = 24.6$, find CD to the nearest tenth.

B

5. In $\triangle ABC$, CD is perpendicular to the hypotenuse AB.



a. If $BD = 4$ and $BA = 9$, find BC.
b. If $BC = 8$ and $BD = 4$, find DA.
c. If $CA = 12$ and $DB = 10$, find AB.

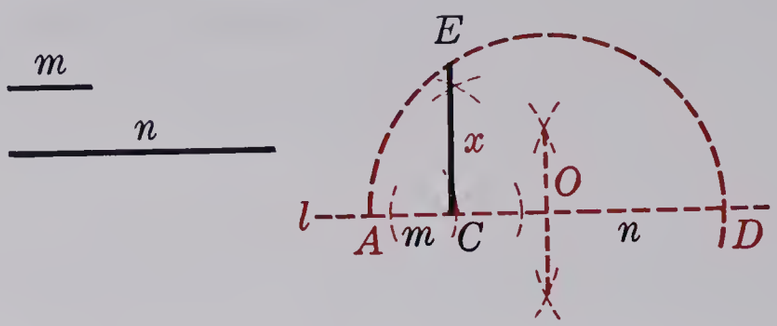
6. From any point C on a circle whose center is O, CE is drawn perpendicular to a diameter AB. Then line segment BC is drawn. Prove that $\overline{BC}^2 = 2\ AO \times BE$.

7. Prove that the line which joins the midpoints of two sides of a triangle cuts off a triangle similar to the given triangle.

8. The diameter of a circle is 15 inches long and is divided into five equal segments. How long are the chords drawn perpendicular to the diameter at the points of division?

★★ Problem 23

429. To construct the mean proportional between two given line segments.



Given the line segments m and n.

To construct the segment x so that $\frac{m}{x} = \frac{x}{n}$.

Construction

STATEMENTS	REASONS
1. On a line l construct $AC = m$ and $CD = n$.	1. Why possible?
2. Construct a semicircle having AD as a diameter.	2. Why possible?
3. At C construct $CE \perp AD$, meeting the semicircle at E.	3. Why possible?

Then CE is the required mean proportional.

Proof: (The proof is left to the student.)

EXERCISES

A

1. Construct the mean proportional between a line segment of 2 inches and one of $4\frac{1}{2}$ inches.

2. Construct a line segment equal to $\sqrt{5}$ inches.

SUGGESTION. Construct $m = 1$ inch and $n = 5$ inches.

3. Use Problem 23 to construct the third proportional to 3 in. and 4 in.

B

4. Find the square root of 6 geometrically; arithmetically. Which method did the early Greeks use?

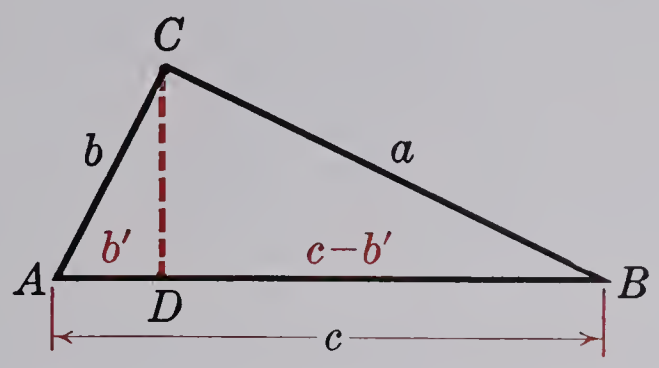
5. Given line segments a and b. Find the segment x so that $x = \frac{b^2}{a}$.

HINT. Which segment is the mean proportional?

6. Use Problem 23 to construct a square equal to a rectangle with length l and width w.

★ Theorem 69 (Pythagorean Theorem)

430. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.



Given rt. $\triangle ABC$ having the hypotenuse AB denoted by c, side AC denoted by b, and side BC denoted by a.

To prove that $c^2 = a^2 + b^2$.

Planning the Proof: 1. We can prove relations between sides of ◬ by §§ 420, 425, 427.

2. We shall use § 427, and Asmt. 1.

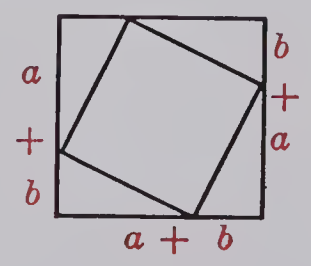
Proof

STATEMENTS	REASONS
1. $\angle ACB$ is a rt. $\angle$.	1. Why?
2. Draw $CD \perp AB$.	2. Why possible?
3. Let $AD = b'$. Then $DB = c - b'$.	3. Why?
4. $\frac{c}{b} = \frac{b}{b'}$ and $\frac{c}{a} = \frac{a}{c - b'}$.	4. § 427.
5. $b^2 = cb'$.	5. Why?
6. $a^2 = c(c - b')$, or $a^2 = c^2 - cb'$.	6. Why?
7. $\therefore a^2 + b^2 = c^2$.	7. Why?

NOTE. The above proof is attributed to Hindu mathematicians.

EXERCISES

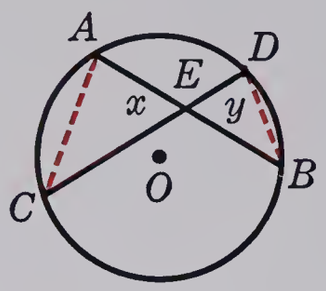
1. See if you can prove Th. 69, using the figure at the right. The proof is similar to that of Garfield (§ 251).



2. Show that the numbers n, $\frac{n^2 - 4}{4}$, and $\frac{n^2 + 4}{4}$ are Pythagorean numbers, that is, that they can represent the numerical values of the sides of a right triangle.

★★ Theorem 70

431. If two chords intersect within a circle, the product of the segments of one chord is equal to the product of the segments of the other.



Given AB and CD two chords of $\odot O$ intersecting at E.
To prove that $AE \cdot EB = CE \cdot ED$.

Planning the Proof: 1. We can prove products of line segments = by § 421.
2. We shall use § 421 a.

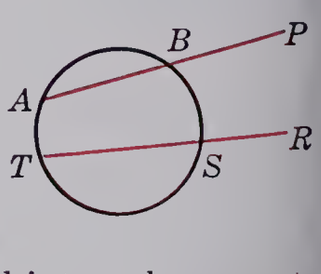
Proof

STATEMENTS	REASONS
1. Draw AC and DB.	1. Why possible?
In $\triangle AEC$ and DEB,	
2. $\angle x = \angle y$ and $\angle A = \angle D$.	2. §§ 50 d, 334.
3. $\therefore \triangle AEC \sim \triangle DEB$.	3. Why?
4. $\therefore \frac{AE}{ED} = \frac{CE}{EB}$.	4. Why?
5. $\therefore AE \cdot EB = CE \cdot ED$.	5. Why?

When applying this theorem, some students may wish to use Step 4 of the proof, $\frac{AE}{ED} = \frac{CE}{EB}$, or $\frac{AE}{CE} = \frac{ED}{EB}$.

432. Segment of a Secant

In § 277 a secant of a circle was defined as a line which intersects the circle in two points. If a secant drawn from a point P intersects a circle in point B and meets it in point A, as in the figure, it is customary to speak of segment PA as the secant from P to the circle or simply as the secant to the circle. PB is the *external* segment of the secant and BA is the *internal* segment, or chord, of the secant. Name the external and internal segments of the secant RT.



EXERCISES

A

1. The segments of one of two intersecting chords of a circle are 4 inches and 8 inches respectively. One segment of the other chord is 6 inches. Compute the length of the other segment.

2. The segments of one of two intersecting chords of a circle are 15 inches and 6 inches respectively. The length of the second chord is 23 inches. Find the lengths of the segments of the second chord.

3. A chord 16 inches long is 6 inches from the center of the circle. Find the length of the diameter of the circle.

4. Two chords intersect within a circle. The segments of one chord are 6 inches and 14 inches respectively. The length of the second chord is 25 inches. Find the lengths of its segments.

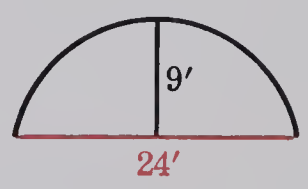
5. The distance from the center of a chord 36 inches long to the midpoint of its arc is 9 inches. Find the radius of the circle.

6. Chords RS and EM intersect in point H. If $RH \times HS = 56$ and $EH = 6$, find HM.

B

7. The radius of the larger of two concentric circles is 30 inches, and a chord of the larger circle is tangent to the smaller circle. Find the radius of the smaller circle if the length of this chord is 36 inches.

8. A contractor wishes to construct a circular arch having a span of 24 feet and a height of 9 feet. Find the radius of the circle of which the arch is an arc.



9. A chord is drawn through a point 5 inches from the center of a circle having a radius of 15 inches. What is the product of the segments into which the chord is divided?

C

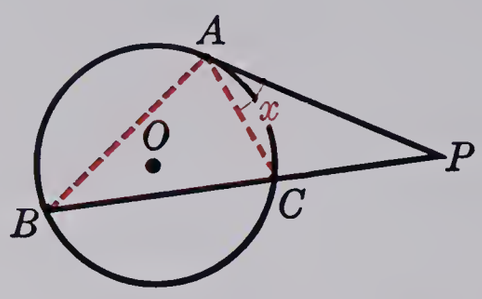
10. Find the height of a circular arch having a radius of 5 feet if the height of the arch is one fourth of its span.

11. A chord 24 inches long is 9 inches from the center of the circle. Find the length of another chord of the circle which is 3 inches from the center.

12. Using Th. 70, construct the fourth proportional to three given line segments.

★Theorem 71

433. If a tangent and a secant are drawn to a circle from the same point, the tangent is the mean proportional between the secant and its external segment.



Given the tangent PA touching $\odot O$ at A and the secant PB intersecting the $\odot$ at C.

To prove that $\frac{PB}{PA} = \frac{PA}{PC}$, or $\overline{PA}^2 = PB \cdot PC$.

Planning the Proof: 1. We can prove a line segment the mean proportional by §§ 420 c, 425, 427, 428.
2. We shall use § 420 c.

Proof

STATEMENTS	REASONS
1. Draw AB and AC.	1. Why possible?
In $\triangle PAC$ and PBA,	
2. $\angle P = \angle P$,	2. Why?
3. $\angle x \doteq \frac{1}{2}\overset{\frown}{AC}$, and	3. Why?
4. $\angle B \doteq \frac{1}{2}\overset{\frown}{AC}$.	4. Why?
5. $\therefore \angle x = \angle B$.	5. Why?
6. $\therefore \triangle PAC \sim \triangle PBA$.	6. Why?
7. $\frac{PB}{PA} = \frac{PA}{PC}$, or $\overline{PA}^2 = PB \cdot PC$.	7. Why?

EXERCISES

A

1. From a point outside a circle a tangent 7 inches long and a secant are drawn to the circle. If the secant is 10 inches long, what is the length of its external segment? of its chord?

2. Find the length of a tangent drawn to a circle from an external point if a secant from this point is 18 inches long and its external segment is 8 inches long.

3. Find the length of a tangent drawn to a circle from an external point if a secant from this point is 36 inches long and the chord of the secant is 10 inches.

4. A tangent and a secant are drawn to a circle from the same point. If the tangent is 24 inches in length and the secant is 20 inches longer than its external segment, find the length of the secant.

5. A point is 9 inches from a circle, and the length of the tangent from this point is 24 inches. Find the diameter of the circle.

6. The chord AB of one of two concentric circles is tangent to the other circle. Find the diameters of the circles if AB is 10 inches and the circles are 1 inch apart.

B

7. Prove that tangents to two intersecting circles from any point in their common chord produced are equal.

8. The product of any secant from a fixed point without a circle and its external segment is constant.

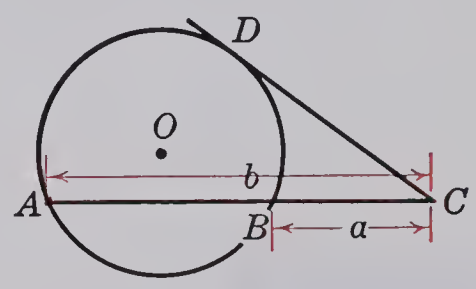
SUGGESTION. The length of a tangent from the fixed point is constant.

9. Two circles are tangent internally. From any point in their common tangent secants are drawn to the two circles respectively. Prove that the product of one secant and its external segment is equal to the product of the other secant and its external segment.

C

10. Construct the mean proportional between two line segments, a and b, making use of Th. 71.

SUGGESTION. Does center O lie on the perpendicular bisector of AB?



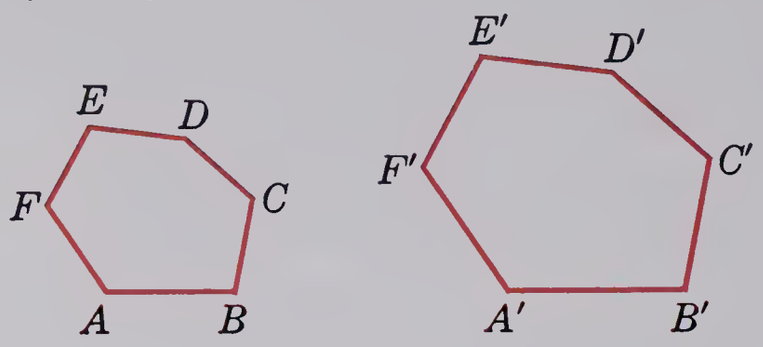
11. Construct a circle passing through two given points and tangent to a given line.

12. If the radius of a circle intersects a chord of the circle, the product of the segments of the chord is equal to the square of the radius minus the square of the segment of the radius adjacent to the center.

13. If a secant CBA and a tangent CE are drawn to a circle from point C and AE is a diameter of the circle, prove that $\overline{AE}^2 = AC \cdot AB$.

Theorem 72

434. The perimeters of two similar polygons have the same ratio as any two corresponding sides.



Given polygon $ABCDEF \sim$ polygon $A'B'C'D'E'F'$ with vertex A corresponding to A', B to B', etc., and with the perimeters denoted by P and P' respectively.

To prove that $\frac{P}{P'} = \frac{AB}{A'B'}$.

Planning the Proof: 1. We can prove proportions by §§ 387, 390, 391, 392, 393, 409, 413, 417.
2. We shall use §§ 413 and 390.

Proof

STATEMENTS	REASONS
1. $ABCDEF \sim A'B'C'D'E'F'$.	1. Why?
2. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots$.	2. Why?
3. $\therefore \frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'}$.	3. Why?
4. $\therefore \frac{P}{P'} = \frac{AB}{A'B'}$.	4. Why?

EXERCISES

A

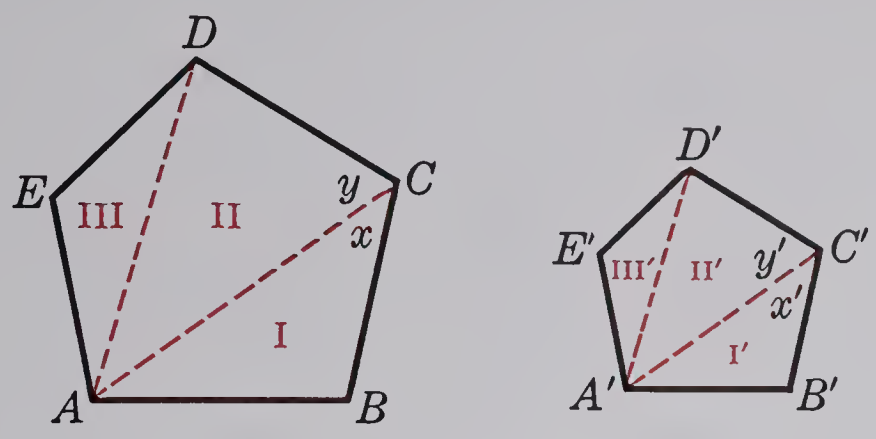
1. Does Th. 72 apply to triangles?

2. The sides of one polygon are 3, 7, 6, 8, and 15. The longest side of a similar polygon is 20. Find its perimeter.

3. The sides of a pentagon are 3, 4, 5, 6, and 7 inches respectively. Find the sides of a similar pentagon if its perimeter is 45 inches.

★ Theorem 73

435. If two polygons are similar, they can be separated into the same number of triangles, similar each to each.



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$ with AB corresponding to $A'B'$, BC to $B'C'$, etc.

To prove that each polygon can be separated into the same number of △ which are similar each to each.

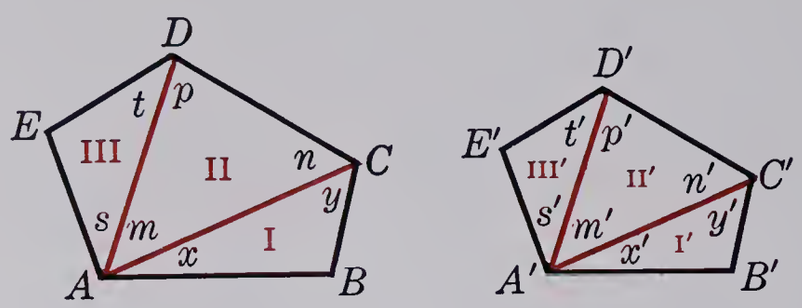
Planning the Proof: 1. We can prove △ $\sim$ by §§ 413, 415, 416, 418, 422, 423, 424.
2. We shall use § 422.

Proof

STATEMENTS	REASONS
1. Draw AC, AD, $A'C'$, and $A'D'$.	1. Why possible?
2. In △ I and I′, $\angle B = \angle B'$, and	2. Why?
3. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.	3. Why?
4. $\therefore$ △ I $\sim$ △ I′.	4. Why?
5. $\angle BCD = \angle B'C'D'$.	5. Why?
6. $\angle x = \angle x'$.	6. Why?
7. $\therefore \angle y = \angle y'$.	7. Asmt. 2.
8. Since △ I $\sim$ △ I′, $\dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	8. Why?
9. But $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$.	9. Why?
10. $\therefore \dfrac{AC}{A'C'} = \dfrac{CD}{C'D'}$.	10. Why?
11. From (7) and (10), △ II $\sim$ △ II′.	11. Why?
12. In like manner, △ III $\sim$ △ III′.	12. Why?

Theorem 74

436. If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.



Given polygons $ABCDE$ and $A'B'C'D'E'$ with $\triangle$ I $\sim \triangle$ I′, $\triangle$ II $\sim \triangle$ II′, and $\triangle$ III $\sim \triangle$ III′.

To prove that $ABCDE \sim A'B'C'D'E'$.

Planning the Proof: 1. We can prove polygons $\sim$ by § 413.
2. We shall use § 413.

Proof

STATEMENTS	REASONS
1. Since $\triangle$ I $\sim \triangle$ I′, $\angle B = \angle B'$, $\angle x = \angle x'$, and $\angle y = \angle y'$.	1. Why?
2. Since $\triangle$ II $\sim \triangle$ II′, $\angle m = \angle m'$, $\angle n = \angle n'$, and $\angle p = \angle p'$.	2. Why?
3. Since $\triangle$ III $\sim \triangle$ III′, $\angle s = \angle s'$, $\angle t = \angle t'$, and $\angle E = \angle E'$.	3. Why?
4. $\therefore \angle EAB = \angle E'A'B'$, $\angle BCD = \angle B'C'D'$, and $\angle CDE = \angle C'D'E'$.	4. Asmt. 1.
5. $\therefore$ the $\measuredangle$s of $ABCDE$ = corr. $\measuredangle$s of $A'B'C'D'E'$.	5. St. 1–4.
6. Since $\triangle$ I $\sim \triangle$ I′, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \left(\frac{AC}{A'C'}\right)$.	6. Why?
7. Since $\triangle$ II $\sim \triangle$ II′, $\left(\frac{AC}{A'C'}\right) = \frac{CD}{C'D'} = \left(\frac{AD}{A'D'}\right)$.	7. Why?
8. Since $\triangle$ III $\sim \triangle$ III′, $\left(\frac{AD}{A'D'}\right) = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.	8. Why?
9. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.	9. Why?
10. $\therefore ABCDE \sim A'B'C'D'E'$.	10. Why?

To the Teacher. If it is desired to save time, Theorems 73, 74, and 76 may be postulated, and Problem 24 may be omitted.

EXERCISES

A

1. The sides of a polygon are 6, 10, 12, and 14 inches respectively. Find the perimeter of a similar polygon whose shortest side is 9 inches.

2. The sides of a quadrilateral are 9, 10, 12, and 15 inches respectively. The perimeter of a similar quadrilateral is $61\frac{1}{3}$ inches. Find its sides.

3. What theorems have you studied that bear on map-making?

4. A certain township is a square 6 miles on a side. Make a map of this township, using the scale of 1 to 100,000. In this township two post offices are 4 miles apart. What is the distance between the locations of the post offices on the map? How long on your map is the diagonal of the township?

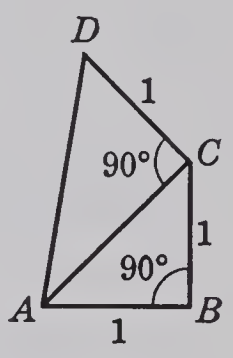
5. The diagonals of a rhombus are 15 and 20 inches respectively. Find the perimeter of the rhombus.

6. The dimensions of one rectangle are $a + 3$ and $2a - 1$, and the dimensions of another rectangle are $3a + 2$ and $2a - 1$. What is the ratio of the area of the first rectangle to the area of the second?

B

7. Find the altitude of an equilateral triangle whose sides are 8 inches in length.

8. Find the length of each side of an equilateral triangle whose altitude is 20 inches.



9. How long is AC in the figure? How long is AD? How would you construct a line segment $\sqrt{3}$ inches in length if you had given a line segment 1 inch long?

10. Find the side of a regular hexagon having an area of $150\sqrt{3}$.

C

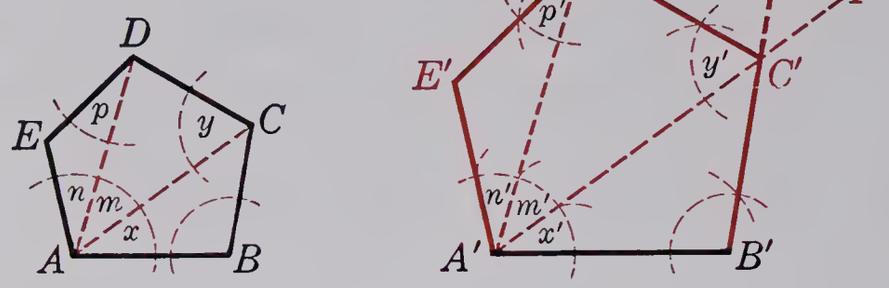
11. Given a line segment m, construct a line segment $m\sqrt{2}$ by using the method of Ex. 9.

12. Given a line segment m, construct a line segment $m\sqrt{2}$ by using § 429.

13. Find the distance between two parallel sides of the rhombus in Ex. 5.

Problem 24

437. Upon a given line segment corresponding to one side of a given polygon to construct a polygon similar to the given polygon.



Given polygon $ABCDE$ and $A'B'$ corresponding to AB.

To construct on $A'B'$ a polygon $\sim ABCDE$.

Construction

STATEMENTS	REASONS
1. From A draw all the diagonals, as AC and AD.	1. Why possible?
2. At A' construct $A'F$, making $\angle x' = \angle x$, and then at B' construct $B'G$, making $\angle B' = \angle B$.	2. Why possible?
3. $A'F$ and $B'G$ will intersect at some point C'.	3. Prove it.
4. In the same manner, using $A'C'$ as a side, construct $\angle m' = \angle m$ and $\angle y' = \angle y$, forming $\triangle A'C'D'$.	4. Why possible?
5. Likewise, construct $\angle n' = \angle n$ and $\angle p' = \angle p$, forming $\triangle A'D'E'$.	5. Why possible?

Then $A'B'C'D'E' \sim ABCDE$.

Proof: (The proof is left to the student.)

SUGGESTIONS. Prove $\triangle A'B'C' \sim \triangle ABC$, $\triangle A'C'D' \sim \triangle ACD$, etc., and then use § 436.

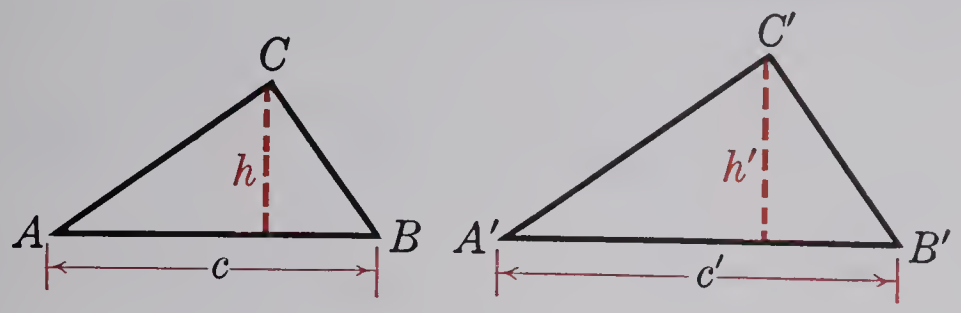
EXERCISES

1. Draw a polygon having four sides, one side being of length b. Draw a segment of length b' and upon it construct a polygon similar to the first polygon.

2. Draw a hexagon and construct a similar hexagon such that the ratio of similitude of the new hexagon to the other is $\frac{2}{1}$.

★★ Theorem 75

438. The areas of two similar triangles have the same ratio as the squares of any two corresponding sides.



Given $\triangle ABC \sim \triangle A'B'C'$ having c and c' any pair of corresponding sides.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c^2}{c'^2}$.

Planning the Proof: 1. We can compare areas of polygons by §§ 272, 274.
2. We shall use § 272 c.

Proof

STATEMENTS	REASONS
1. Draw the altitudes h and h' to the sides c and c' respectively.	1. Why possible?
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{ch}{c'h'}$.	2. § 272 c.
3. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c}{c'} \times \dfrac{h}{h'}$.	3. Why?
4. But $\dfrac{h}{h'} = \dfrac{c}{c'}$.	4. Why?
5. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c}{c'} \times \dfrac{c}{c'} = \dfrac{c^2}{c'^2}$.	5. Why?

439. Corollary. The areas of two similar triangles have the same ratio as the squares of any two corresponding altitudes.

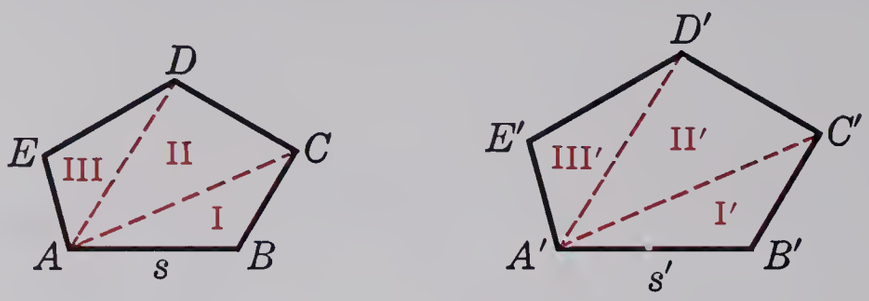
EXERCISES Oral

1. The corresponding sides of two similar triangles are 4 and 5. What is the ratio of their areas?

2. The areas of two similar triangles are 64 and 100. What is the ratio of their corresponding sides?

★★ Theorem 76

440. The areas of two similar polygons have the same ratio as the squares of any two corresponding sides.



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$ having any pair of corresponding sides s and s'.

To prove that $\dfrac{ABCDE}{A'B'C'D'E'} = \dfrac{s^2}{s'^2}$.

Planning the Proof: 1. We can compare areas of polygons by §§ 272, 274, 438, 439.
2. We shall use § 438.

Proof

STATEMENTS	REASONS
1. From any two corresponding vertices, A and A', draw the diagonals of each polygon.	1. Why possible?
2. $\triangle$ I $\sim \triangle$ I′, $\triangle$ II $\sim \triangle$ II′, and $\triangle$ III $\sim \triangle$ III′.	2. Why?
3. $\dfrac{\triangle \text{ I}}{\triangle \text{ I}'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\triangle \text{ II}}{\triangle \text{ II}'} = \dfrac{\overline{AD}^2}{\overline{A'D'}^2} = \dfrac{\triangle \text{ III}}{\triangle \text{ III}'}$.	3. Why?
4. $\therefore \dfrac{\triangle \text{ I}}{\triangle \text{ I}'} = \dfrac{\triangle \text{ II}}{\triangle \text{ II}'} = \dfrac{\triangle \text{ III}}{\triangle \text{ III}'}$.	4. Asmt. 7
5. $\therefore \dfrac{\triangle \text{ I} + \triangle \text{ II} + \triangle \text{ III}}{\triangle \text{ I}' + \triangle \text{ II}' + \triangle \text{ III}'} = \dfrac{\triangle \text{ I}}{\triangle \text{ I}'}$.	5. Why?
6. But $\triangle$ I + $\triangle$ II + $\triangle$ III = polygon $ABCDE$ and $\triangle$ I′ + $\triangle$ II′ + $\triangle$ III′ = polygon $A'B'C'D'E'$.	6. Asmt. 9
7. $\dfrac{\triangle \text{ I}}{\triangle \text{ I}'} = \dfrac{s^2}{s'^2}$.	7. § 438.
8. $\therefore \dfrac{ABCDE}{A'B'C'D'E'} = \dfrac{s^2}{s'^2}$.	8. Asmt. 6

EXERCISE

In the figure of § 440, if $s' = 3\,s$, how are the areas related?

EXERCISES Similar Polygons

A

1. The corresponding sides of two similar triangles have the ratio 1 : 2. What is the ratio of their altitudes? perimeters? areas?

2. Two corresponding sides of two similar polygons are 6 and 18 respectively. What is the ratio of their areas? of their perimeters?

3. The area of a triangle is 16 times that of a similar one. Find the ratio of two corresponding sides.

4. Construct a triangle that is similar to a given triangle and equal to one ninth of it.

5. The areas of two similar polygons are 64 square inches and 100 square inches. If a side of the first is 6 inches, find the corresponding side of the second.

6. Two similar polygons have areas of 200 square inches and 1152 square inches respectively. If a side of the first is 20 inches, find the corresponding side of the second.

B

7. The ratio of the sides of two squares is 2 : 3, and the difference of the areas of the squares is 405. What is the area of each of the squares?

8. The altitude of one equilateral triangle is equal to the side of another equilateral triangle. What is the ratio of their areas?

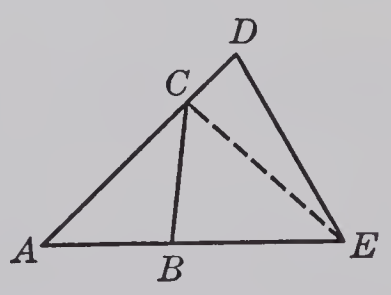
9. In $\triangle DEF$, $DE = 6$ inches and $EF = 4$ inches. G and H are points on DE and EF respectively such that $DG = 4$ inches and $EH = 3$ inches. Is $\triangle DEF \sim \triangle EHG$? Give reason for your answer.

10. AED and BEC are two chords of a circle. $AE = 12''$ and $EC = 6''$. Find the ratio of the area of $\triangle ECD$ to the area of $\triangle ABE$.

C

11. Divide a triangle into three equal parts by drawing lines parallel to the base.

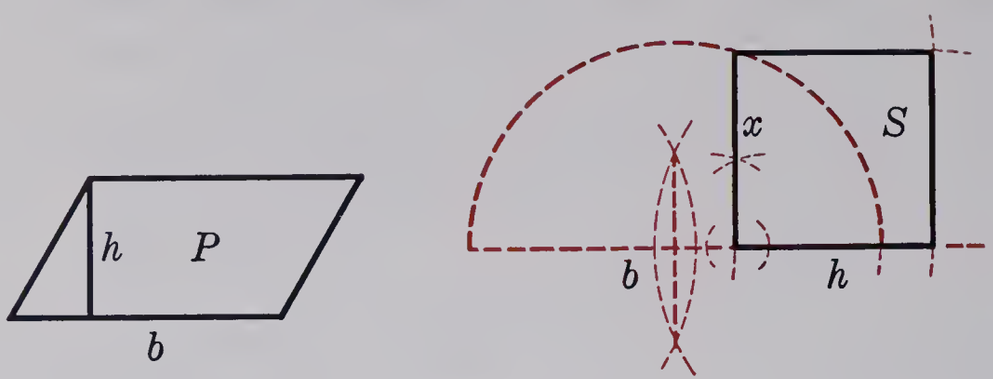
12. *Prove*: Two triangles which have an angle of one equal to an angle of the other have the same ratio as the products of the sides including the equal angles.



SUGGESTIONS. $\frac{\triangle ABC}{\triangle AEC} = ?$ $\frac{\triangle AEC}{\triangle AED} = ?$ $\frac{\triangle ABC}{\triangle AEC} \times \frac{\triangle AEC}{\triangle AED} = ?$

Problem 25

441. To construct a square equal to a given parallelogram.



Given the $\square$ P having base b and altitude h.
To construct a square $S = P$.

Construction

STATEMENTS	REASONS
1. Construct x, the mean proportional between h and b.	1. § 429.
2. Construct a square S having x as a side.	2. § 173.
Then $S = P$.	

Proof

1. $x^2 = hb$.	1. Const.
2. But $x^2 = S$.	2. Why?
3. $P = hb$.	3. Why?
4. $\therefore S = P$.	4. Why?

442. Solving Problems

At this time you should review the directions in § 175 for the solution of construction problems. In addition to these directions, if the construction exercise involves area or other numerical relations, you will often be able to discover the method of construction by the use of algebra. For example, in Prob. 25, $S = x^2$ and $P = hb$. Since S must equal P, then $x^2 = hb$. Then x is the mean proportional between h and b. Also the area of a rectangle with base b and altitude h, and equal in area to a square with side s, is given by $bh = s^2$, or $\frac{b}{s} = \frac{s}{h}$.

EXERCISES — Construction

A

1. Construct a square equal to the sum of two given squares whose sides are 3 inches and 4 inches respectively.

2. Construct a square equal to a given rectangle.

3. Construct a square equal to a given triangle.

SUGGESTIONS. Let x be a side of the required square, b the base of the given triangle, and a its altitude. Then $x^2 = \frac{1}{2}\, a \cdot b$, or $\frac{\frac{1}{2}\, a}{x} = \frac{x}{b}$.

4. Construct a right triangle equal to a given triangle and having the same base.

B

5. Construct a square equal to a given quadrilateral.

SUGGESTION. Construct a triangle equal to the quadrilateral and then construct a square equal to the triangle.

6. Construct a square equal to a given polygon.

7. Construct a rectangle which has a given base and is equal in area to another rectangle. (Use proportion.)

8. Construct an isosceles triangle on a given line segment as a base and equal to a given triangle.

9. Construct a square having $\frac{9}{16}$ of the area of a given square.

10. Construct a triangle equal to the sum of two given triangles.

11. Construct a right triangle equal to a given triangle and having its hypotenuse equal to a given line segment.

12. Construct a square equal to a given trapezoid.

13. Divide a parallelogram into three equal parts by lines from one vertex.

14. Construct a line parallel to the base of a given triangle dividing the triangle into two parts which have the ratio 4 : 5.

C

15. Construct $\triangle ABC$ similar to a given $\triangle DEF$ and having a perimeter equal to a given line segment m.

16. Construct an equilateral triangle equal to a given triangle.

SUGGESTION. $\frac{1}{4}\, s^2\sqrt{3} = \frac{1}{2}\, bh$. Then $s^2 = \frac{2}{3}\sqrt{3}\, bh$.

EXERCISES — Miscellaneous

A

1. A chord 6 inches long is 4 inches from the center of the circle. What is the length of the radius of the circle?

2. A chord 30 inches long is in a circle with a radius of 34 inches. How far is the chord from the center of the circle?

3. Construct a polygon similar to a given hexagon and having its sides one half as long.

4. The diagonal of a square is $7\sqrt{2}$. Find the length of a side.

5. Find the altitude of an isosceles triangle whose base is 14 inches and whose perimeter is 50 inches.

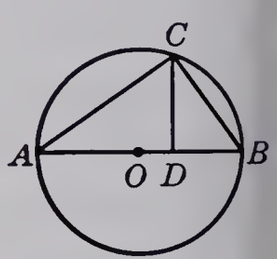
6. Construct a rectangle similar to a given rectangle and having its base three times as long as the given rectangle. What is the ratio of the two rectangles?

7. The radii of two concentric circles are 9 inches and 15 inches respectively. Find the length of a chord of the larger circle that is tangent to the smaller circle.

8. The corresponding sides of two similar polygons are 6 inches and 9 inches. If the area of the first polygon is 48 square inches, what is the area of the second?

9. The base of one rectangle is 56 feet and its altitude is 27 feet. The base of another rectangle is 18 feet and its altitude is 24 feet. Are the rectangles similar? Why?

10. In the figure, AB, a diameter of circle O, is 26 inches long and DB is 8 inches long. Find the length of the perpendicular CD; of AC; of BC.



11. If the ratio of the areas of two similar triangles is 16 : 25, what is the ratio of their perimeters?

12. Two corresponding sides of two similar polygons are 15 feet and 18 feet. What is the ratio of their perimeters? of their areas?

13. $\triangle ABC$ has a right angle at C. CA is extended through A to D. $DE \perp BA$ extended at E. Prove that $DA : AB = DE : CB$.

14. Find the side of a square equal to the sum of three squares whose sides are 3, 4, and 12 respectively.

15. The diagonals AC and BD of trapezoid $ABCD$ intersect in O; $AB \parallel DC$; $AB = 3\,DC$. Compare $\triangle AOB$ and COD; $\triangle AOD$ and BOC.

B and C

16. *Prove:* If two triangles have their sides respectively parallel, they are similar.

17. *Prove:* If two triangles have their sides respectively perpendicular, they are similar.

18. *Prove:* A line parallel to the base of a triangle and intersecting the other two sides forms a triangle similar to the given triangle.

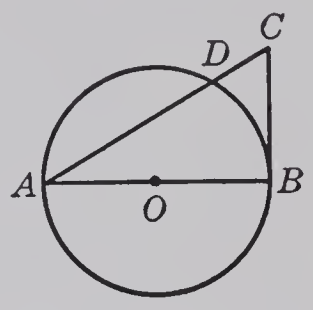
19. If one leg of a right triangle is $2\,ab$ and the other leg is $a^2 - b^2$, show that the hypotenuse is $a^2 + b^2$.

20. The product of the two legs of a right triangle is equal to the product of the hypotenuse and the altitude on the hypotenuse.

21. The legs of a right triangle are 16 feet and 20 feet respectively. Find the length of the radius of the circumcircle.

22. *Given* $\odot O$, diameter AB, tangent BC, and secant CA.

Prove that $AB = \sqrt{AD \times AC}$.



23. Construct the locus of a point if the sum of the squares of its distances from two given perpendicular lines is 25.

SUGGESTION. How far is the point from the intersection of the two lines?

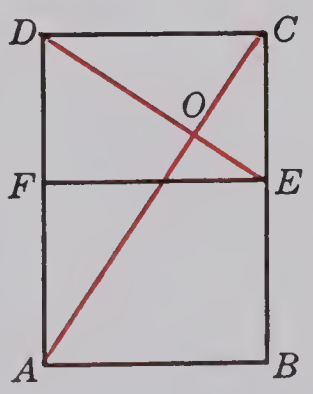
24. Two circles having radii of 10 inches and 20 inches are tangent externally. Find the length of their common external tangents.

25. Find the area of a triangle whose medians are 30 inches, 30 inches, and 48 inches.

26. The area of a rectangle is 320 square inches and the perimeter is 72 inches. Find the dimensions of the rectangle.

27. Find the area of a rectangle having diagonal d and length l.

28. *Given* rectangle $ABCD$ with $DE \perp AC$ and $FE \perp BC$.



a. *Prove:* $\triangle AOD \sim \triangle COE$.

b. *Prove:* $\triangle AOD \sim \triangle ADC$.

c. *Prove:* $\triangle ADC \sim \triangle DCE$.

d. *Prove:* $\dfrac{AD}{DC} = \dfrac{DC}{CE}$.

e. *Prove:* rectangle $ABCD \sim$ rectangle $FECD$.

29. Find the area of a triangle whose sides are b, b, and c.

30. Through P, a point within $\angle ABC$, construct a line segment whose end points are in AB and CB and which is divided by P into segments having the ratio 1 : 2.

31. The sum of the three perpendiculars from any point within an equilateral triangle to the sides is constant.

SUGGESTION. Prove the sum equal to the altitude of the triangle.

32. Find the locus of the midpoints of the chords of a given circle if the chords pass through a given point within the circle.

33. If PA is a secant which intersects a circle at A and B, determine the locus of point P so that $PA \cdot PB$ is constant.

34. Prove that in a 30°-60° right triangle the altitude on the hypotenuse divides the hypotenuse in the ratio 1 : 3.

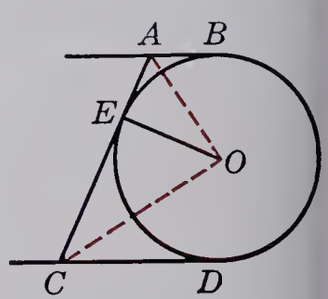
35. Prove that if one leg of a right triangle is twice the other, the altitude on the hypotenuse divides the hypotenuse in the ratio 1 : 4.

36. If AB is the hypotenuse of a right triangle and the leg BC is bisected in D, $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

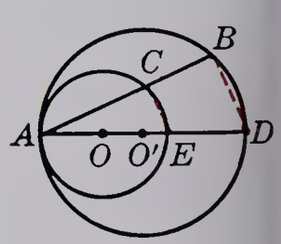
37. In the right triangle ABC, BE and AF bisect the legs AC and BC in the points E and F. Prove that $4\,\overline{BE}^2 + 4\,\overline{AF}^2 = 5\,\overline{AB}^2$.

38. Two chords intersect within a circle whose center is O. The shorter chord is divided into segments $2x$ and $3x$ and the longer chord into segments y and $6y$. Find the lengths of the chords if one exceeds the other by 8 inches.

39. Construct $\sqrt{7}$ geometrically.



40. If a third tangent is included between two parallel tangents to a circle, the radius of the circle is the mean proportional between the segments of the third tangent.

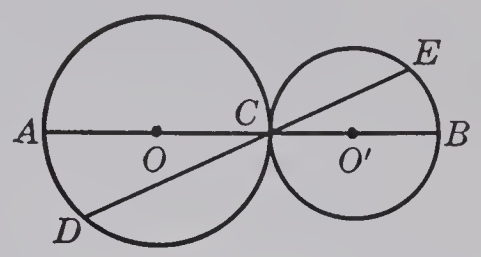


41. If two circles are tangent internally, any chord of the larger circle drawn from the point of tangency is divided by the smaller circle in a constant ratio.

SUGGESTION. Prove that $\dfrac{AB}{AC} = \dfrac{AD}{AE}$.

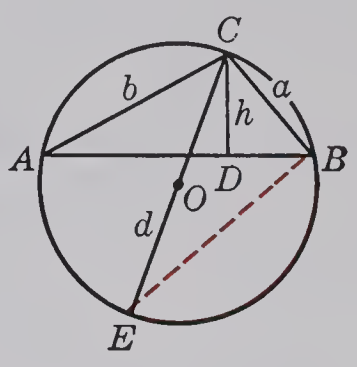
42. Construct a polygon similar to a given polygon and equal to a given square.

43. *Given* $\odot O$ and $\odot O'$ tangent externally at C, with DE a straight line segment through C terminated by the circles and AB a straight line segment passing through the centers O and O'.



Prove that $CE : CD = CB : CA$.

44. In a triangle the product of two sides is equal to the product of the altitude on the third side and the diameter of the circumcircle.

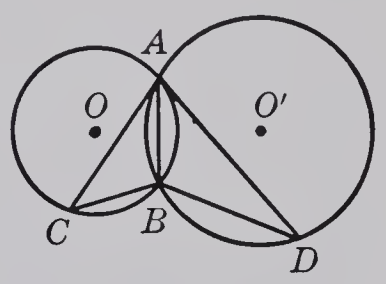


SUGGESTION. Prove that $\triangle ACD \sim \triangle ECB$.

45. Prove that the area of $\triangle ABC$ in the figure is given by the formula $A = \dfrac{abc}{4R}$. $(R = \frac{1}{2}d.)$

46. Construct x, if $ax = b^2$.

47. *Given* $\odot O$ and $\odot O'$ with the common chord AB, chord AC of $\odot O$ tangent to $\odot O'$ at A, chord AD of $\odot O'$ tangent to $\odot O$ at A, and chords BC and BD.



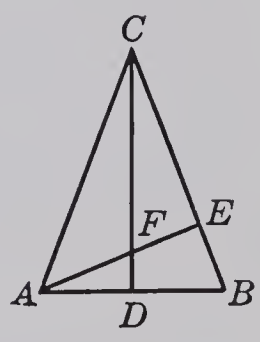
Prove that AB is the mean proportional between BC and BD.

48. In a given circle inscribe a triangle similar to a given triangle.

49. Inscribe a square in a semicircle.

50. Solve $x^2 = 6$ geometrically by § 429.

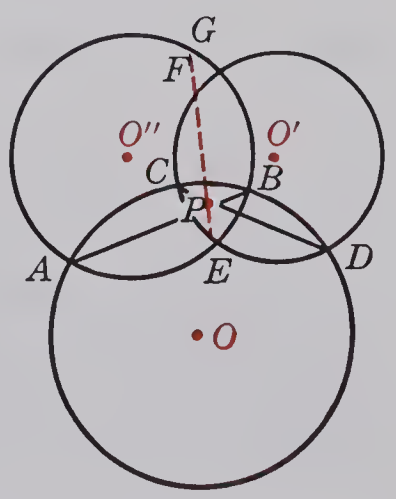
51. *Given* the isosceles $\triangle ABC$ with $AC = BC$ and the altitudes CD and AE intersecting at F.



Prove that $AC : AF = CD : DB$.

52. Inscribe a square in a given triangle so that one side of the square lies on the base of the triangle.

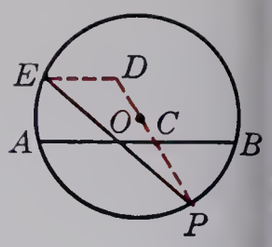
53. If three circles intersect one another, their common chords are concurrent.



SUGGESTIONS. AB and CD will intersect at some point P. Join E to P. Produce EP and let it meet ⓢ O' and O'' at F and G. Prove that $PF = PG$ by § 431, and hence F and G coincide.

54. Given the altitude, construct a triangle similar to a given triangle.

55. Through a given point in the arc of a given chord, construct another chord which shall be bisected by the given chord.

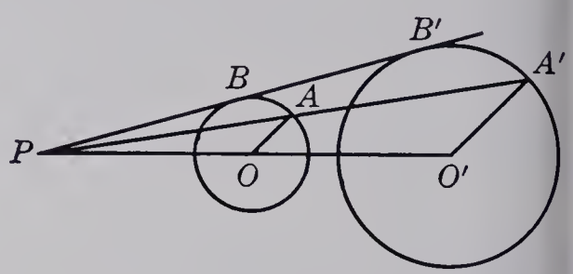


SUGGESTIONS. On radius OP produced take $CD = CP$ and construct $DE \parallel AB$. Then EP is the required chord.

56. Construct a triangle, given two angles and the sum of the included side and the altitude on the included side.

SUGGESTIONS. Construct any triangle with the two given angles and from it determine the ratio of the base and the altitude for the required triangle.

57. *Given* PB' tangent to ⓢ O and O' at B and B' respectively, POO' through the line of centers, PAA' a common secant, and radii OA and $O'A'$.



Prove that $OA \parallel O'A'$.

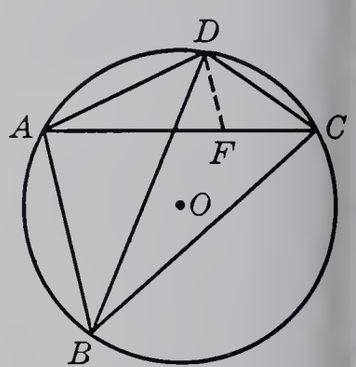
58. Find a point such that tangents from it to two given circles are equal.

59. Prove that the line segment joining the midpoints of two adjacent sides of a parallelogram forms with the parallel diagonal a trapezoid whose area is three eighths the area of the parallelogram.

60. *Ptolemy's Theorem.*

Given quadrilateral $ABCD$ inscribed in $\odot O$ and having diagonals AC and BD.

Prove: $AC \times BD = AD \times BC + AB \times DC$.



SUGGESTIONS. Draw DF, making $\angle FDC = \angle ADB$.

Prove $\triangle ABD \sim \triangle FDC$ and $\triangle ADF \sim \triangle BDC$.

443. Scale Drawings

A *map* or *plan* is a scale drawing. Its lines are proportional to the lines they represent. By Ths. 67 and 74 a map or plan of any plane figure is similar to the figure.

The *scale of a drawing* is the ratio of any line of the drawing to the corresponding line of the object. Thus when 1 inch of the drawing represents 40 feet (480 inches) of the object, the scale is 1 : 480 (usually written in the form *Scale* $1'' = 40'$).

Adapted from *Living on the Level* by Royal Barry Wills. Courtesy of Houghton Mifflin Company

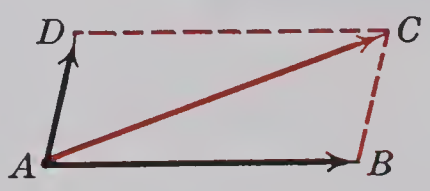
EXERCISES

A

1. Above is an architect's plan for a one-story house, with the dimensions erased, drawn to the scale $\frac{1}{16}'' = 1'$. The living room is 22 feet wide. How long is it? What are the dimensions of each of the other rooms?

2. Excluding the end zones, a football field is 300 feet long and 160 feet wide. It is marked off with lines 5 yards apart and parallel to the end lines. Make a plan of the field using the scale $1'' = 50'$.

3. It is shown in physics that when two concurring forces act at an angle upon an object, their *resultant* (the single force which will produce the same effect on the motion of the object) is represented by the diagonal of a parallelogram constructed on the two segments representing the two given forces. In the figure, AB and AC represent two forces acting at A, and AD represents the resultant of these forces. Two forces, of 30 and 40 pounds respectively, act upon a body at an angle of 90°. Find their resultant by constructing the parallelogram of forces to scale.



4. Construct the resultant of two forces, of 100 and 120 pounds respectively, which act at an angle of 60°; at an angle of 90°; at an angle of 120°. Which resultant is the greatest?

Theorem 77 [Optional]

444. In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides and the projection of the other side on it.

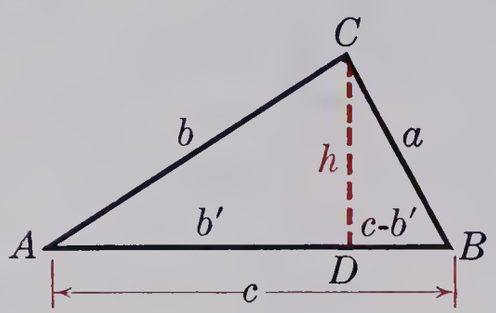


Fig. 1

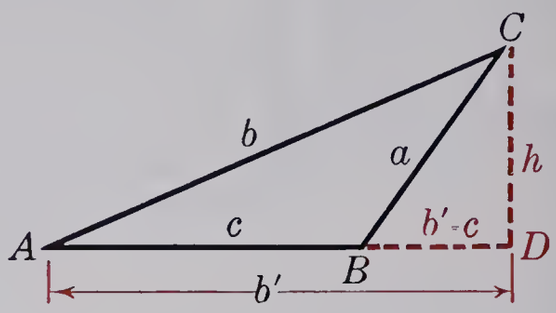


Fig. 2

Given $\triangle ABC$ with $\angle A$ acute and $AD = b'$ the projection of b on c.
To prove that $a^2 = b^2 + c^2 - 2cb'$.

Proof

STATEMENTS	REASONS
1. In Fig. 1, $a^2 = h^2 + (c - b')^2$ $= h^2 + c^2 - 2cb' + b'^2$.	1. Why?
2. In Fig. 2, $a^2 = h^2 + (b' - c)^2$ $= h^2 + c^2 - 2cb' + b'^2$.	2. Why?
3. But $h^2 + b'^2 = b^2$.	3. Why?
4. Then, for either triangle, $a^2 = b^2 + c^2 - 2cb'$.	4. Why?

EXERCISES

1. In the theorem above, if $b = 13$, $c = 14$, and $b' = 5$, find a.

2. In the theorem above, if $a = 96$, $b = 58$, and the projection of c on $b = 22$, find c.

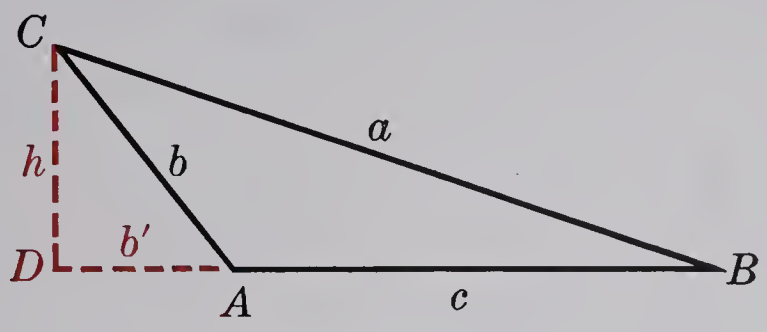
3. In $\triangle ABC$, $AB = 10$, $AC = 8$, and $b' = 6.95$. Find BC.

4. In $\triangle ABC$, $AC = 12$, $AB = 17$, and $BC = 19$. Find the projection of AC on AB.

5. In $\triangle DEF$, $DE = 15$, $DF = 12$, and $EF = 10$. Find the projection of EF on DE.

Theorem 78 [Optional]

445. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides and the projection of the other side on it.*



Given $\triangle ABC$ with $\angle A$ obtuse and b' the projection of b on c.
To prove that $a^2 = b^2 + c^2 + 2cb'$.

Proof

STATEMENTS	REASONS
1. $a^2 = h^2 + (b' + c)^2$ $= h^2 + b'^2 + 2b'c + c^2$.	1. Why?
2. But $h^2 + b'^2 = b^2$.	2. Why?
3. $a^2 = b^2 + c^2 + 2b'c$.	3. Why?

Theorem 79

446. In $\triangle ABC$, if $a^2 < b^2 + c^2$, $\angle A$ is acute; and if $a^2 > b^2 + c^2$, $\angle A$ is obtuse.

EXERCISES

1. The sides of a triangle are 6 inches, 7 inches, and 9 inches. Is the largest angle of the triangle acute, right, or obtuse?

2. The sides of a triangle are 7, 24, and 25. Is the largest angle acute, obtuse, or right?

3. In $\triangle ABC$, $AC = 10$, $AB = 12$, and the projection of AC on $AB = 3\frac{1}{3}$; $\angle A$ is obtuse. Find BC.

4. *Prove:* The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides.

*The law of cosines states that $a^2 = b^2 + c^2 - 2bc \cos A$. This law holds for $\angle A$ being an acute angle, an obtuse angle, or a right angle. See Ex. 30, page 502.

447.* Formulas for the Median and Angle Bisector of a Triangle (Optional)

The median of a triangle (Fig. 1) is given by the formula

$$m_c = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$

The bisector of an angle of a triangle can be expressed by $t_c = \sqrt{ab - mn}$ where m and n are the segments of c made by the bisector t_c. (Fig. 2.)

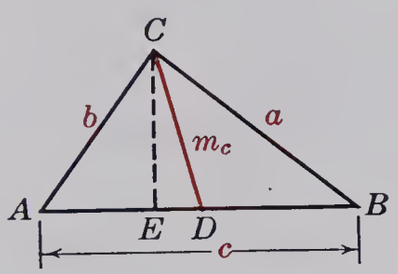


Fig. 1

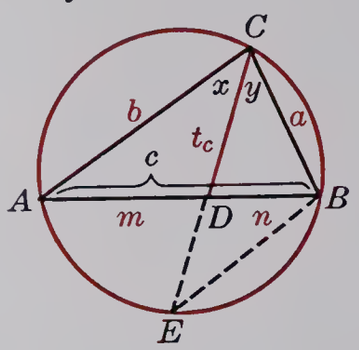


Fig. 2

The bisector of an angle of a triangle in terms of its sides is given by $t_c = \dfrac{2}{a+b}\sqrt{abs(s-c)}$ where $s = \tfrac{1}{2}(a+b+c)$. (Fig. 2.)

EXERCISES

1. The sides of a triangle are 18, 24, and 30. Find the median to the side 24.

2. The sides of a triangle are 8, 15, and 17. Find the altitude to the side 15.

3. In triangle ABC, $AB = 30$, $AC = 15$, and $BC = 18$. Find the segments of AB made by the bisector of angle C. Find the length of the bisector of angle C.

4. In $\triangle ABC$, $AB = 17$, $AC = 8$, and $BC = 15$. Find the lengths of the three medians.

5. In $\triangle ABC$, $AB = 18$, $BC = 20$, and $AC = 16$. Find the lengths of the bisectors of the three angles.

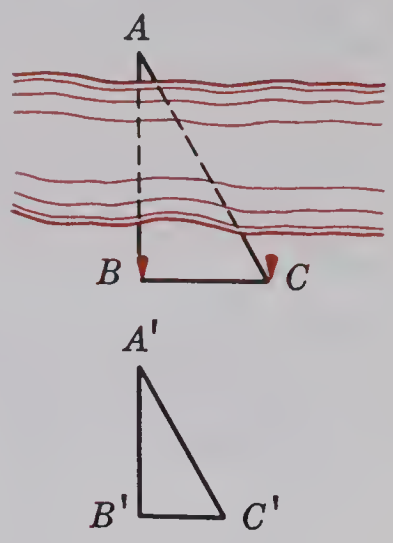
6. In $\triangle RST$, $RS = 8$, $ST = 10$, and $RT = 12$. Find the three altitudes.

7. In $\triangle ABC$, $AB = 10$, $AC = 13$, and $BC = 13$. Find the lengths of the medians; of the altitudes; of the bisectors of the angles.

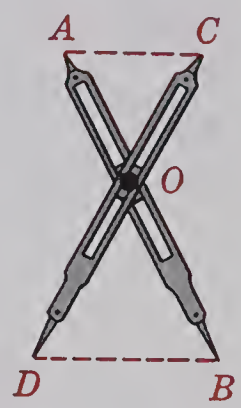
*The formulas in § 447 are given without proof because of the time element.

EXERCISES Applied Problems

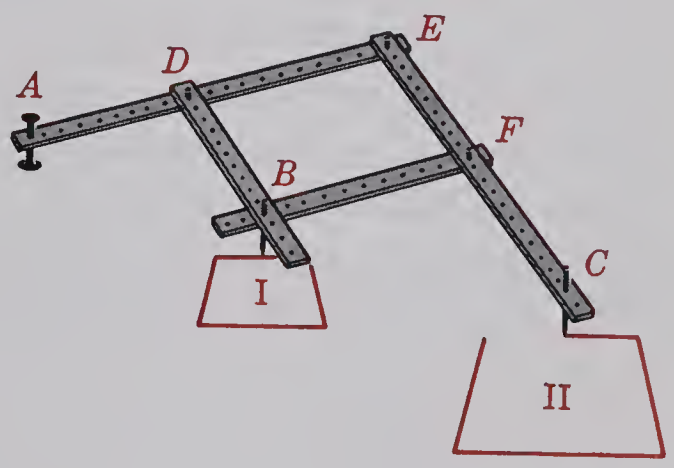
1. Some boys at the point B wished to find the distance across a stream to A. They proceeded as follows: First they set a stake at the point C so that $BC = 100$ feet. From C they sighted the point A. Then on paper they constructed $\triangle A'B'C'$, making $B'C' = 8$ inches, $\angle B' = \angle B$, and $\angle C' = \angle C$. Then they measured $A'B'$ and found it to be 15.5 inches. See if you can find the length of AB by using their measurements.



2. This figure represents a pair of *proportional dividers*. They are adjusted by a thumbscrew at O so that $\frac{AO}{OB} = \frac{CO}{OD}$. Prove that $\triangle AOC \sim \triangle DOB$. If the dividers are adjusted so that $\frac{AO}{OB} = \frac{3}{2}$, how will any measurement made with the points A and C compare with a measurement made by D and B? How could you use the dividers to make a line segment two thirds as large as another line segment?



3. The *pantograph* is an instrument used to draw a plane figure similar to a given plane figure. With it, drawings and maps may be enlarged or reduced in size. There are many different forms of pantographs, but they are all based on the same principle. The instrument consists of two pairs of parallel bars joined to form a parallelogram. The point A is fixed and the points B and C are movable. All three points, A, B, and C, lie in a straight line and no two of them are on the same bar.



a. Prove that $\triangle ABD \sim \triangle ACE$.

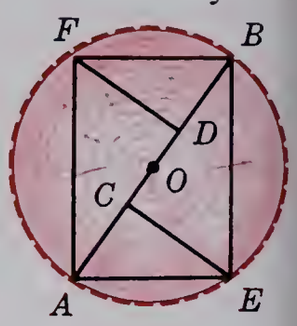
b. Prove that $\frac{AD}{AE} = \frac{EF}{EC}$.

c. Prove that quadrilateral I $\sim$ quadrilateral II. (Draw diagonals.)

4. See if you can discover a practical method of finding the height of a flagpole by using similar triangles.

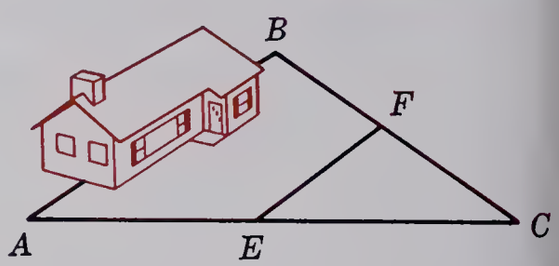
5. An engineer wishes to construct a circular arch having a span of 12 feet and a height of 4 feet. Find the radius of the arc.

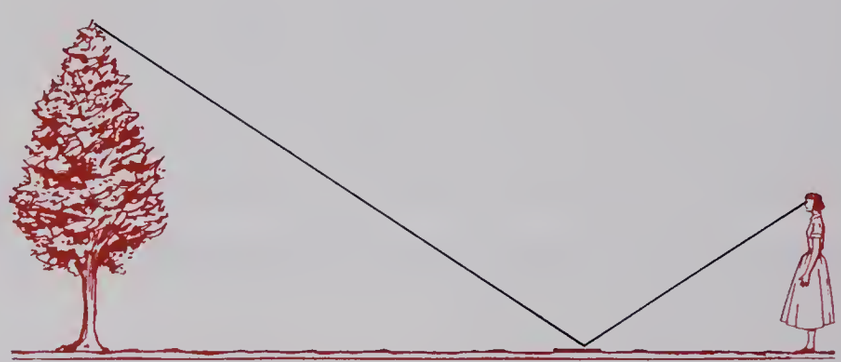
6. To lay out the strongest rectangular beam (one that will carry the heaviest load without breaking) that can be cut from a circular log, the sawyer trisects the diameter of a cross section. At C and D, the points of trisection, he then draws CE and DF, each perpendicular to AB. Next he draws BF, FA, AE, and EB.



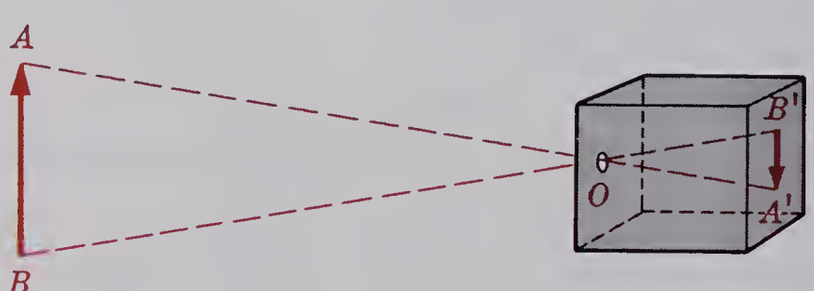
Prove that $\frac{AE}{BE} = \frac{1}{\sqrt{2}}$.

7. A geometry class was asked to find the distance between two points A and B, which were on opposite sides of a building. Show how the figure $AECFB$ can be used to find AB.





8. A girl found the height of a tree by placing a mirror horizontally on the ground and walking backwards until she could see the reflection of the top of the tree in it. Her eye was 4 feet 3 inches from the ground when her feet were 6 feet from the mirror. What was the height of the tree if it was 42 feet from the mirror?

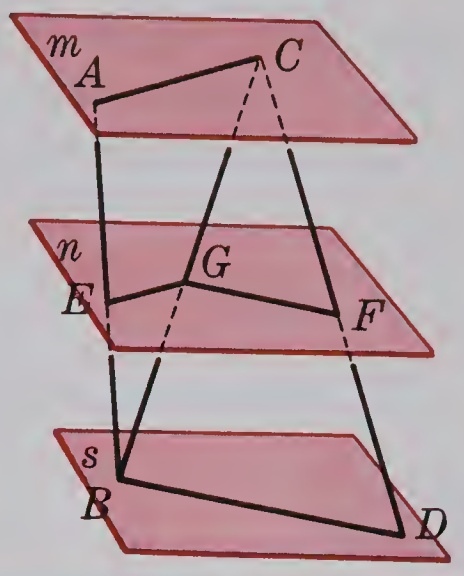


9. $A'B'$ is the image of AB in a pinhole box camera. Notice that the image is inverted. The object AB is 10″ long and the camera is 8″ long. How far should the camera be from AB to make the image 1″ long?

SPACE GEOMETRY

[Optional]

448. Theorem. If two lines are cut by three parallel planes, their corresponding segments are proportional.



Given AB and CD cut by the $\parallel$ planes m, n, and s at A, E, B, and C, F, D respectively.

To prove that $\dfrac{AE}{EB} = \dfrac{CF}{FD}$.

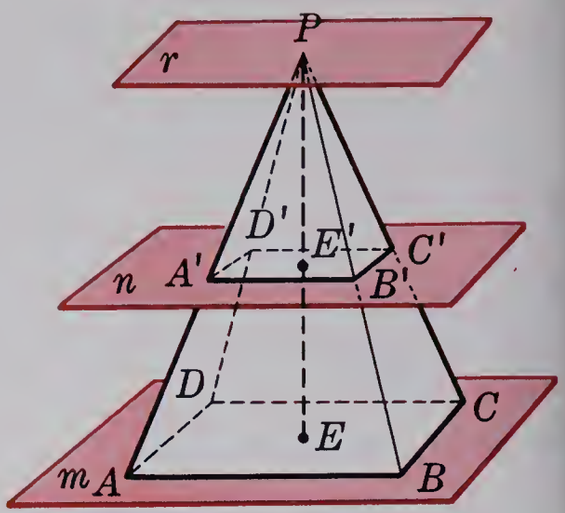
Proof

STATEMENTS	REASONS
1. Draw BC, intersecting n in G.	1. Why possible?
2. The plane of BA and BC intersects n in EG and m in AC. Also the plane of BC and DC intersects s in BD and n in GF.	2. Why?
3. $EG \parallel AC$ and $GF \parallel BD$.	3. Why?
4. $\therefore \dfrac{AE}{EB} = \dfrac{CG}{GB}$ and $\dfrac{CG}{GB} = \dfrac{CF}{FD}$.	4. Why?
5. $\therefore \dfrac{AE}{EB} = \dfrac{CF}{FD}$.	5. Why?

EXERCISES

1. In the figure above, if $AE = 24$, $EB = 15$, and $CF = 36$, find FD.
2. In the figure above, if $AE = 12$, $EB = 18$, and $CD = 40$, find CF.
3. In the figure above, find EG if $AE = 10$, $EB = 6$, and $AC = 8$.
4. In the figure above, find BD if $CD = 20$, $CF = 8$, and $GF = 6$.
5. Find AB and CD in the figure above if $BG = GC$, $AE = 4$, and $CF = 6$.
6. What is the locus of a point whose distances from two parallel planes are in the ratio 1 : 2?
7. What is the locus of a line which intersects a given line at a given point and makes an angle of 45 degrees with the given line?

449. Theorem. If a pyramid is cut by a plane parallel to the base, the lateral edges and the altitude are divided proportionally.



Given the pyramid $P\text{-}ABCD$, and plane n parallel to plane m of the base and cutting the lateral edges in A', B', C', D', and the altitude PE in E'.

To prove that

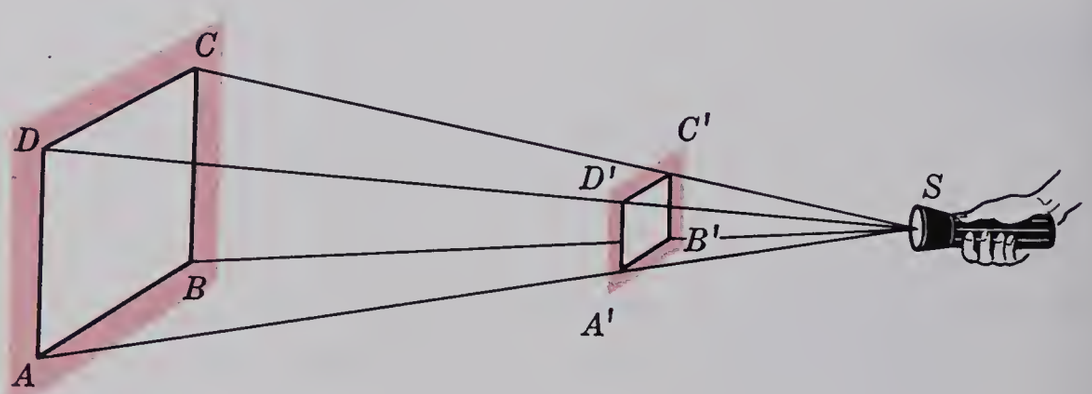
$$\frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = \frac{PD'}{PD} = \frac{PE'}{PE}.$$

Proof

STATEMENTS	REASONS
1. Pass plane r through $P \parallel m$.	1. Why possible?
2. Then $r \parallel n \parallel m$.	2. Why?
3. Then $\frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = \frac{PD'}{PD} = \frac{PE'}{PE}$.	3. Why?

EXERCISES

1. In the figure above, polygon $ABCD \sim$ polygon $A'B'C'D'$, and the area of $A'B'C'D'$ is to the area of $ABCD$ as $\overline{PE'}^2$ is to $\overline{PE}^2$. Find the area of $ABCD$ if $PE' = 2$, $PE = 4$, and the area of $A'B'C'D'$ is 12.



2. The intensity of light on a screen varies inversely as the square of the distance from the source of light S. If the distance from S to $A'B'C'D'$ is 2 inches and from S to $ABCD$ is 6 inches, compare the intensities of the light on the two screens.

REVIEW QUESTIONS — Chapter 13

1. When is the projection of one line upon another a point?

2. What is a ratio? a proportion?

3. State five methods of proving triangles similar.

4. What are the two conditions necessary for the similarity of two polygons?

5. How many pairs of similar triangles are formed when an altitude is drawn to the hypotenuse of a right triangle?

6. State two theorems in which a mean proportional is mentioned.

7. If the altitude is drawn to the hypotenuse of a right triangle, state three geometric truths which follow.

8. For what purposes are triangles proved congruent?

9. For what purposes are triangles proved similar?

10. What fact do you know about the altitudes of similar triangles?

11. Is the side of a square the mean proportional between the whole diagonal and half the diagonal?

12. What is the most common method of proving triangles similar?

13. What special kind of triangles are similar?

14. What special kind of quadrilaterals are similar?

15. If the altitude of a triangle is 4, what is the altitude of a similar triangle whose perimeter is five times as great?

16. Are two polygons always similar if their corresponding sides are proportional? Are two triangles always similar if their corresponding sides are proportional?

17. State the Pythagorean Theorem.

18. State the theorem concerning a tangent and a secant from a point outside a circle.

19. What is the ratio of a diagonal of a square to one of its sides?

20. State a theorem in which the product of two line segments equals the product of two other line segments.

21. How can you find a leg of a right triangle when the hypotenuse and the other leg are known?

22. Must similar polygons have the same number of sides?

SUMMARY of Principal Bases for Proof

450. Two triangles are similar

a. If two angles of one triangle are equal respectively to two angles of another.
b. If they have their sides respectively proportional.
c. If an angle of one triangle equals an angle of the other and the including sides are proportional.
d. If they are each similar to a third triangle.
e. If they are corresponding triangles of similar polygons.
f. If they are right triangles and have an acute angle of one equal to an acute angle of the other.
g. If they are right triangles and one of them is formed by the altitude on the hypotenuse of the other.
h. If they are formed by the altitude on the hypotenuse of a right triangle.

451. Line segments in proportion

a. Two corresponding sides of similar polygons have the same ratio as any other two corresponding sides.
b. Two corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.
c. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

452. Products of line segments

If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

453. A line segment the mean proportional

a. The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.
b. Either leg of a right triangle is the mean proportional between the hypotenuse and its projection on the hypotenuse.
c. The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the segments of the diameter.
d. If a tangent and a secant are drawn to a circle from the same point, the tangent is the mean proportional between the secant and its external segment.

454. Two polygons are similar

a. If their corresponding angles are equal and their corresponding sides are proportional.

b. If they are composed of the same number of triangles, similar each to each and similarly placed.

455. Ratios of areas

a. The areas of two similar triangles have the same ratio as the squares of any two corresponding sides or as the squares of any two corresponding altitudes.

b. The areas of any two similar polygons have the same ratio as the squares of any two corresponding sides.

456. Constructions

a. To construct the mean proportional between two given line segments.

b. Upon a given line segment corresponding to one side of a given polygon construct a polygon similar to the given polygon.

c. To construct a square equal to a given parallelogram.

TEST 32 — True-False Statements

[6 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number. Do not guess.

1. Two congruent polygons are similar.

2. All squares are similar.

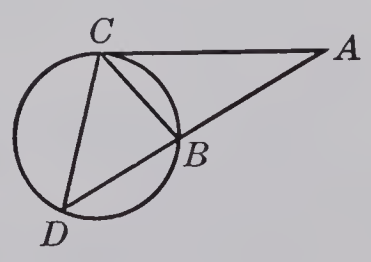
3. All rectangles are similar.

4. If the sides of one quadrilateral are respectively proportional to the sides of another quadrilateral, the corresponding angles are equal.

5. The altitude upon the hypotenuse of a right triangle forms two similar right triangles.

6. Two isosceles triangles are similar if an angle of one is equal to a corresponding angle of the other.

7. In the figure, $\triangle ABC \sim \triangle BCD$.



8. Either leg of a right triangle is a mean proportional between the hypotenuse and the other leg.

9. Two similar polygons are congruent.

TEST 33 — Completing Statements

[10 Minutes]

On your paper write one word, and only one, for each blank to make the following statements true:

1. Two polygons are similar if their corresponding sides are __?__ and their corresponding angles are __?__.

2. If two triangles have two angles of one equal respectively to two angles of the other, the __?__ are __?__.

3. The corresponding altitudes of two similar triangles have the same ratio as any two __?__ __?__.

4. If two triangles have an angle of one equal to an angle of the other and the including __?__ __?__, the triangles are similar.

5. The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the __?__ of the __?__.

6. Not all rectangles are similar, because their __?__ __?__ are not necessarily __?__.

7. Not all rhombuses are similar, because their __?__ __?__ are not necessarily __?__.

8. The __?__ of a line segment upon a line is the segment that is cut off on the line by the perpendiculars drawn from the end points of the given line segment.

9. If two polygons can be separated into the same number of triangles, similar each to each and similarly placed, the polygons __?__ __?__.

10. If a tangent and a secant are drawn to a circle from the same point, the tangent is the mean proportional between the __?__ and its __?__ __?__.

TEST 34 — Constructions

[12 Minutes]

Make the following geometric constructions accurately.

1. Construct a triangle similar to a given triangle and having its sides twice as long.

2. Given two line segments, m and n, construct a line segment p so that $p = \sqrt{mn}$.

3. Construct the fourth proportional to 3 in., 2 in., and 4 in.

TEST 35 — Applications

[30 Minutes]

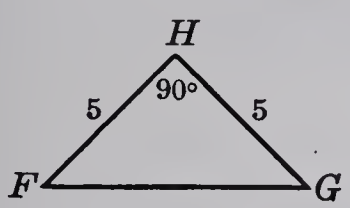
1. The sides of one triangle are 8, 10, and 12. The smallest side of a similar triangle is 20. Find the length of the greatest side.

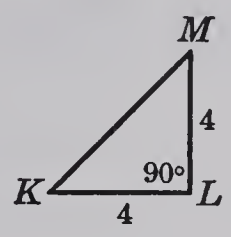
2. In a circle, chords AB and CD intersect in E. If $CE = 8$, $ED = 6$, and $EB = 3$, find AE.

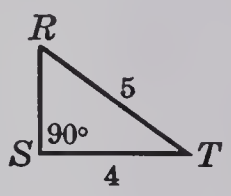
3. The perimeter of a rectangle is 50 inches and the width is 8 inches. Find the perimeter of a similar rectangle whose width is 12 inches.

4. $\triangle ABC \sim \triangle A'B'C'$, $\angle A$ corresponds to $\angle A'$, and $\angle B$ corresponds to $\angle B'$. Find $B'C'$ if $CB = 7$, altitude $CD = 4$, and altitude $C'D' = 6$.

5. The diagonal of a square is $\sqrt{8}$. Find the length of a side.

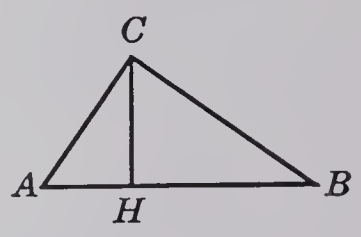






6. Which two triangles are similar in the figures above? Find FG. Find RS.

7. In the figure at the right, CH is the altitude upon hypotenuse AB of rt. $\triangle ABC$. If $AH = 4$ and $CH = 6$, find HB.

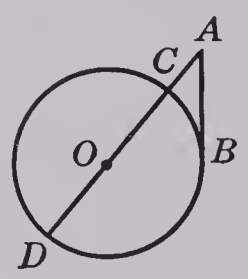


8. Using the figure in Ex. 7, if $AH = 5$ and $HB = 45$, find CH.

9. Using the figure in Ex. 7, if $AH = 4$ and $HB = 12$, find AC.

10. AB is a diameter of $\odot O$, C is a point on the circle, and $CD \perp AB$ at D. Find CD if AD is 9 inches and AB is 25 inches.

11. In the figure at the right, AB is a tangent to $\odot O$ and secant ACD passes through O. If $AC = 4$ and $AB = 6$, find AD. If $AC = 3$ and $CD = 24$, find AB.



12. The sides of a polygon are 7, 10, 12, and 14 respectively. Find the perimeter of a similar polygon whose longest side is 3.

13. A line segment 10 inches long makes an angle of 60° with another line. Find the length of the projection of the first line on the second.

14. What is the ratio of similitude of the similar triangles in Ex. 6 above?

CHAPTER 14

Regular Polygons and the Circle

In this chapter
you will study the relations of a circle
to the regular polygons inscribed in
and circumscribed about the circle.
From these relations
you will be able to determine
the approximate value of π
and to express the area of a circle
in terms of its radius.

457. Need for a Review

A large amount of this chapter is based upon Chapters 7, 9, 10, and 13. You should review these chapters at this time and test the thoroughness of your review by the following exercises.

EXERCISES — Review

A

1. What two conditions are necessary for a polygon to be regular? to be similar to another polygon?

2. What is the name of a regular polygon of three sides? of four sides? of five sides? of six sides?

3. When is a polygon said to be inscribed in a circle? circumscribed about a circle? In each case what can you say about the circle?

4. State a theorem concerning the perimeters of two similar polygons.

5. State a theorem concerning the areas of two similar polygons.

6. What have you proved concerning two equal chords of a circle? about equal arcs of a circle?

7. What arc has the same number of degrees as a central angle? as an inscribed angle? as an angle formed by a tangent and a chord intersecting on a circle?

8. Name a quadrilateral which is equilateral but not regular.

9. State the theorem concerning the sum of the exterior angles of a polygon.

10. State the theorem concerning the sum of the interior angles of a polygon.

11. *Complete:* An equilateral triangle is a regular __?__; and a square is a regular __?__.

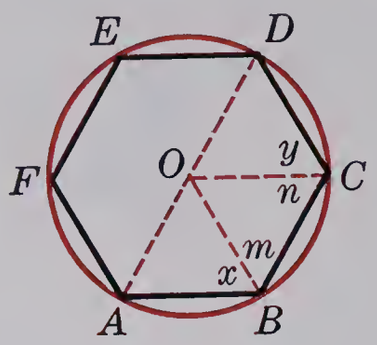
12. In any triangle, is there a point which is equidistant from its vertices? equidistant from its sides?

13. In any quadrilateral, do you think that there is a point that is equidistant from its vertices? equidistant from its sides?

14. Thus far have you proved that a circle can be circumscribed about any triangle? about any polygon?

★ Theorem 80

458. A circle can be circumscribed about any regular polygon.



Given the regular polygon $ABCDEF$.

To prove that a $\odot$ can be circumscribed about $ABCDEF$.

Planning the Proof. 1. We can prove that a circle is circumscribed by §§ 288, 364.

2. We shall use §§ 288, 364.

Proof

STATEMENTS	REASONS
1. Construct $\odot O$ through A, B, and C.	1. § 364.
2. Draw OA, OB, OC, and OD.	2. Why possible?
3. $OB = OC$.	3. Why?
4. $\therefore \angle m = \angle n$.	4. Why?
5. But $\angle ABC = \angle BCD$.	5. Why?
6. $\therefore \angle x = \angle y$.	6. Asmt. 2.
7. $AB = CD$.	7. Why?
8. From (3), (6), (7), $\triangle AOB \cong \triangle COD$.	8. Why?
9. $\therefore OA = OD$, or $OD =$ a radius.	9. Why?
10. $\therefore \odot O$ passes through D.	10. § 281 b.
11. In like manner, we can prove that the $\odot$ passes through the remaining vertices.	11. Reasons 3–10.
12. $\therefore$ the $\odot O$ is circumscribed about $ABCDEF$.	12. § 288.

EXERCISES

A

1. Construct a square given one side s; then circumscribe a circle about it.

2. Construct a regular hexagon given one side, and circumscribe a circle about it.

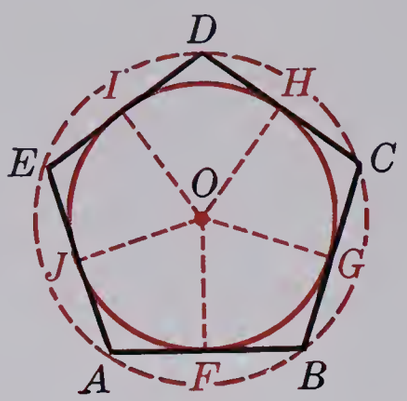
Kaiser Aluminum

Photo courtesy of Better Homes and Gardens magazine

Auditorium at the Hawaiian Village Hotel, Honolulu, so designed that it needs no interior support. The principle employed is that of the geodesic dome, originated by Richard Buckminster Fuller. Mr. Fuller is shown here with models of several of his inventions.

Theorem 81

459. A circle can be inscribed in any regular polygon.



Given the regular polygon $ABCDE$.

To prove that a $\odot$ can be inscribed in $ABCDE$.

Planning the Proof: 1. We can prove that a circle is inscribed by §§ 325, 369.
2. We shall use § 325.

Proof

STATEMENTS	REASONS
1. Circumscribe $\odot O$ about $ABCDE$.	1. § 458.
2. $AB = BC = CD = DE = EA$.	2. Why?
3. Construct $\perp$s from O to each side of $ABCDE$.	3. Why possible?
4. $OF = OG = OH = OI = OJ$.	4. § 293.
5. With O as a center and OF as a radius construct a $\odot$.	5. Why possible?
6. The sides of $ABCDE$ are tangent to this $\odot$.	6. §§ 281 b, 298.
7. $\therefore$ this $\odot$ is inscribed in $ABCDE$.	7. § 325.

460. Corollary I. The circumscribed circle and the inscribed circle of a regular polygon are concentric.

461. Names Associated with a Regular Polygon

The *center* of a regular polygon is the common center of the circumcircle and the incircle.

The *radius* of a regular polygon is the radius of the circumcircle. It connects the center to a vertex of the polygon.

The *apothem* (ăp′ō thĕm) of a regular polygon is the radius of the incircle—the perpendicular from the center to a side.

A *central angle* of a regular polygon is an angle formed by two radii drawn to the extremities of a side.

462. Corollary II. The central angle of a regular polygon of n sides is $\frac{360°}{n}$.

EXERCISES

A

1. Find the number of degrees in the central angle and in each interior angle of each of the following regular polygons:

a. Triangle. c. Quadrilateral. e. Octagon.
b. Pentagon. d. Hexagon. f. Heptagon.

2. A central angle of a regular polygon is the supplement of an interior angle of the polygon.

3. A central angle of a regular polygon is equal to an exterior angle of the polygon.

4. The apothem of a regular polygon bisects the side to which it is drawn. (Draw the circumcircle.)

5. Each angle of a regular polygon is bisected by the radius of the polygon drawn to its vertex.

6. Can a circle be inscribed in any square? in any rhombus? in any parallelogram?

B

7. Show that the apothem of a regular inscribed hexagon is $\frac{1}{2} R\sqrt{3}$ and that the area of the polygon is $\frac{3}{2} R^2\sqrt{3}$, where R is the radius of the polygon.

8. Find the area of a regular hexagon with a side of 4 inches.

9. *Given* a regular hexagon $ABCDEF$ with diagonals AC, CE, and AE.

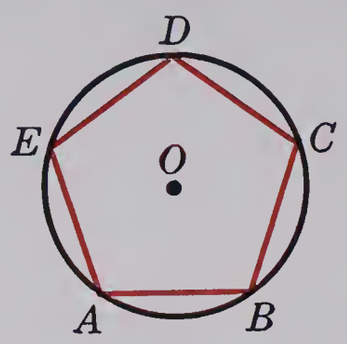
Prove that $\triangle ACE$ is equilateral.

10. Show that the bisectors of the angles and the perpendicular bisectors of the sides of any regular polygon meet at the center of the inscribed circle.

11. Find the side of an equilateral triangle whose area is equal to that of the triangle whose sides are 8 in., 15 in., and 17 in.

Theorem 82

463. If a circle is divided into three or more equal arcs, the chords of these arcs form a regular inscribed polygon.



Given $\odot O$, with $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$ and chords AB, BC, CD, DE, and EA.

To prove that $ABCDE$ is a regular inscribed polygon.

Planning the Proof: 1. We can prove polygons regular by § 184.
2. We shall use § 184.

Proof

STATEMENTS	REASONS
1. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$.	1. Why?
2. $\therefore AB = BC = CD = DE = EA$.	2. Why?
3. $\overset{\frown}{BCDE} = \overset{\frown}{CDEA} = \overset{\frown}{DEAB} =$ etc.	3. Asmt. 1.
4. $\angle A \doteq \frac{1}{2}\overset{\frown}{BCDE}$, $\angle B \doteq \frac{1}{2}\overset{\frown}{CDEA}$, $\angle C \doteq \frac{1}{2}\overset{\frown}{DEAB}$, etc.	4. Why?
5. $\therefore \angle A = \angle B = \angle C = \angle D = \angle E$.	5. Why?
6. $\therefore ABCDE$ is a regular inscribed polygon.	6. §§ 184, 288.

464. Corollary I. An equilateral polygon inscribed in a circle is a regular polygon.

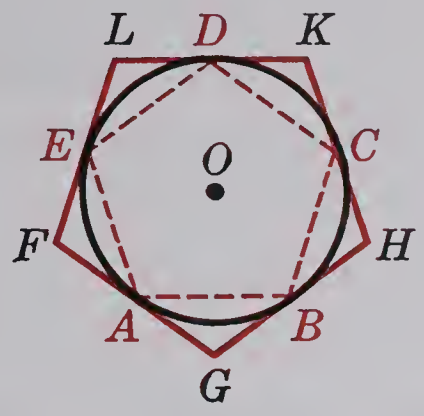
465. Corollary II. If the midpoints of the arcs of a regular inscribed polygon are joined to the extremities of the respective sides, a regular inscribed polygon of double the number of sides is formed.

EXERCISE

How can you divide a circle into four equal arcs? six equal arcs?

Theorem 83

466. If a circle is divided into three or more equal arcs, the tangents drawn to the circle at the successive points of division form a regular circumscribed polygon.



Given $\odot O$ with $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$ and FG, GH, HK, KL, and LF tangent to the $\odot$ at A, B, C, D, and E respectively.

To prove that $FGHKL$ is a regular circumscribed polygon.

Planning the Proof: 1. We can prove polygons regular by §§ 184, 463, 464, 465.

2. We shall use § 184.

Proof

STATEMENTS	REASONS
1. Draw the chords AB, BC, CD, DE, and EA.	1. Why possible?
2. $AB = BC = CD = DE = EA$.	2. Why?
3. $\angle BAG = \angle ABG = \angle CBH = \angle BCH = \angle DCK = \angle CDK =$ etc.	3. § 322, Asmt. 7.
4. $\therefore \triangle AGB \cong \triangle BHC \cong \triangle CKD \cong$ etc.	4. Why?
5. $\therefore \angle G = \angle H = \angle K =$ etc.	5. Why?
6. $\triangle AGB$, BHC, CKD, etc. are isosceles.	6. Why?
7. $AG = GB = BH = HC =$ etc.	7. Why?
8. $\therefore FG = GH = HK = KL =$ etc.	8. Asmt. 1.
9. $\therefore FGHKL$ is a regular circumscribed polygon.	9. §§ 184, 325.

EXERCISES

B

1. Prove that an equilateral pentagon circumscribed about a circle is regular.

2. Prove that an equiangular hexagon circumscribed about a circle is regular.

EXERCISES Review

A

1. Inscribe a square in a given circle.
2. Inscribe a regular hexagon in a given circle.
3. How would you inscribe a regular octagon in a circle?
4. How would you inscribe a regular polygon of 12 sides in a circle?
5. Circumscribe a square about a circle.
6. Circumscribe a circle about a given square.
7. Draw a hexagon which is equiangular but not regular.
8. Draw a hexagon which is equilateral but not regular.

B

9. Draw a regular polygon. Prove that the lines which join the midpoints of its sides in order form another regular polygon.
10. The apothem of an equilateral triangle is one half the radius.
11. An equiangular polygon circumscribed about a circle is regular.
12. Prove that the diagonals of a regular pentagon are equal.
13. Find the perimeter of a regular hexagon circumscribed about a circle whose radius is 4 inches.
14. Show that when a circle is inscribed in a right triangle, the sum of the lengths of the two legs diminished by the length of the hypotenuse equals the length of the diameter.

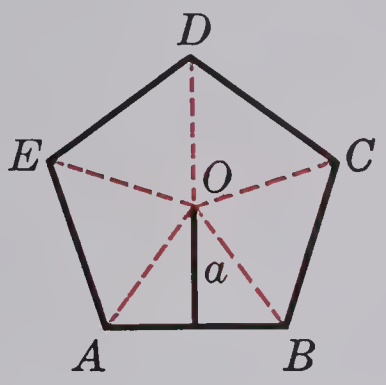
C

15. If s is a side of an equilateral triangle inscribed in a circle whose radius is R, show that $s = R\sqrt{3}$.
16. If s is a side of an equilateral octagon inscribed in a circle whose radius is R, show that $s = R\sqrt{2 - \sqrt{2}}$.
17. If two diagonals of a regular pentagon intersect, prove that the greater segment of each is equal to a side of the pentagon.
18. *Given* regular pentagon $MNOPQ$ inscribed in a circle and having its diagonals NQ and MO intersecting in R.

Prove that $\dfrac{MR}{RO} = \dfrac{RO}{MO}$.

★ Theorem 84

467. The area of a regular polygon is equal to one-half the product of its apothem and its perimeter.



Given the regular polygon $ABCDE$, with apothem a, perimeter p, and area S.

To prove that $S = \frac{1}{2}\,ap$.

Planning the Proof: 1. We can find areas of polygons by § 273.
2. We shall use § 273 c.

Proof

STATEMENTS	REASONS
1. Draw the radii $OA, OB, OC, \cdots$.	1. Why possible?
2. The altitude of each triangle $= a$.	2. Why?
3. $\triangle OAB = \frac{1}{2}\,a \cdot AB$, $\triangle OBC = \frac{1}{2}\,a \cdot BC$, $\triangle OCD = \frac{1}{2}\,a \cdot CD$, etc.	3. § 273 c.
4. $\triangle OAB + \triangle OBC + \triangle OCD + \cdots = \frac{1}{2}\,a(AB + BC + CD + \cdots)$.	4. Why?
5. $\therefore S = \frac{1}{2}\,ap$.	5. Why?

EXERCISES

A

1. Prove that the area of a polygon circumscribed about a circle is equal to one half the product of the perimeter of the polygon and the radius of the circle.

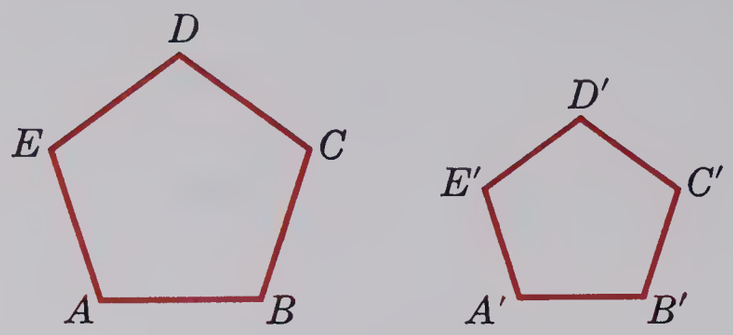
2. Find the apothem of a regular hexagon whose perimeter is 12 inches.

3. Show that the area of a square circumscribed about a circle having radius r is $4\,r^2$.

4. Show that the area of a square inscribed in a circle having radius r is $2\,r^2$.

Theorem 85

468. Two regular polygons of the same number of sides are similar.



Given the regular polygons $ABCDE$ and $A'B'C'D'E'$, each having n sides.

To prove that $ABCDE \sim A'B'C'D'E'$.

Planning the Proof. 1. We can prove polygons $\sim$ by § 454.
2. We shall use § 454 a.

Proof

STATEMENTS	REASONS
1. $\angle A = \angle B = \angle C = \cdots = \frac{180(n-2)^\circ}{n}$.	1. § 211.
2. $\angle A' = \angle B' = \angle C' = \cdots = \frac{180(n-2)^\circ}{n}$.	2. Why?
3. $\therefore \angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\cdots$.	3. Why?
4. $AB = BC = CD = \cdots$.	4. Why?
5. $A'B' = B'C' = C'D' = \cdots$.	5. Why?
6. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots$.	6. Asmt. 4.
7. $\therefore ABCDE \sim A'B'C'D'E'$.	7. § 454 a.

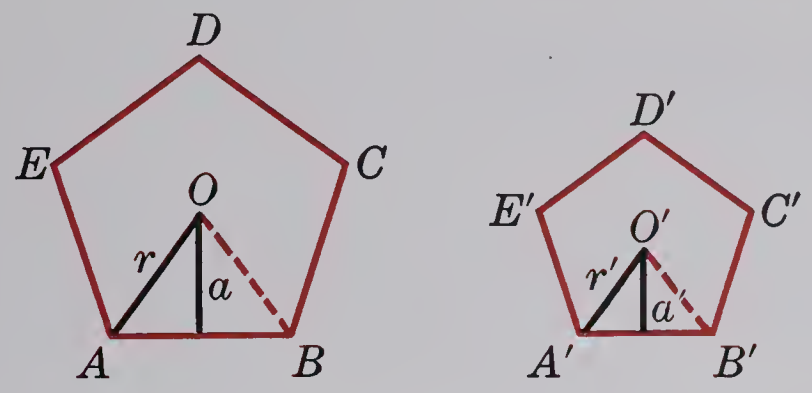
469. Corollary. The areas of two regular polygons of the same number of sides have the same ratio as the squares of any two corresponding sides. (See § 440.)

EXERCISE

A circle is divided into four equal arcs by the points A, B, C, and D. The points are joined to form polygon $ABCD$ and tangents are drawn to the circle at these points to form polygon $EFGH$. If the area of polygon $ABCD$ is 64 square inches, what is the area of polygon $EFGH$?

Theorem 86

470. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or as their apothems.



Given the regular polygons $ABCDE$ and $A'B'C'D'E'$ each having n sides, with centers O and O', radii r and r', apothems a and a', and perimeters p and p'.

To prove that $\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$.

Planning the Proof: 1. We can prove line segments proportional by §§ 409, 451.
2. We shall use § 451 c, b.

Proof

STATEMENTS	REASONS
1. Draw OB and $O'B'$.	1. Why possible?
2. $ABCDE \sim A'B'C'D'E'$.	2. Why?
3. $\therefore \frac{p}{p'} = \frac{AB}{A'B'}$.	3. Why?
4. Each polygon has n sides.	4. Why?
5. In $\triangle AOB$ and $A'O'B'$, $\angle AOB = \frac{360°}{n}$ $= \angle A'O'B'$.	5. Why?
6. $AO = OB$ and $A'O' = O'B'$.	6. Why?
7. $\therefore \frac{AO}{A'O'} = \frac{OB}{O'B'}$.	7. Asmt. 4.
8. $\therefore \triangle AOB \sim \triangle A'O'B'$.	8. § 422.
9. $\therefore \frac{a}{a'} = \frac{r}{r'} = \frac{AB}{A'B'}$.	9. § 451 b.
10. From (3) and (8), $\frac{p}{p'} = \frac{a}{a'} = \frac{r}{r'}$.	10. Asmt. 7.

471. Corollary. The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.

EXERCISES — Regular Polygons

A

1. Find the apothem of an inscribed square whose side is 10 inches.

2. The corresponding sides of two regular decagons are 2 inches and 6 inches respectively. Find the ratio of their perimeters and the ratio of their areas.

3. The corresponding sides of two regular octagons are 3 inches and 5 inches respectively. What is the ratio of their radii? of their apothems? of their perimeters? of their areas?

4. Find the side of a square inscribed in a circle whose radius is 10 inches.

5. Find the area of a regular hexagon if each side is 16 inches.

6. The perimeters of two regular polygons having the same number of sides are 36 inches and 63 inches respectively. If the radius of the first polygon is 6 inches, what is the radius of the other?

7. The area of one regular nonagon is 112 square inches, and the area of another regular nonagon is 28 square inches. Find the ratio of their radii; of their apothems.

B

8. Find the ratio of the perimeters of two regular octagons whose areas are 25 square feet and 50 square feet respectively.

9. The altitudes of two equilateral triangles are 3 inches and 8 inches respectively. Find the ratio of their perimeters; the ratio of their areas.

10. The area of the cross section of a steel beam 3 inches thick is 28 square inches. What is the area of a cross section of a beam of similar proportions and $4\frac{1}{2}$ inches thick?

11. What is the area of the cross section of the largest possible square piece of timber that can be sawed from a round log 18 inches in diameter?

12. Find the apothem of an equilateral triangle whose side is 12 feet.

13. Find the radius of an equilateral triangle whose side is 16 feet.

14. The area of one regular pentagon is $2\frac{1}{4}$ times that of another. What is the ratio of their perimeters?

15. The area of one equilateral triangle is 9 times that of another. Find the ratio of their altitudes.

16. Compare the radii of the circumcircle and incircle of an equilateral triangle.

17. Compare the radii of the circumcircle and incircle of a square.

18. The side of a square inscribed in a circle is 6 inches. Find the diameter of the circle.

19. Find the area of a square whose radius is 8.

20. The radius of a circle is 8 inches. Find the areas and the perimeters of the inscribed and circumscribed squares.

C

21. Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides. Is this true if the polygon has an even number of sides?

22. The side of an inscribed regular hexagon is twice the apothem of an inscribed equilateral triangle.

23. Find the area of a rhombus whose shorter diagonal and whose sides are each 18 inches.

24. Each side of the regular hexagon $ABCDEF$ is $8\sqrt{3}$ inches. AE and AD are diagonals. Find the area of $\triangle AED$.

472. Method of Measuring the Circle

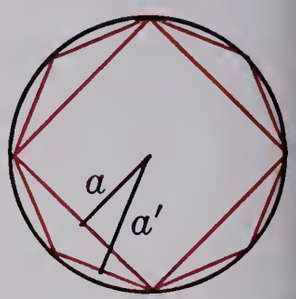
A line segment is measured by finding how many times a unit line segment is contained in it. An arc of a circle is measured in terms of a unit of arc of that circle by finding how many times it contains the unit of arc. It is often desirable to measure a circle, or an arc of it, in terms of a straight-line segment. Since we cannot directly apply a straight-line segment as a unit of measure to a curve line, we shall resort to a process known as the *method of limits* in the measurement of the circle.

By doubling the number of sides of a regular circumscribed polygon, a polygon may be obtained whose area will be almost the same as the area of the circle. The *area of a circle* is the area of the plane surface enclosed by the circle. In this manner it will be shown that the approximate area of the circle can be expressed in terms of the radius.

Before studying the theorems on the circumferences and areas of circles, students should have a thorough understanding of the terms "variable," "constant," and "limit of a variable."

473. Constants and Variables

A *constant* is a quantity which has a fixed value throughout a given discussion; and a *variable* is a quantity which may have different successive values during a discussion.



If we inscribe a regular polygon in a circle and then continue to double the number of its sides, the successive perimeters become greater and greater, and the successive apothems become greater and greater. In this case, the polygon is a variable, for it changes; the perimeter is a variable, for it has different values; but the circle is a constant, for it does not change. Is the apothem a constant or a variable? Is the radius a constant or a variable?

474.* Limits

When a variable so approaches a constant that the difference between the two *becomes* and *remains* less than any assigned positive value, however small, the constant is called the *limit of the variable.*

Consider the decimal which has the successive values .3, .33, .333, · · ·. As the digit 3 is continually annexed, the value of the decimal becomes greater and greater but always remains less than $\frac{1}{3}$. This decimal is a variable quantity and approaches the fraction $\frac{1}{3}$ as a limit.

The symbol for "approaches as a limit" is $\rightarrow$. Thus $x \rightarrow c$ is read "x approaches c as a limit."

EXERCISES — Oral Exercises, Limits

1. Two regular polygons, of 16 and 32 sides respectively, are inscribed in a circle. Which polygon has the greater apothem? Why? Which polygon has the greater perimeter? Why?

2. As the number of sides of a regular inscribed polygon is indefinitely increased, what do you think will be the limit of its apothem? of its perimeter?

*If desired, §§ 473–478 may be omitted and Theorems 87 and 88 may be treated as assumptions.

3. As the number of 6's in the decimal .666 is increased indefinitely, what common fraction will the decimal approach as a limit?

4. As the number of 7's in the decimal .777 is increased indefinitely, what common fraction will the decimal approach as a limit?

5. As the number of 8's in the decimal .888 is increased indefinitely, what common fraction will the decimal approach as a limit?

6. If the number of sides of a regular circumscribed polygon is increased indefinitely,

a. What is the limit of the perimeter of the polygon?

b. What is the limit of the area of the polygon?

c. Is the apothem of the polygon a variable?

7. Zeno, one of the most prominent investigators of problems of infinite series in the fifth century B.C., argued that Achilles could not pass a tortoise even though he went faster than the tortoise. He argued that if Achilles could run ten times as fast as a tortoise and if the tortoise had a start of 1000 yards, when Achilles had gone the 1000 yards, the tortoise would be 100 yards ahead of him. When Achilles had covered these 100 yards, the tortoise would be 10 yards ahead of him. This would continue forever. Thus Achilles would come nearer and nearer the tortoise but would never reach it. Was Zeno right in his reasoning?

475. Theorems on Limits

The two theorems which follow are used in proving equalities involving the circumferences of circles. Their proofs are too difficult for high-school students, and they will be accepted as true without proof.

476. Theorem. If a variable x approaches a limit k and if c is a constant, then cx approaches ck as a limit, and $\frac{x}{c}$ approaches $\frac{k}{c}$ as a limit.

Thus, if x is the variable decimal $.333\cdots$ and if c is 2, and it is known that $.333\cdots \to \frac{1}{3}$, the theorem states that $.666\cdots \to \frac{2}{3}$.

477. Theorem. If two variables are always equal while approaching their respective limits, their limits are equal.

If the limits were not equal, could the variables be equal when they are very near their limits?

ORAL EXERCISES Limits

1. If the repeating decimal $.333\cdots \to \frac{1}{3}$, why will the repeating decimal $.04166\cdots \to \frac{1}{24}$?

2. If $x = y$, $x \to m$, and $y \to n$, what do you know of m and n? Why?

3. If $p \to c$ and if r is a constant, what do you know of pr?

4. If the number of sides of a regular inscribed polygon is indefinitely increased, what is the limit of each of its angles? of each of its central angles?

478. Assumptions on Limits Related to the Circle

Below are three assumptions on limits expressing relations of the circle to the inscribed and circumscribed regular polygons.

Assumption 50. If the number of sides of a regular inscribed (or circumscribed) polygon is indefinitely increased, the perimeter of the polygon approaches the circumference of the circle as a limit.

Assumption 51. If the number of sides of a regular inscribed (or circumscribed) polygon is indefinitely increased, the area of the polygon approaches the area of the circle as a limit.

Assumption 52. If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as a limit.

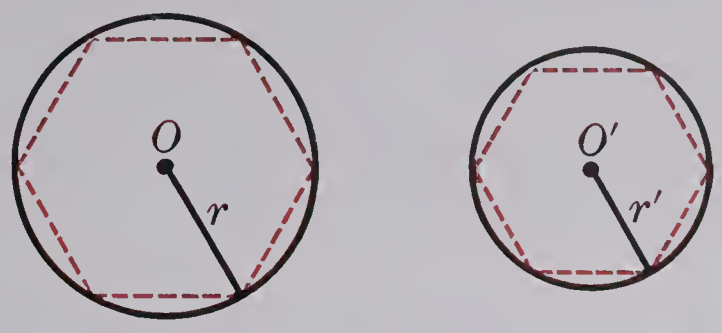
ORAL EXERCISES Limits

1. A circle is inscribed in one square and circumscribed about another. Which square has the greater area? the greater apothem?

2. A square, a regular octagon, and a regular polygon of 16 sides are circumscribed about a circle. Of the three polygons, which has an area nearest that of the circle?

★ **Theorem 87**

479. The circumferences of two circles have the same ratio as their radii.



Given the Ⓢ O and O' with circumferences c and c' and radii r and r' respectively.

To prove that $\frac{c}{c'} = \frac{r}{r'}$.

Planning the Proof: 1. We can prove line segments proportional by §§ 409, 451, 470.
2. We shall use § 470.

Proof

STATEMENTS	REASONS
1. Inscribe regular hexagons in the Ⓢ and denote their perimeters by p and p'.	1. § 332.
2. Then $\frac{p}{p'} = \frac{r}{r'}$.	2. § 470.
3. $\therefore pr' = p'r$.	3. Why?
4. Form regular inscribed polygons of double the number of sides and continue this process indefinitely, keeping the number of sides the same in each $\odot$.	4. § 465.
5. Then $p \to c$ and $p' \to c'$.	5. Asmt. 50.
6. $\therefore pr' \to cr'$ and $p'r \to c'r$.	6. § 476.
7. $\therefore cr' = c'r$.	7. § 477.
8. $\therefore \frac{c}{c'} = \frac{r}{r'}$.	8. § 387.

480. Corollary I. The circumferences of two circles have the same ratio as their diameters.

Since $\frac{c}{c'} = \frac{r}{r'}$, $\frac{c}{c'} = \frac{2r}{2r'}$. Why? Then $\frac{c}{c'} = \frac{d}{d'}$.

481. The Ratio of the Circumference of a Circle to Its Diameter

Since $\frac{c}{c'} = \frac{d}{d'}$, then $\frac{c}{d} = \frac{c'}{d'}$. Why? Since $\frac{c}{d}$ of one circle is equal to $\frac{c'}{d'}$ of any other circle, Corollary I may be stated thus: *The ratio of the circumference of a circle to its diameter is constant.* This constant is represented by the Greek letter π (pī).

★ **482. Corollary II.** The circumference of a circle is expressed by the formula $c = \pi d$, or $c = 2\pi r$.

483. Historical Note on π

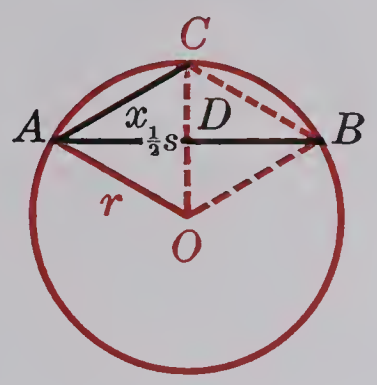
The early Babylonians and Hebrews used 3 as the value of π. Later Ahmes, an Egyptian, found the area of a circle by squaring eight ninths of the diameter, which is approximately the same as using 3.1605 as the value of π. Archimedes (287–212 B.C.) inscribed in and circumscribed about a circle regular polygons of ninety-six sides. He then calculated their perimeters and assumed the circumference of the circle to lie between them. From these results he found the value of π to lie between $3\frac{1}{7}$ and $3\frac{10}{71}$. To make these calculations, Archimedes must have had some method of finding square roots of numbers. About 150 A.D., Ptolemy gave π the value $3\frac{17}{120}$ or 3.16166. The value of π was worked out by Shanks correct to 707 decimal places. The value of π to ten decimal places is 3.1415926535. This value of π will give the circumference of the earth correct to within a fraction of an inch. In 1766 Lambert proved that π is not rational and in 1882 Lindermann proved that π is a transcendental number—that is, it cannot be the root of an algebraic equation. The fact that π is not the root of an algebraic equation makes it impossible, using only the straightedge and compasses, to construct a line segment equal to the circumference of a circle, or to square a circle (to construct a square whose area is equal to that of a given circle).

484. Finding the Approximate Value of π. [Optional]

If you are an exceptional student, you may wish to compute an approximate value of π. A formula for this purpose is developed in § 485. The computation for finding the approximate value of π is given in § 486.

Problem 26 [Optional]

485. Given a side of a regular inscribed polygon of n sides, to find the side of a regular inscribed polygon of $2n$ sides.



Given $\odot O$ with radius r, AB (s) a side of a regular inscribed polygon of n sides, and AC (x) a side of a regular inscribed polygon of $2n$ sides.

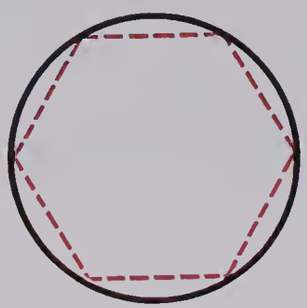
To find the value of x.

Solution

STATEMENTS	REASONS
1. Draw CO, intersecting AB at D.	1. Why possible?
2. Draw BC and BO.	2. Why possible?
3. $AC = BC$.	3. Give proof.
4. $AO = BO$.	4. Why?
5. CO is the $\perp$ bisector of AB.	5. Why?
6. Then $AD = \frac{s}{2}$.	6. Why?
7. $OD = \sqrt{r^2 - \frac{s^2}{4}}$.	7. By § 430.
8. $CD = r - \sqrt{r^2 - \frac{s^2}{4}}$.	8. Why?
9. $x = \sqrt{\frac{s^2}{4} + \left(r - \sqrt{r^2 - \frac{s^2}{4}}\right)^2}$.	9. By § 430.
10. $x = \sqrt{\frac{s^2}{4} + r^2 - 2r\sqrt{r^2 - \frac{s^2}{4}} + r^2 - \frac{s^2}{4}}$.	10. Why?
11. $x = \sqrt{2r^2 - 2r\sqrt{r^2 - \frac{s^2}{4}}}$, or	11. Why?
12. $x = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.	12. Why?

Problem 27 [Optional]

486. To compute the approximate value of π.



Solution. Inscribe a regular hexagon in a $\odot$ having a radius of 1 inch and denote a side by s_6. Then the perimeter of the hexagon is 6. Why? Doubling the number of sides to obtain a regular polygon of 12 sides and denoting each of its sides by s_{12}, we obtain

$$s_{12} = \sqrt{2\,r^2 - r\sqrt{4\,r^2 - {s_6}^2}} = \sqrt{2 - \sqrt{3}} = .51763809$$

by § 485. The perimeter of the polygon of 12 sides $= 12 \times .51763809 = 6.21165708$. Continuing this process we obtain the following:

No. of Sides	One Side *	Perimeter	$P \div D$
6	1	6	3
12	$\sqrt{2-\sqrt{4-(1)^2}} = .51763809$	6.21165708	3.10582854
24	$\sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238$	6.26525722	3.13262861
48	$\sqrt{2-\sqrt{4-(.26105238)^2}} = .13080626$	6.27870041	3.13935020
96	$\sqrt{2-\sqrt{4-(.13080626)^2}} = .06543817$	6.28206396	3.14103198
192	$\sqrt{2-\sqrt{4-(.06543817)^2}} = .03272346$	6.28290510	3.14145255
384	$\sqrt{2-\sqrt{4-(.03272346)^2}} = .01636228$	6.28311544	3.14155772
768	$\sqrt{2-\sqrt{4-(.01636228)^2}} = .00818121$	6.28316941	3.14158470

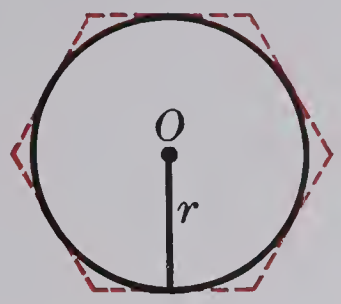
Hence $\pi = 3.1416$, approximately.

There are ways shorter than the one just described to find the value of π, but they are not geometric. For example, by calculus it can be shown that $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots)$. Another method uses the theory of probability.

*The length of each side has been computed to 17 decimal places but is given correct to only 8 decimal places.

★★ Theorem 88

487. The area of a circle is equal to one half the product of its radius and its circumference.



Given the $\odot O$ with circumference c, radius r, and area S.

To prove that $S = \frac{1}{2}rc$.

Planning the Proof: 1. We can find areas by §§ 273, 467, 478.
2. We shall use §§ 467, 478.

Proof

STATEMENTS	REASONS
1. Circumscribe a regular polygon about the $\odot$ and let S' denote its area and p its perimeter.	1. § 466.
2. Then $S' = \frac{1}{2}rp$.	2. § 467.
3. Form a regular circumscribed polygon of double the number of sides and continue this process indefinitely.	3. § 466.
4. Then $S' \to S$ and $p \to c$.	4. Asmts. 51, 50.
5. $\therefore \frac{1}{2}rp \to \frac{1}{2}rc$.	5. § 476.
6. From (2), (4), and (5), $S = \frac{1}{2}rc$.	6. § 477.

★ **488. Corollary I.** The area of a circle is given by the formula $S = \pi r^2$.

489. Corollary II. The areas of two circles have the same ratio as the squares of their radii or as the squares of their diameters.

490. Corollary III. The area of a sector of a circle is to the area of the circle as the angle of the sector is to 360°. (In the figure of § 491, OACB is a sector.)

Example 1. Find the circumference of a circle whose diameter is 10 inches.

Solution. If 10 is an exact number, $c = \pi d = 3.1416 \times 10 = 31.4160$. Then we round off 31.4160 so that it has the same number of significant figures as 3.1416, getting 31.416. Then the circumference to five significant figures is 31.416 in.

If 10 is an approximate number, we round off 3.1416 so that it has one more significant figure than the number 10, getting 3.14.

Then $c = \pi d = 3.14 \times 10 = 31.4$. Finally, we round off 31.4 so that it has the same number of significant figures as the number 10, getting 31.

Then $c = 31$ inches.

Example 2. Find the radius of a circle whose circumference is 24 inches.

Solution. We assume that 24 is an exact number.

$$2\pi r = 24$$
$$2 \times 3.1416\, r = 24$$
$$r = 3.8197$$

The radius is 3.8197 inches.

Example 3. The area of a circle is 15 sq. in. Find the diameter of the circle correct to two decimal places.

Solution. If 15 is an exact number,

$$\pi r^2 = 15$$
$$3.1416\, r^2 = 15$$
$$r^2 = 4.7746$$
$$r = 2.185$$
$$d = 4.370.$$

To the nearest hundredth of an inch, $d = 4.37$ inches.

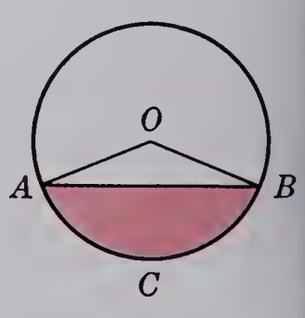
If 15 is an approximate number,

$$\pi r^2 = 15$$
$$3.14\, r^2 = 15$$
$$r^2 = 4.777$$
$$r = 2.186$$
$$d = 4.372.$$

The diameter is 4.4 inches.

491. Area of a Segment

The area of segment ACB of circle O (§371) may be found by subtracting the area of $\triangle AOB$ from the area of the sector $OACB$.



Unless we use trigonometry we can find the area of a segment for only certain values of $\angle AOB$.

Example. Find the area of a segment of a circle formed by one side of an inscribed equilateral triangle and its arc if the radius of the circle is 12.

Solution. $\angle AOB = 120°$. Why?

Area of sector $OAEB = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi\ 12^2 = 150.8$.

$\angle AOD = 60°$. Why?

$\therefore \angle DAO = 30°$ and $DO = 6$. Why?

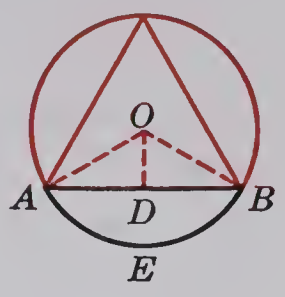
$\overline{AD}^2 + \overline{DO}^2 = \overline{AO}^2$, or $\overline{AD}^2 + 36 = 144$.

Solving, $AD = 6\sqrt{3} = 10.4$.

Then $AB = 20.8$.

Area of $\triangle ABO = \frac{1}{2} \times DO \times AB = \frac{1}{2} \times 6 \times 20.8 = 62.4$.

$\therefore$ area of segment $AEB = 150.8 - 62.4 = 88.4$.



Unless stated otherwise, consider the numbers expressing the data in the exercises as exact.

EXERCISES — Circles

A

1. Find the circumference of a circle whose radius is

a. 4. **b.** 6. **c.** 12. **d.** 125.

2. Find the area of a circle whose radius is

a. 3. **b.** 15. **c.** 42. **d.** 8π.

3. Find the diameter of a circle whose circumference is

a. 12.5664. **b.** 78.54. **c.** 34.5576. **d.** 100.

4. Find the radius of a circle whose area is

a. 13.0946. **b.** 18. **c.** 324π.

5. Find the radius of a circle whose circumference is

a. 30. **b.** 45. **c.** 68.

6. Find the area of a semicircle whose diameter is 12.

7. The radii of two circles are 5 inches and 10 inches respectively. Find the ratio of their circumferences; of their areas.

8. Find the radius of a circle equal in area to a square whose area is 16.

9. Find the area of a circular ring formed by two concentric circles of radii 6 inches and 8 inches respectively.

10. By measurement the diameter of a circle is 34.15 inches. Find its circumference.

11. Find the length of a belt which connects two pulleys having radii of 4 inches, if their centers are 20 inches apart.

12. The radius of one circle is three times that of another. If the area of the smaller circle is 36 square inches, find the area of the larger circle.

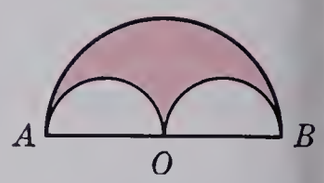
B

13. A circular pond with an area of two acres is surrounded by a walk 2 yards wide. Find the cost of graveling the walk at 9 cents a square yard.

14. Find the length of a belt connecting two pulleys having radii of 3 inches and 26 inches respectively if their centers are 46 inches apart.

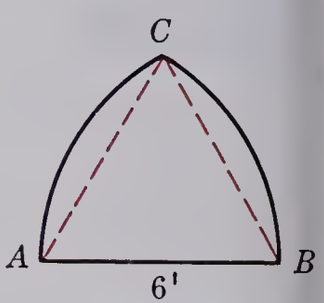
15. A square is inscribed in a circle whose diameter is 10 inches. Find the difference between the area of the circle and that of the square.

16. In the figure, $AO = OB$. Prove that the area of the shaded portion is equal to the sum of the areas of the smaller semicircles.



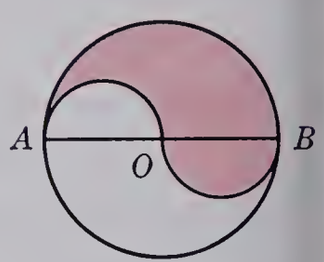
17. Find the diameter of a circle whose circumference and area are numerically equal.

18. Find the area of the equilateral arch ACB if the radius of each arc is 6 feet.



19. The area of the cross section of a $\frac{1}{2}$-inch wire is how many times the area of the cross section of a $\frac{1}{8}$-inch wire?

20. In the figure, $AO = OB$ and semicircles are constructed on AO and OB as diameters. Prove that the area of the shaded portion of the circle equals the area of the unshaded portion.

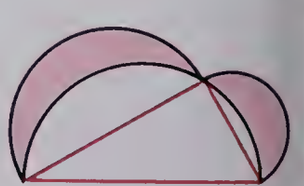


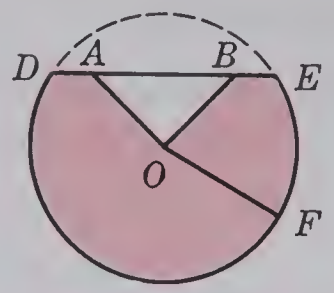
21. If the drive wheels of a locomotive are 60 inches in diameter, find the number of revolutions a minute they make when the engine is going 60 miles an hour.

22. Prove that the area of a circle constructed upon the hypotenuse of a right triangle as a diameter is equal to the sum of the areas of the circles constructed upon the legs as diameters.

C

23. If upon the three sides of a right triangle three semicircles (as shown in the figure) are drawn, the area of the right triangle is equal to the sum of the areas of the two crescents (shaded portions).

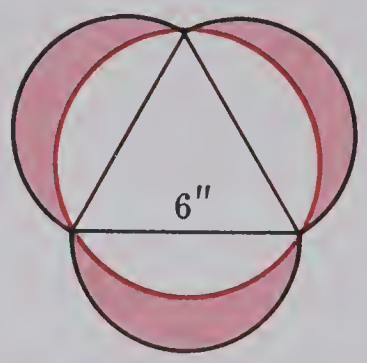




24. In this figure, $DABE$ is a st. line; $\angle AOB$ is a rt. $\angle$; radius $OF = 20$; $AO = BO = 10\sqrt{2}$.

Find the area of the shaded portion of the figure.

25. Prove that the incircle of a square divides the area of the circumcircle into two equal parts.



26. If one side of an equilateral triangle is 6 inches, compute the area of the three crescents bounded by the circumcircle of the triangle and the semicircles constructed on the sides of the triangle as diameters.

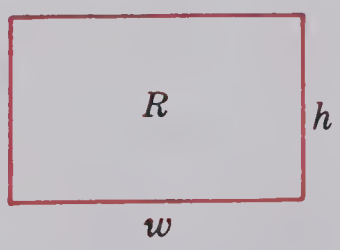
27. Prove that the area of the regular hexagon inscribed in a circle is a mean proportional between the area of the inscribed equilateral triangle and the area of the circumscribed equilateral triangle.

492. Extreme and Mean Ratio

A line segment is divided in *extreme and mean ratio* if it is separated into two parts such that the square of one part is equal to the product of

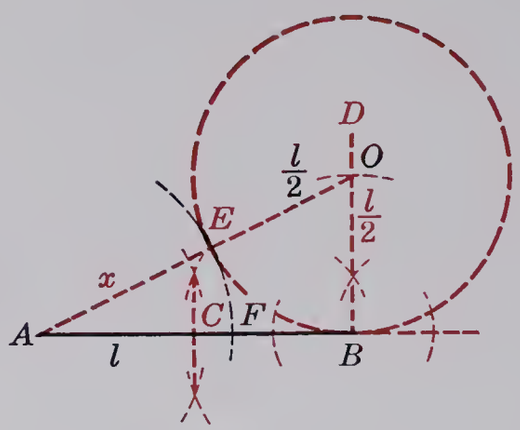
the whole segment and the other part. Thus the line segment AB is divided in extreme and mean ratio by the point C if $\overline{AC}^2 = AB \times CB$, or $\dfrac{AB}{AC} = \dfrac{AC}{CB}$.

When a line is divided in extreme and mean ratio, the division is called the *golden section*, or *divine proportion*. It is claimed that a line segment is most harmoniously divided when it is bisected or when it is divided in extreme and mean ratio by the golden section. Nature has made use of this fact in the construction of many plants. Artists often place the central figure of their paintings in accordance with the principle of the golden section. Picture frames and book covers are pleasing to the eye when the width and length have the ratio of the segments of a line divided in extreme and mean ratio. Thus the rectangle R has good proportions when $w^2 = h(w + h)$.



Problem 28

493. To divide a given line segment in extreme and mean ratio.



Given the line segment AB, or l.
To divide AB in extreme and mean ratio.

Construction

STATEMENTS	REASONS
1. Bisect AB, locating C, the midpoint of AB.	1. Why possible?
2. Construct $BD \perp AB$.	2. Why possible?
3. On BD construct $BO = BC = \frac{l}{2}$.	3. Why possible?
4. Construct the $\odot$ having center O and radius $= BO$, or $\frac{l}{2}$.	4. Why possible?
5. Draw AO, meeting $\odot O$ at E.	5. Why possible?
6. On AB construct $AF = AE = x$.	6. Why possible?

Then F divides AB in extreme and mean ratio, or $\overline{AF}^2 = AB \times FB$.

Proof

1. $\left(x + \frac{l}{2}\right)^2 = \left(\frac{l}{2}\right)^2 + l^2$.	1. § 430.
2. $x^2 + lx + \frac{l^2}{4} = \frac{l^2}{4} + l^2$.	2. Why?
3. $x^2 = l^2 - lx$.	3. Why?
4. $x^2 = l(l - x)$, or	4. Why?
5. $\overline{AF}^2 = AB \times FB$.	5. Why?

EXERCISES

C

1. A line segment 8 inches long is divided in extreme and mean ratio. Find the lengths of its segments.

Solution. Let x = the number of inches in the longer segment.
Then $8 - x$ = the number of inches in the shorter segment.

$$x^2 = 8(8 - x). \quad \text{Why?}$$
$$x^2 = 64 - 8x.$$
$$x^2 + 8x = 64.$$

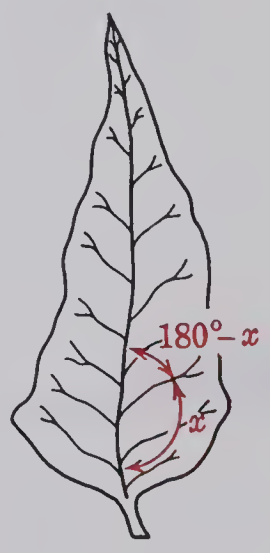
Solving, $x = 4.94+$, the number of inches in one segment.
$8 - x = 3.06-$, the number of inches in the other segment.

Check. Does $(4.94)^2 = 8 \times 3.06$? $24.4 = 24.5$, approximately.

2. Find the lengths of the parts of a line segment 5 inches long when the segment is divided in extreme and mean ratio.

3. Using straightedge and compasses, divide a line segment 5 inches long in extreme and mean ratio.

4. A line segment of length a is divided in extreme and mean ratio. Show that the lengths of the two parts are approximately .62 a and .38 a.

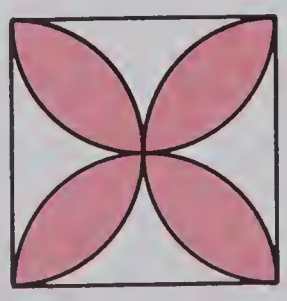


5. The veins of a certain fern leaf meet the main stem so as to divide the straight angle in extreme and mean ratio. Find the angles formed by a vein and the main stem. $[x^2 = 180(180 - x).]$

6. Divide a line segment 6 inches long in extreme and mean ratio. Measure the lengths of the two segments. Find the lengths by algebra.

7. Find the area of a circular segment formed by a chord 8 inches long and an arc of 106° 16′ if the radius of the circle is 5 inches.

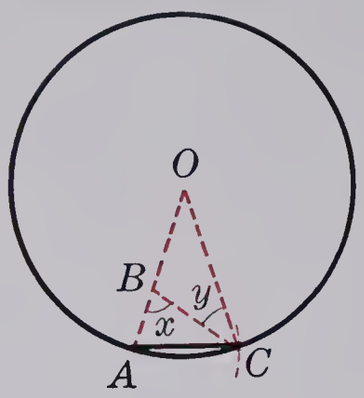
8. Find the area of a circular segment formed by a chord 6 inches long and its arc if the radius of the circle is 6 inches.



9. If one side of the adjoining square is 8 inches, find the area of the shaded portion of the square formed by semicircles constructed on the sides as diameters.

Problem 29

494. To inscribe a regular decagon in a given circle.



Given the $\odot O$.

To inscribe a regular decagon in $\odot O$.

Construction

STATEMENTS	REASONS
1. Draw a radius OA.	1. Why possible?
2. Divide OA in extreme and mean ratio so that $\frac{AO}{BO}=\frac{BO}{AB}$.	2. § 493.
3. With A as a center and BO as a radius construct an arc intersecting the $\odot$ at C.	3. Why possible?
4. Draw AC.	4. Why possible?

Then AC is a side of the required decagon.

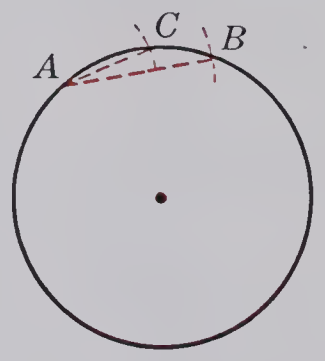
Proof

1. Draw CB and CO.	1. Why possible?
2. $\frac{AO}{BO}=\frac{BO}{AB}$.	2. Const.
3. But $AC=BO$.	3. Const.
4. $\therefore \frac{AO}{AC}=\frac{AC}{AB}$.	4. Why?
5. In $\triangle AOC$ and ACB, $\angle A=\angle A$.	5. Why?
6. $\therefore \triangle AOC \sim \triangle ACB$.	6. Why?
7. Since $\triangle AOC$ is isos., $\triangle ACB$ is isos. and $AC=BC$.	7. Why?
8. But $BO=AC$.	8. Why?
9. $\therefore BC=BO$.	9. Why?

10. $\therefore \angle O = \angle y$.	10. Why?
11. $\angle x = \angle O + \angle y$, or $\angle x = 2\ \angle O$.	11. § 123.
12. From (7), $\angle A = \angle x$.	12. Why?
13. $\therefore \angle A = 2\ \angle O$.	13. Why?
14. $\angle ACO = \angle A$.	14. Why?
15. $\therefore \angle ACO = 2\ \angle O$.	15. Why?
16. $\angle A + \angle ACO + \angle O = 180°$.	16. Why?
17. $\therefore 2\ \angle O + 2\ \angle O + \angle O = 180°$.	17. Why?
18. $\therefore 5\ \angle O = 180°$.	18. Why?
19. $\therefore \angle O = 36°$.	19. Why?
20. $\therefore \overset{\frown}{AC} = 36°$, or $\frac{1}{10}$ of the $\odot$.	20. Why?
21. $\therefore$ chord AC is one side of the required decagon.	21. Why?

495. Corollary I. A regular pentagon can be inscribed in a circle by joining the alternate vertices of a regular inscribed decagon.

496. Corollary II. A regular pentadecagon (a polygon of 15 sides) can be inscribed in a circle.



SUGGESTIONS. Construct AB, the side of a regular inscribed hexagon, and AC, the side of a regular inscribed decagon (see § 494).

$$\overset{\frown}{AB} - \overset{\frown}{AC} = 60° - 36° = \overset{\frown}{BC} = 24°.$$

Chord BC is a side of the required pentadecagon.

The division of the circle into any number of equal arcs by means of the straightedge and compasses can be done only in special cases. Gauss (1777–1855), when nineteen years of age, proved that a regular polygon of 17 sides could be inscribed in a circle. He also proved that a polygon of $(2^{2^n} + 1)$ sides can be inscribed in a circle if n is an integer and if $(2^{2^n} + 1)$ is a prime number.

What is the value of $(2^{2^n} + 1)$ if $n = 1$? if $n = 2$? if $n = 3$? if $n = 4$? Why can a regular polygon of $2^n(2^{2^n} + 1)$ sides, where n is a positive integer, be inscribed in a circle?

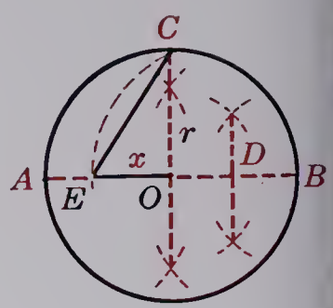
EXERCISES

C

1. Construct a five-pointed star.
2. Construct a regular inscribed polygon of 20 sides.

3. Construct a regular inscribed polygon of 30 sides.

4. The draftsman has the following method of inscribing a regular pentagon and decagon in a circle. Draw a diameter AB. Construct radius $CO \perp AB$. Bisect OB at D. With D as a center and CD as a radius draw arc CE. Then chord CE is a side of the regular pentagon inscribed in the circle and EO is a side of the regular decagon inscribed in the circle.



a. *Prove* that EO, or x, is a side of the regular inscribed decagon.

SUGGESTIONS. Let $OB = r$. Then $OD = \frac{1}{2}r$. $CD = \sqrt{r^2 + \frac{r^2}{4}} = \frac{r}{2}\sqrt{5}$. Why?

$ED = \frac{r}{2}\sqrt{5}$. $x = \frac{r}{2}\sqrt{5} - \frac{r}{2} = \frac{r}{2}(\sqrt{5} - 1)$.

In Problem 29 it was proved that when a radius r of a circle is divided into extreme and mean ratio, the larger segment of the radius is equal to a side d of an inscribed regular decagon.

Then $\frac{r}{d} = \frac{d}{r - d}$.

Solving for d^2, $d^2 = r^2 - rd$.

$d^2 + rd = r^2$.

Solving for d, $d = \frac{r}{2}(\sqrt{5} - 1)$.

$\therefore x = d$.

b. *Prove* that CE is a side of the regular inscribed pentagon.

SUGGESTIONS. Let p be a side of the regular inscribed pentagon.

Then $d = \sqrt{2r^2 - r\sqrt{4r^2 - p^2}}$, by § 485. But $d = \frac{r}{2}(\sqrt{5} - 1)$.

$\therefore \frac{r}{2}(\sqrt{5} - 1) = \sqrt{2r^2 - r\sqrt{4r^2 - p^2}}$.

Solving for p^2, $p^2 = r^2 + \frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5}$.

But $d^2 = \frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5}$.

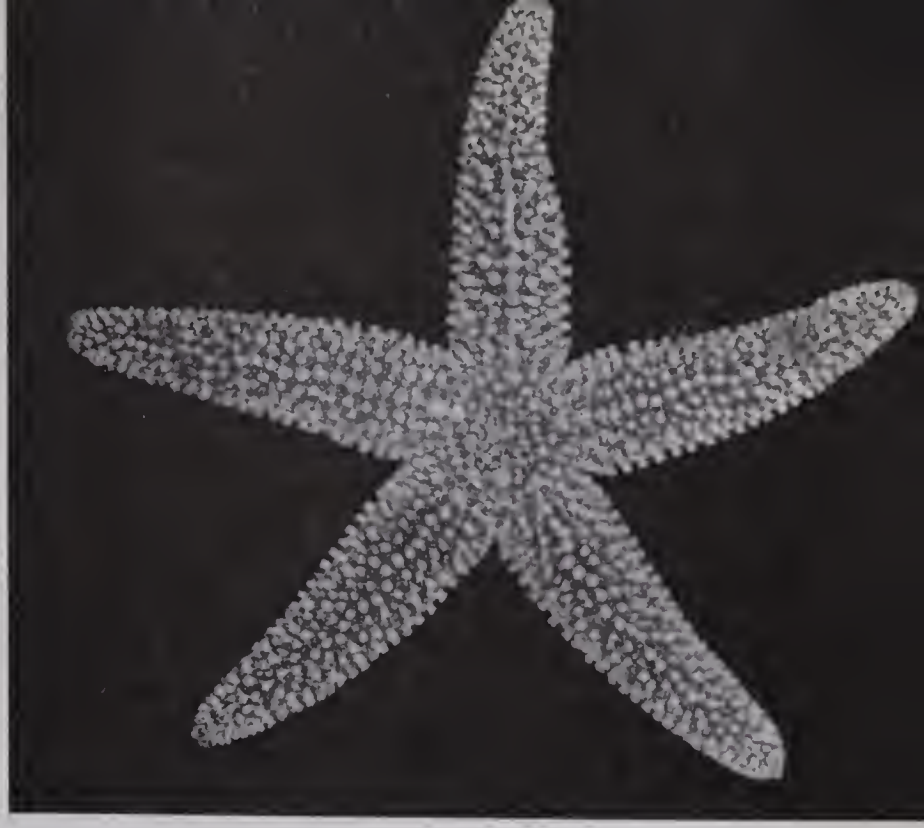
Then $p^2 = r^2 + d^2$.

But $\overline{CE}^2 = r^2 + d^2$.

$\therefore \overline{CE}^2 = p^2$ and $CE = p$, a side of the regular inscribed pentagon.

5. Using the method given in Ex. 4, inscribe a regular pentagon and a regular decagon in a circle.

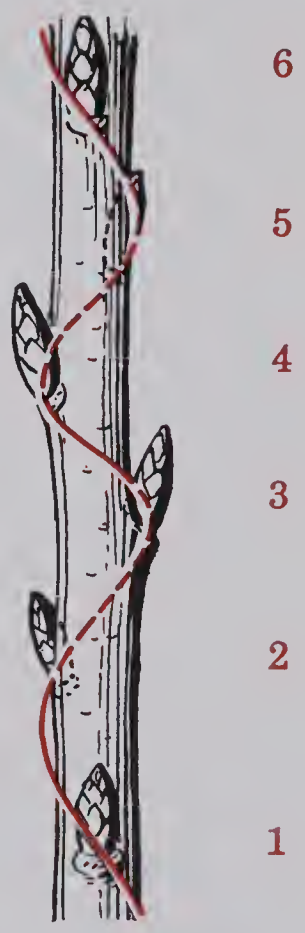
Rutherford Platt



Clarence Flaten

497. How Nature Makes Use of the Golden Section

Nature has used the golden section in both vegetable and animal life. When a flower has five or ten petals the principle of the golden section is applied. The starfish is a common example of the use of the golden section in animal life.



The stems of leaves are often arranged on the branch in the form of a spiral. In such an arrangement each stem appears at regular intervals on the spiral, each one being a little higher on the branch and farther around on it than the preceding one.

In the diagram of an oak twig at the right, notice that leaf bud 6 is directly above bud 1 on the stem. If 1 is the first bud of one cycle, then 6 is the first bud of the next cycle. A complete cycle consists of five leaves arranged equally distant apart in two revolutions of the spiral. We can represent this arrangement by the fraction $\frac{2}{5}$. The elm tree has the $\frac{1}{2}$ arrangement and the beech tree has the $\frac{1}{3}$ arrangement. Some trees have the $\frac{3}{8}$ arrangement, and some, including certain bushes, have the $\frac{5}{13}$ arrangement.

In most if not all the spiral arrangements of leaves on stems, both the numerator and the denominator of the fraction are members of the series 1, 2, 3, 5, 8, 13, 21, 34, · · ·. In this series of numbers, called the Fibonacci series, each number is equal to the sum of the two numbers immediately preceding it. As the numbers increase in size, the

quotient obtained by dividing any number by the next consecutive one becomes nearer and nearer to $\frac{\sqrt{5}-1}{2}$, or .618+. The fraction .618+ is the ratio of the larger segment to the whole when a segment of a line is divided into the golden section. The nth term of the Fibonacci series can be found by

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

The sunflower illustrates the use of the golden section in two ways. The fruit sockets of a sunflower head form a series of intersecting curves which seem to be logarithmic spirals. A very small sunflower head has 21 curves crossing 34 curves, another has 34 curves crossing 55 curves, and a large head has 55 curves crossing 89 curves. The numbers of these curves are also of the series 1, 2, 3, 5, 8, 13, 21, 34, · · ·.

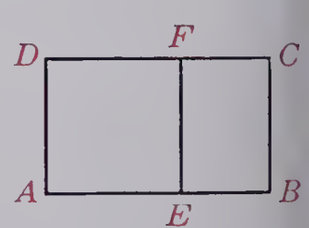
Rutherford Platt

The pine cone and the common teasel have a similar arrangement of the seed pods, except that their heads are not so flat as those of the sunflower.

The Fibonacci numbers appear in other flowers. There are 8 ray-florets in the single dahlia, 13 in the ragwort, and 21 in the oxeye daisy.

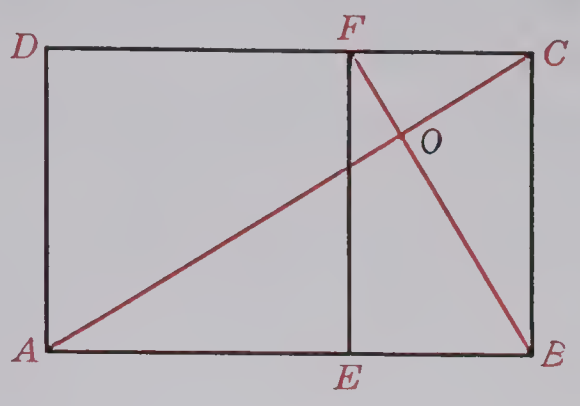
498. The Most Beautiful Rectangle

We all know that some rectangles are more pleasing to the eye than others. The most beautiful rectangle is constructed as follows:



Divide the base AB into extreme and mean ratio so that $\frac{AB}{AE}=\frac{AE}{EB}$. Then construct the rectangle $ABCD$ having the height $AD=AE$. This rectangle has many interesting properties. It consists of the similar rectangle $EBCF$ and the square $AEFD$. Its diagonal AC is perpendicular to BF,

the diagonal of $EBCF$. As explained below, it is sometimes called the *rectangle of whirling squares*. Since its numerical properties are related to plant and animal life, including the human skeleton, and since it has a property of dynamic symmetry, its use helps to give life to painting and sculpture. See how many of the following properties of the rectangle you can prove, and determine whether or not they apply to any rectangle.



EXERCISES

C

1. The square AF is equal to a rectangle whose base is AB and whose altitude is equal to EB.

2. Rectangle $AC \sim$ rectangle EC.

SUGGESTIONS. $\frac{AB}{AE}=\frac{AE}{EB}$. Then $\frac{AB}{BC}=\frac{BC}{EB}$.

3. AB is the mean proportional between BC and $AB+BC$.

4. $\frac{AO}{OB}=\frac{OB}{OC}=\frac{OC}{OF}$.

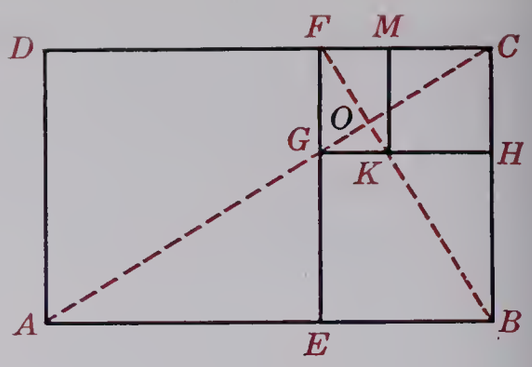
5. If $AD=1$, show that rectangle $EBCF$ is the reciprocal of rectangle $ABCD$, or that $EBCF=\frac{1}{ABCD}$.

499. The Rectangle of the Whirling Squares

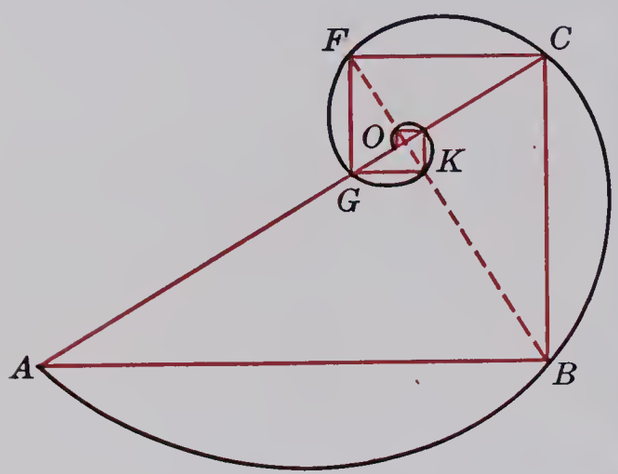
If the length of the most beautiful rectangle $ABCD$ is represented by $\frac{\sqrt{5}+1}{2}$ units or 1.618, its width is represented by 1 unit. The rectangle $ABCD$ is composed of the reciprocal rectangle $EBCF$ and the square $AEFD$.

The reciprocal rectangle $EBCF$ can be constructed by drawing $BF \perp AC$ and then drawing $EF \perp DC$. The reciprocal rectangle $EBCF$ cuts off the square $AEFD$. Likewise the rectangle $GHCF$ (see p. 484) is the reciprocal of the rectangle $EBCF$ and cuts the square $EBHG$ from

rectangle $EBCF$. Again, the rectangle $GKMF$ cuts the square $KHCM$ from rectangle $GHCF$. If this process is continued indefinitely, a series is formed consisting of squares which continually get smaller and arrange themselves about O at intervals of 90°.



If the longer side of each rectangle is drawn as shown below, a broken line is formed which winds itself about the point O, but never reaches O. The points $ABCFGK\cdots$ lie on a



curve called the logarithmic spiral. This curve appears to be the growth curve of plants. Observe that the chambered nautilus has the shape of this curve.

SPACE GEOMETRY

[Optional]

500. Area and Volume of a Circular Cylinder

In § 487 we saw that the theorem on the area of a circle was based on the area of a regular polygon. In solid geometry the theorems on the area and volume of a circular cylinder (see page 312) are based on the theorems on the area and volume of a prism having regular polygons as bases. These theorems are:

The lateral area of a right circular cylinder is equal to the product of the altitude and the circumference of the base, or $S = 2\pi rh$.

The volume of a right circular cylinder is equal to the product of the base and altitude, or $V = \pi r^2 h$.

EXERCISES

1. Show that the total area, T, of a right circular cylinder is given by the formula $T = 2\,\pi r(r + h)$.

SUGGESTIONS. Let A = area of one base. $A = \pi r^2$, $2\,A = 2\,\pi r^2$, and $S = 2\,\pi rh$. $S + 2\,A = T$.

2. Find the lateral areas of the following right circular cylinders, given

a. Altitude 6 inches, radius of base 4 inches.
b. Altitude 10 feet, radius of base 2 feet.
c. Altitude 4 feet, diameter of base 7 feet.
d. Altitude 14 inches, circumference of base 32 inches.

3. Find the volume of the following right circular cylinders, given

a. Radius of base 5 inches, altitude 12 inches.
b. Radius of base 11 feet, altitude 20 feet.
c. Diameter of base 10 inches, altitude 16 inches.
d. Circumference of base 80 inches, altitude 6 feet.

4. Find the total area and volume of a right circular cylinder if the radius of the base is 5 feet and the altitude is 20 feet.

5. The circumference of the base of a right circular cylinder is 28 inches, and its height is 15 inches. What is the lateral area of the cylinder?

6. What is the height of a cylindrical pail 8 inches in diameter if it holds a gallon?

501. Area and Volume of a Right Circular Cone

The theorems from solid geometry on the area and volume of a right circular cone are:

The lateral area of a right circular cone is equal to half the product of its slant height and the circumference of its base, or $S = \pi rl$.

The volume of a right circular cone is equal to one third of the product of its base and altitude, or $V = \frac{1}{3}\,\pi r^2 h$.

EXERCISES

1. Show that the total area of a right circular cone is given by the formula $T = \pi r(l + r)$.

2. Find the slant height and lateral area of a right circular cone if the radius of the base is 5 inches and the altitude is 12 inches.

3. Find the volume and lateral area of a right circular cone if the altitude is 8 inches and the radius of the base is 6 inches.

4. Find the lateral area and the volume of a right circular cone if the slant height is 17 feet and the radius of the base is 8 inches.

5. Find the total area of a right circular cone if the altitude is 24 inches and the diameter of the base is 14 inches.

502. Area and Volume of a Sphere

The theorems on the area and volume of a sphere are:

The area of a sphere is equal to the area of four great circles of the sphere, or $S = 4\ \pi r^2$.

The volume of a sphere with radius r is given by the formula $V = \frac{4}{3}\ \pi r^3$.

EXERCISES

1. Find the area and volume of the following spheres:

a. Radius 6 inches.
b. Radius 4 inches.
c. Radius 3 feet.
d. Radius 5 feet.

2. A sphere 12 inches in diameter is inscribed in a right circular cylinder. What is the area of the sphere? the lateral area of the cylinder? What is the volume of the sphere? of the cylinder?

3. Solve the formula $S = 4\ \pi r^2$ for r.

4. A right circular cylinder and a right circular cone have equal bases and equal altitudes. What is the ratio of their areas? the ratio of their volumes?

5. Find the cost at \$3 a square foot of gilding a dome in the shape of a hemisphere whose diameter is 25 feet.

6. A sphere 14 inches in diameter is inscribed in a cylinder. What is the area of the sphere? the lateral area of the cylinder? What is the volume of the sphere? of the cylinder?

503. Reasoning in a Circle. [Optional]

When Frank gave his teacher Tuesday's homework, which consisted of writing the proofs of Theorem 80 on page 452 and Theorem 81 on page 454, he told her he had a new proof of Theorem 80. His new proof began as follows: "Inscribe $\odot O$ in $ABCDEF$"; reason, "A circle can be inscribed in any regular polygon." Frank followed the plan of the text in proving Theorem 81.

His teacher told him that he used Theorem 81 to prove Theorem 80 and used Theorem 80 to prove Theorem 81. She said that he used *reasoning in a circle* and that this type of reasoning was useless. He had only proved that Theorem 80 is true if Theorem 81 is true, and that Theorem 81 is true if Theorem 80 is true.

To prevent reasoning in a circle, you should remember the sequence of the theorems in the text you are using and you should never use a theorem as a reason in a proof before you have proved it.

Let us take as an example a small child's use of reasoning in a circle.

Mother. Where is Clara?
Billy. She is with Betty.
Mother. Well, where is Betty?
Billy. Oh! She is with Clara.

EXERCISES

Optional

Discuss the following:

1. A right angle is an angle formed by two perpendicular lines. Two lines are perpendicular to each other if they form a right angle.

2. Mrs. Jones wished to buy a certain brand of cheese from her grocer but found that he did not have it in stock. She asked him why he did not sell this particular brand and he replied that there was not enough demand for it. Mrs. Jones said that this brand was excellent and asked why more people did not call for it. He answered, "They do not call for it because I do not have it for sale."

3. When studying Theorem 20, page 207, Celia said, "I have an original proof of the theorem." Her teacher said, "How would you prove the theorem?" Celia replied, "I would draw BD and prove the triangles congruent by § 85. Then $\angle m = \angle n$ by § 78. Then $AD \parallel BC$ by § 104. Then $ABCD$ is a $\square$ by § 198."

REVIEW QUESTIONS — Chapter 14

1. What are the names of the four centers of a triangle?

2. Can a circle be circumscribed about

   a. A triangle?
   b. A quadrilateral?
   c. A rhombus?
   d. A square?
   e. A rectangle?
   f. A regular pentagon?

3. What two conditions are necessary for a polygon to be regular?

4. In what polygon that is not regular can a circle be inscribed?

5. When are two regular polygons similar?

6. If the number of sides of a regular inscribed polygon is increased indefinitely, what is the limit

   a. Of the apothem?
   b. Of each side?
   c. Of the perimeter?
   d. Of the area of the polygon?
   e. Of an angle of the polygon?
   f. Of a central angle?

7. How many of the regular polygons having less than twenty-five sides can be constructed with compasses and straightedge?

8. Why is the radius of a regular polygon so named?

9. What inscribed polygon has a side equal to the radius?

10. If the radius of a circle is doubled,

    a. Is the area doubled?
    b. Is the circumference doubled?

11. What is a constant? a variable? Does a constant have a limit?

12. The ratio of the radii of two circles is 2 : 5. What is the ratio of their circumferences? of their areas?

13. Is the ratio of the area of a circle to its radius constant?

In exercises 14, 15 select the best answer,—a, b, c, or d.

14. The difference between the areas of two circles with diameters 4.0 inches and 6.0 inches is:

    a. 2 sq. in.
    b. 4 sq. in.
    c. 4π sq. in.
    d. 5.0π sq. in.

15. The area of a square inscribed in a 10-inch circle is:

    a. 50 sq. in.
    b. $50\pi\sqrt{2}$ in.
    c. 25π inches.
    d. Not (a), (b), or (c).

SUMMARY of Principal Bases for Proof

504. Regular polygons

a. If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon and the tangents at the points of division form a regular circumscribed polygon.
b. An equilateral polygon inscribed in a circle is regular.
c. If the midpoints of the arcs of a regular inscribed polygon are joined to the extremities of the respective sides, a regular inscribed polygon of double the number of sides is formed.

505. Areas

a. The area of a regular polygon is equal to one half the product of its apothem and its perimeter.
b. The area of a circle is equal to one half the product of its radius and its circumference.
c. The area of a circle is given by the formula $S = \pi r^2$.

506. Proportions involving polygons and circles

a. The areas of two regular polygons of the same number of sides have the same ratio as the squares of any two corresponding sides or as the squares of their radii or as the squares of their apothems.
b. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or as their apothems.
c. The circumferences of two circles have the same ratio as their radii or as their diameters.
d. The areas of two circles have the same ratio as the squares of their radii or as the squares of their diameters.
e. The area of a sector of a circle is to the area of the circle as the angle of the sector is to 360°.

507. Similar polygons

Two regular polygons of the same number of sides are similar.

508. Circumference of a circle

The circumference of a circle is expressed by the formula $c = \pi d$ or $c = 2\ \pi r$.

509. Constructions

a. A circle can be circumscribed about any regular polygon.
b. A circle can be inscribed in any regular polygon.
c. To divide a given line segment in extreme and mean ratio.
d. To inscribe a regular decagon in a given circle.
e. A regular pentagon can be inscribed in a circle by joining the alternate vertices of a regular inscribed decagon.
f. A regular pentadecagon can be inscribed in a circle.

WORD LIST

apothem	circumscribed	inscribed	polygon
arc	complementary	limit	radii
arch	diameter	measure	sector
central	divided	pentagon	similar
circumference	exercises	pi	variable

TEST 36 — True-False Test

[12 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number. Do not guess.

1. All regular polygons are similar.
2. The apothem of a regular polygon is the radius of the incircle.
3. The radius of a regular polygon is the radius of the circumcircle.
4. An equilateral polygon inscribed in a circle is a regular polygon.
5. A circle can be circumscribed about any polygon.
6. If a regular pentagon and a square have the same area, the pentagon has the greater perimeter.
7. The circumcircle and the incircle of a regular polygon are concentric.
8. The area of a regular polygon is equal to one half the product of its radius and its perimeter.
9. The perimeters of two regular polygons have the same ratio as their radii.
10. The areas of two circles have the same ratio as their radii.
11. The ratio of the circumference of a circle to its diameter is 3.1416.

12. The area of a circle is equal to one half the product of its radius and its circumference.

13. If the diameter of one circle is twice the diameter of a second circle, the area of the first circle is twice the area of the second circle.

14. The area of a sector of a circle is to the area of the circle as the angle of the sector is to 180°.

15. The apothem of an equilateral triangle is equal to one half the radius of the circumcircle.

16. If the perimeter of a regular polygon is doubled, its area is doubled.

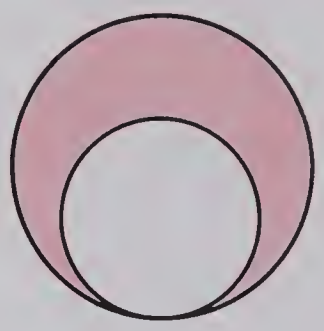
17. Two polygons are similar if their sides are parallel each to each.

18. Two polygons are similar if their corresponding sides are proportional.

TEST 37 — Applications

[25 Minutes]

1. What is the ratio of the areas of two circles whose diameters are 4 and 5 inches respectively?



2. Find the area of the shaded portion in the figure if the diameter of the smaller circle is 2 inches and the diameter of the larger circle is 3 inches.

3. If the radius of a circle is 4, what is the area of an inscribed equilateral triangle?

4. The radius of a circle is $\sqrt{2}$. Find the area of an inscribed square.

5. The area of a circle is 180 sq. ft. Find the area of a sector of 80°.

6. One side of a regular hexagon is 6 inches. Find the length of the apothem.

7. How large is the central angle of a regular quadrilateral?

8. Find the circumference of a circle whose radius is 8 inches.

9. A pentagon is circumscribed about a circle of radius 5. Find the area of the pentagon if its sides are 4, 6, 8, 9, and 10.5 respectively.

10. The area of one regular heptagon is $6\frac{1}{4}$ times that of another. What is the ratio of their perimeters?

11. Find the area of a segment of a circle formed by one side of an inscribed square and its arc if the radius of the circle is 10 inches.

12. Two tangents to a circle form an angle of 105°. Find the length of the minor arc if the radius of the circle is 10 inches.

CHAPTER 15

Elements of Trigonometry

In this chapter you will learn
how the relationships
between the parts of similar triangles
become the basis
for a system of indirect measurement.

The word "trigonometry" is derived from two Greek words which mean "triangle measurement." It is that branch of mathematics which deals with the relations between the sides and the angles of triangles.

One of the main purposes of trigonometry is to obtain formulas by means of which distances and angles may be measured indirectly. You have seen how to construct a triangle when three parts are given, one of which must be a side. When the numerical values of these parts are given, trigonometry will enable you to compute the values of the unknown parts. With the steel tape the engineer measures distances, and with the transit he measures angles. Then, using trigonometry, he computes other distances and angles and is able to make plans and specifications for construction of various kinds.

Trigonometry is constantly used in surveying, engineering, physics, astronomy, and navigation. Without trigonometry, surveying and navigation would be very difficult and the distances from the earth to the sun, the moon, and the planets would be unknown.

510. Trigonometric Functions

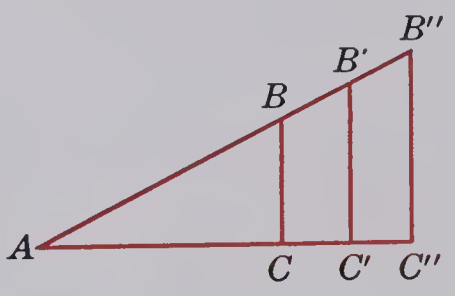
We proved in § 416 that if right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar. In the figure, $\triangle ABC$, $AB'C'$, and $AB''C''$ are right triangles having the same acute $\angle A$. Hence

$$\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''},$$

$$\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{AC''}{AB''},$$

and

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''}. \qquad (\S\ 413.)$$



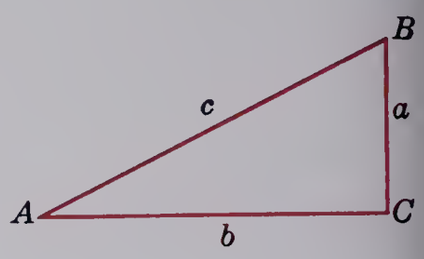
That is, the value of the ratios of the sides of the right triangles depends upon the size of angle A and not upon the size of the right triangles. The values of these ratios change as $\angle A$ changes and for this reason are called *functions* of $\angle A$.

511. Trigonometric Functions of an Acute Angle

1. The sine of an acute angle of a right triangle is the ratio of the opposite leg to the hypotenuse.
2. The cosine of an acute angle of a right triangle is the ratio of the adjacent leg to the hypotenuse.

3. The tangent of an acute angle of a right triangle is the ratio of the opposite leg to the adjacent leg.

It is customary, in the study of trigonometry, to let the capital letters A, B, and C denote the angles of a right triangle, C being the right angle, and the small letters a, b, and c denote the corresponding opposite sides. From the right $\triangle ABC$,



$$\text{sine of } \angle A \text{ (written sin } A) = \frac{a}{c}$$

$$\text{cosine of } \angle A \text{ (written cos } A) = \frac{b}{c}$$

$$\text{tangent of } \angle A \text{ (written tan } A) = \frac{a}{b}$$

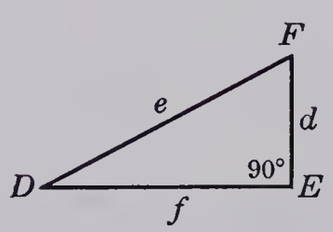
In like manner, $\sin B = \frac{b}{c}$; $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$.

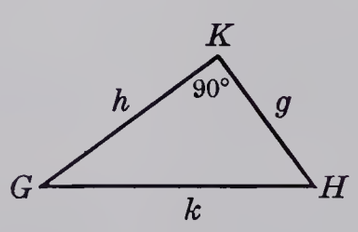
The definitions in this section should be memorized so that the ratio for each function can be read from a figure instantly.

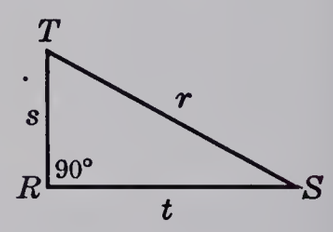
EXERCISES

B

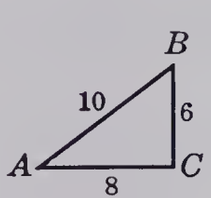
1. From the figures below, give the sine, cosine, and tangent of each acute angle in terms of the sides of the triangle. Thus, $\sin D = \frac{d}{e}$, $\cos D = \frac{f}{?}$, etc.

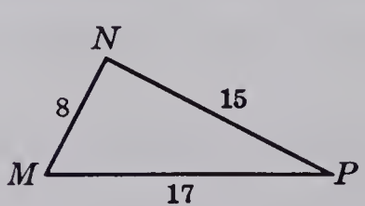


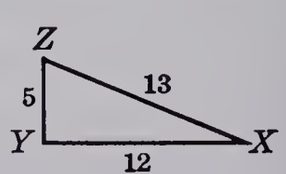




2. In the figures below, the lengths of the sides are indicated. Verify by § 430 that the triangles are right triangles and give the numerical value of the sine, cosine, and tangent of each acute angle to two decimal places.



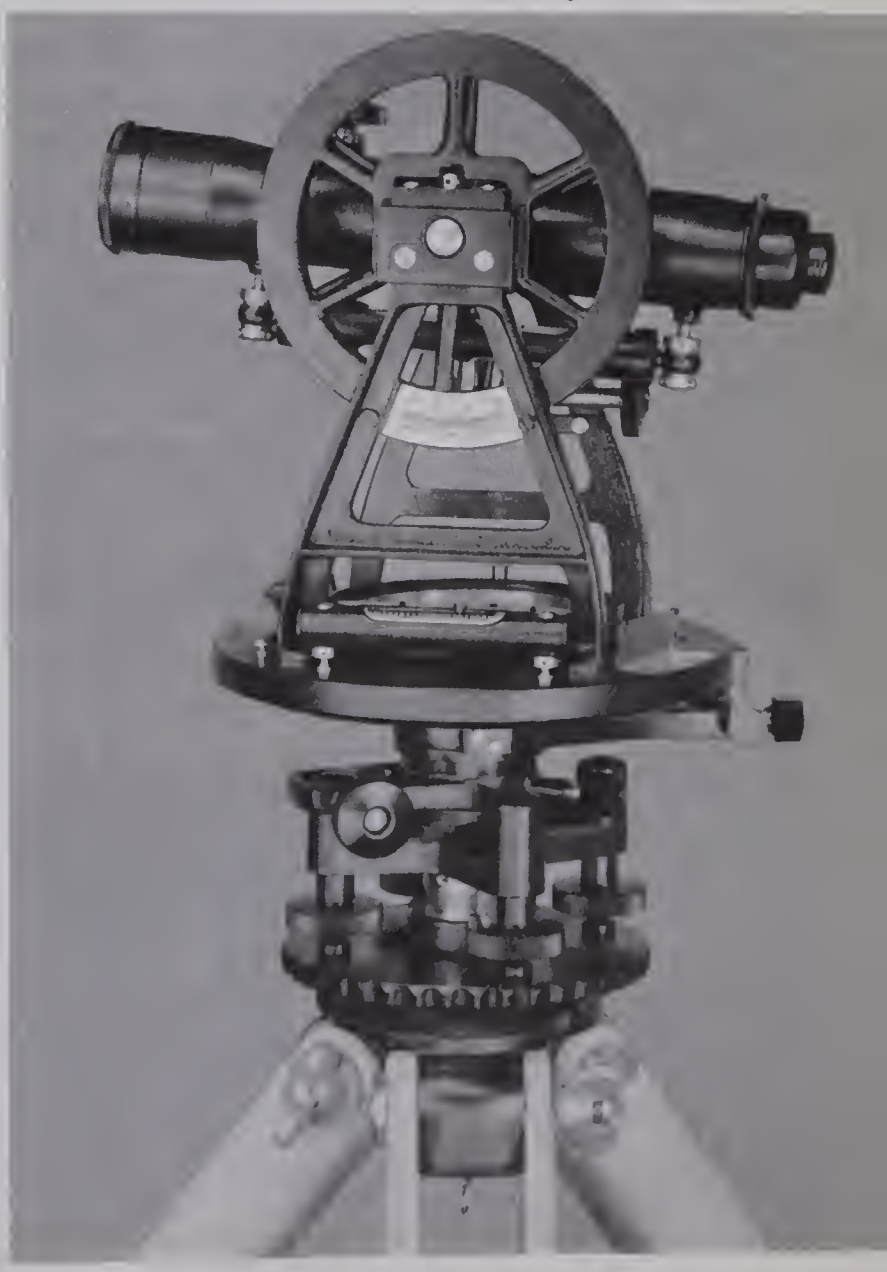




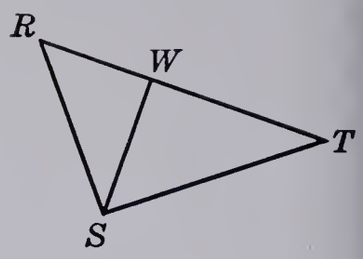
Bob Towers from Black Star

You have probably watched a surveyor at work with his transit. The instrument is used to measure either horizontal or vertical angles. The engineer in the picture is plotting desert wastes for a housing development near Phoenix, Arizona.

W. & L. E. Gurley



3. In rt. $\triangle RST$, $\angle S$ is the rt. $\angle$ and $SW \perp TR$. Give the three functions of $\angle T$ in terms of RS, ST, and TR. Also give these functions in terms of SW, ST, and TW.

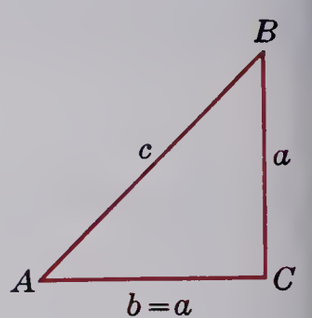


4. In the same figure, give the three functions of $\angle R$ in terms of RS, ST, and TR and also in terms of RS, SW, and RW.

5. With a protractor construct any right triangle having an acute angle of 20°, and by measurement find the numerical values of sin 20°, cos 20°, and tan 20° to two decimal places. From the same figure find the numerical values of sin 70°, cos 70°, and tan 70°.

512. Functions of 45°, 30°, and 60°

In the isosceles rt. $\triangle ABC$, $\angle A = 45°$ and $b = a$. $c^2 = a^2 + b^2$ (§ 430). $c^2 = 2\,a^2$ (Asmt. 6). $\therefore c = a\sqrt{2}$ (Asmt. 8).

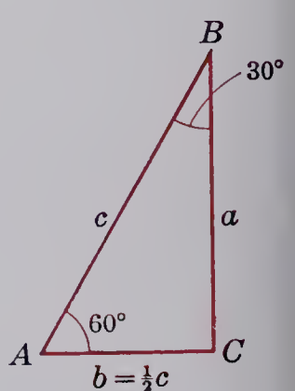


$$\sin 45° = \frac{a}{c} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}.$$

$$\cos 45° = \frac{b}{c} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}.$$

$$\tan 45° = \frac{a}{b} = \frac{a}{a} = 1.$$

In a 30°-60° rt. $\triangle ABC$, $b = \frac{1}{2}\,c$ (§ 136) and $a^2 = c^2 - \frac{1}{4}\,c^2$ (§ 430). $a^2 = \frac{3}{4}\,c^2$. $\therefore a = \frac{1}{2}\,c\sqrt{3}$ (Asmt. 8).



$$\sin 30° = \frac{b}{c} = \frac{\frac{1}{2}\,c}{c} = \frac{1}{2}.$$

$$\cos 30° = \frac{a}{c} = \frac{\frac{1}{2}\,c\sqrt{3}}{c} = \frac{1}{2}\sqrt{3}.$$

$$\tan 30° = \frac{b}{a} = \frac{\frac{1}{2}\,c}{\frac{1}{2}\,c\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

$$\sin 60° = \frac{a}{c} = \frac{\frac{1}{2}\,c\sqrt{3}}{c} = \frac{1}{2}\sqrt{3}.$$

$$\cos 60° = \frac{b}{c} = \frac{\frac{1}{2}\,c}{c} = \frac{1}{2}.$$

$$\tan 60° = \frac{a}{b} = \frac{\frac{1}{2}\,c\sqrt{3}}{\frac{1}{2}\,c} = \sqrt{3}.$$

The sine and cosine of 30°, 45°, and 60° are easily remembered in the following way:

Function	30°	45°	60°
sin	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
cos	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$

513. Trigonometric Tables

The functions of the angles given in § 512 are the only ones we can find by elementary algebra and geometry. We could find the approximate values of the functions of other angles by the method of Ex. 5, p. 496. For practical purposes, however, we refer to more accurate values computed by higher mathematics. In Table II, p. 576, the values of the functions from 0° to 90° are given to four decimal places.

EXERCISES

B

Using Table II, find the value of each of the following functions:

1. sin 20°.
2. cos 14°.
3. sin 76°.
4. cos 70°.
5. tan 17°.
6. tan 63°.
7. sin 12°.
8. cos 54°.
9. tan 85°.
10. Which of the three functions increase as the angles increase and which decreases as the angles increase?

Find the number of degrees in the angle in each of the following exercises:

11. $\sin A = .2250$.
12. $\cos A = .8988$.
13. $\tan A = .2679$.
14. $\sin B = .7771$.
15. $\cos B = .1392$.
16. $\tan B = 7.1154$.

514. Interpolation

In order to find the value of the function of an angle expressed in degrees and a fractional part of a degree, it is necessary to resort to a process known as *interpolation*, which can be illustrated best by the solution of some examples.

Example 1. Find sin 16° 30′.

Solution. The sine of 16° 30′ is found by adding to the sine of 16° half the difference between sin 16° and sin 17°.

Thus
$$\begin{aligned} \sin 17° &= .2924 \\ \sin 16° &= .2756 \\ \text{Difference} &= .0168 \end{aligned}$$

$\frac{30}{60}$, or $\frac{1}{2}$, of .0168 = .0084. Then .2756 + .0084 = .2840.
Therefore sin 16° 30′ = .2840.

In the case of increasing functions, like the sine and tangent, the correction must be added to the value of the function of the smaller angle; but in a decreasing function, like the cosine, the correction must be subtracted.

Example 2. Find cos 32° 10′.

Solution.
$$\begin{aligned} \cos 32° &= .8480 \\ \cos 33° &= .8387 \\ \text{Difference} &= .0093 \end{aligned}$$

$\frac{10}{60}$, or $\frac{1}{6}$, of .0093 = .0016.
Then .8480 − .0016 = .8464.
Therefore cos 32° 10′ = .8464.

Example 3. Find tan 45.3°.

Solution.
$$\begin{aligned} \tan 46° &= 1.0355 \\ \tan 45° &= 1.0000 \\ \text{Difference} &= .0355 \end{aligned}$$

$\frac{3}{10}$ of .0355 = .0107.
Then 1.000 + .0107 = 1.0107.
Therefore tan 45.3° = 1.0107.

EXERCISES

B

Find the values of the following from Table II. You are less likely to make errors if you organize your solutions similarly to those of the examples.

1. sin 36° 20′.
2. sin 47.4°.
3. tan 18° 50′.
4. tan 72.7°.
5. tan 62.3°.
6. cos 11.9°.
7. cos 80° 40′.
8. cos 33.3°.
9. cos 0°.
10. sin 41.6°.
11. cos 53° 5′.
12. tan 90°.
13. tan 60° 7′.
14. sin 15° 15′.
15. sin 0°.
16. cos 61.8°.

515. Inverse Use of Table

Finding the angle when the value of the function is given will now be illustrated.

Example 1. tan B = .2905. Find $\angle B$.

Solution. .2905 lies between .2867 and .3057.

$$\begin{aligned} .3057 &= \tan 17° \\ .2867 &= \tan 16° \\ \hline .0190 &= \text{difference} \end{aligned}$$

$.2905 - .2867 = .0038$. Then $\frac{.0038}{.0190}$ of $60' = 12'$, or .2°, the amount to be added to 16°. Therefore the required angle is 16° 12′, or 16.2°.

Notice that the result is taken to the nearest minute or tenth of a degree, depending upon which method you use in expressing the angle.

Example 2. $\cos B = .3140$. Find $\angle B$.

Solution. .3140 lies between cos 72° and cos 71°.

$$\begin{aligned} .3256 &= \cos 71° \\ .3090 &= \cos 72° \\ \hline .0166 &= \text{difference} \end{aligned}$$

$.3140 - .3090 = .0050$. Then $\frac{.0050}{.0166}$ of $60' = 18'$, or .3°.

$72° - 18' = 71° 42'$, or 71.7°. Therefore $\angle B = 71° 42'$.

Since the cosine is a decreasing function, the correction is subtracted from 72°.

EXERCISES

B

Find the number of degrees in each angle:

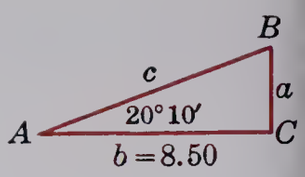
1. $\sin A = .1132$.
2. $\tan X = .3105$.
3. $\cos B = .4664$.
4. $\sin Y = .4664$.
5. $\cos A = .9788$.
6. $\tan B = 3.0841$.
7. $\sin A = .9930$.
8. $\cos A = .1422$.
9. $\tan A = .8233$.
10. $\sin B = .5618$.
11. $\cos B = .9630$.
12. $\tan B = .1500$.
13. $\cos C = .5180$.
14. $\tan C = 1.8045$.
15. $\sin C = .9460$.

516. Finding Parts of a Right Triangle

By the use of trigonometric functions, any part of a right triangle can be found if any two sides or a side and an acute angle are given.

Example 1. Given the rt. $\triangle ABC$ with $\angle A = 20° 10'$ and $b = 8.50$. Find a.

Solution. $\dfrac{a}{b} = \tan A.$



$a = b \tan A.$

$\therefore a = 8.50 \tan 20° 10'.$

$a = 8.50(.3673).$

$a = 3.12205.$

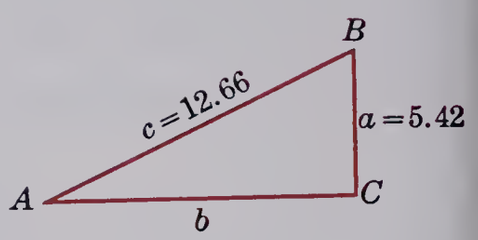
If data are found by measurement, $a = 3.12$.

Example 2. Given $c = 12.66$, $a = 5.42$. Find B.

Solution. $\cos B = \dfrac{a}{c}.$

$\cos B = \dfrac{5.42}{12.66}.$

$\cos B = .4281.$

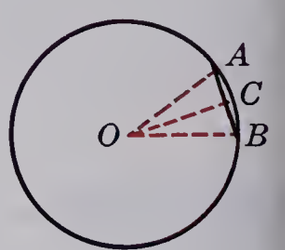


From the table, $B = 64° 39'$, or $64° 40'$.

Example 3. In a circle of 6-inch radius find the length of a chord that has a central angle of 36°.

Solution. Draw $OC \perp AB$.

OC bisects $\angle O$. Why?



$\dfrac{AC}{OA} = \sin \tfrac{1}{2} O.$

$AC = OA \sin \tfrac{1}{2} O.$

$\therefore AC = 6 \sin 18° = 6(.3090).$

$AC = 1.8540$, or 1.85 (inches) approximately.

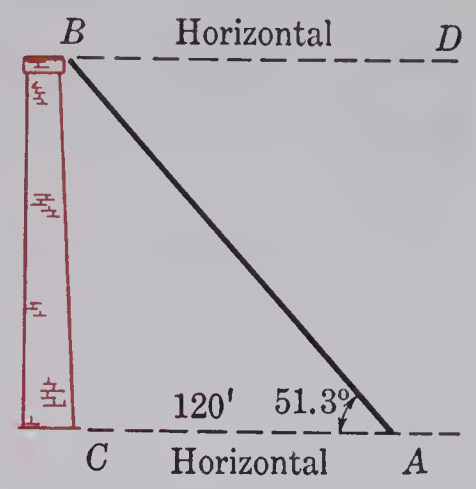
If the 6 inches were obtained by measurement, 1.8540 in. should be rounded off to 2 in.

EXERCISES

B

1. Given $A = 60°$, $b = 40$. Find c.
2. Given $A = 12°$, $b = 14$. Find a.
3. Given $B = 55°$, $b = 10$. Find a.
4. Given $A = 20.5°$, $c = 80$. Find a.
5. Given $B = 56° 10'$, $c = 16.0$. Find a.
6. Given $a = 4.84$, $b = 3.63$. Find A.
7. Given $a = 6.5$, $c = 9.8$. Find B.

8. A man standing 120 feet from the foot of a chimney finds that the angle of elevation of the top of the chimney is 51.3°. Find the height of the chimney.



NOTE. In the figure, $\angle A$ is the *angle of elevation* of B from A. $\angle ABD$ is the *angle of depression* of A from B.

9. From the top of a building 160.2 feet high, the angle of depression of an automobile on a road is 26° 20′. How far is the automobile from the foot of the building?

10. At a horizontal distance of 112.0 feet from the base of a tower, the angle of elevation of the top is 72° 10′. Find the height of the tower.

11. Find the angle of elevation of the sun when a tree whose height is 96 feet casts a shadow 116 feet in length.

12. What is the angle of elevation of an inclined plane if it rises a foot in a horizontal distance of 12 feet?

13. The Washington Monument is 555 feet high. What is the angle of elevation of the top when viewed at a distance of half a mile?

14. Find the angles of an isosceles triangle if the equal sides are each 12 inches and the base 18 inches.

SUGGESTION. Draw the altitude on the base.

15. Find the angles of an isosceles triangle if the altitude is 8 inches and the equal sides are each 10 inches.

16. The average inclination of the bed of a stream is 3° 10′. Find its fall in a distance of 1 mile along the stream.

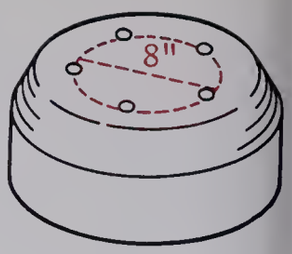
C

17. The sides of a right triangle are 3, 4, and 5. Find the number of degrees in each acute angle.

18. If a hillside has a slope of 5° 0′, how far up the hillside will a dam 36.00 feet high force the water?

19. Find the area of an isosceles triangle if the base is 16 feet and each of the equal sides is 15 feet.

20. Find the area of an isosceles triangle if the vertex angle is 40° and the altitude is 6 feet.



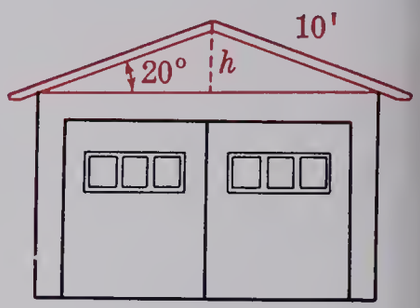
21. In making a cylinder head for a steam engine five holes are to be placed equally about a circle 8 inches in diameter. How far apart will the holes be, center to center?

22. The hypotenuse of a right triangle is 24 feet and one acute angle is 36° 20′. Find the altitude on the hypotenuse.

23. Find the radius of a circle inscribed in an equilateral triangle whose perimeter is 36 inches.

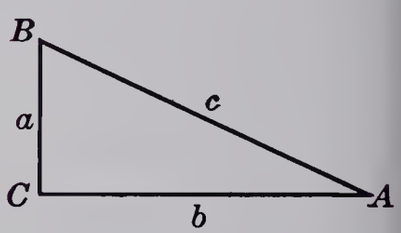
24. Two sides and the included angle of a parallelogram are 40, 60, and 68° respectively. Find the altitude on side 60.

25. On a circular railway curve an arc of 30° has a chord 400 feet long. Find the radius of the curve.



26. Find the number of degrees in an arc whose chord is 18 feet if the radius of the circle is 12.5 feet.

27. Gable rafters 10 feet long, with a pitch of 20°, project one foot beyond the walls of a garage. Find the height h of the ridgepole and the width of the garage.



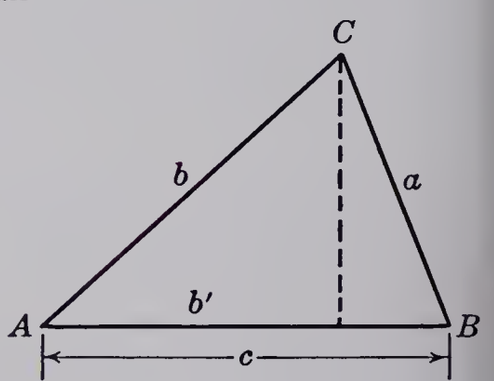
28. *Given* the rt. $\triangle ABC$, A an acute $\angle$.
Prove that $\sin^2 A + \cos^2 A = 1$.*

29. *Prove:* In any right triangle having an acute angle A, $\tan A = \dfrac{\sin A}{\cos A}$.

30. *Prove:* In any triangle

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

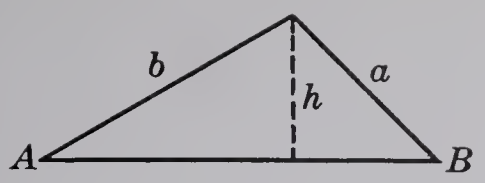
This formula is called the *law of cosines.*

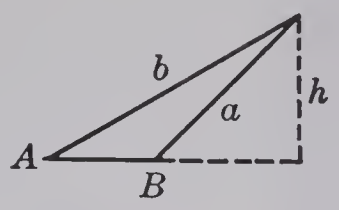


SUGGESTIONS. Use § 444 and § 445. When $\angle A$ is obtuse, $\cos A$ is negative.

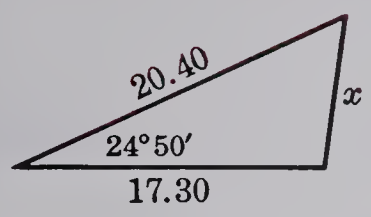
31. *Prove:* In any triangle $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. (Use following diagrams.)

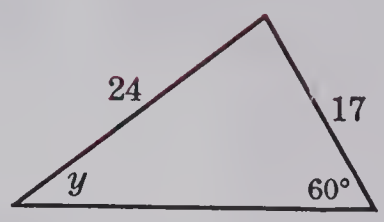
*(Sin A)^2 is written $\sin^2 A$ to avoid use of parentheses.





The formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ is called the *law of sines.*





32. Use the law of cosines to find x.

33. Use the law of sines to find y.

TEST 38 — True-False Statements

[10 Minutes]

Copy the numbers of these statements on your paper. Then if a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number. Do not guess.

1. In a right triangle the sine of an acute angle is the ratio of the opposite side to the adjacent side.

2. $\cos 60° = .5$. **3.** $\tan 30° = \sqrt{3}$. **4.** $\sin A = \frac{1}{\cos A}$.

5. As an angle increases from 0° to 90° the value of its sine increases from 0 to 1.

6. If A and B are the two acute angles of a right triangle, $\sin A = \cos B$.

7. In any right triangle having an acute angle A, $\sin^2 A - \cos^2 A = 1$.

8. The area of a parallelogram is equal to the product of two adjacent sides multiplied by the sine of the included angle.

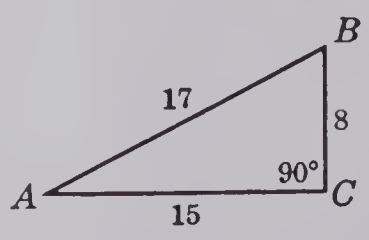
TEST 39 — Applications

[12 Minutes]

1. In $\triangle ABC$ at the right, give the values of $\sin A$ and $\cos A$ to four decimal places.

2. Give the value of $\tan B$ to three decimal places.

3. In a circle of 4-inch radius, find the length of a side of a regular inscribed pentagon.



CHAPTER 16

Inequalities

In our study of geometry thus far
we have learned several methods
of proving line segments equal,
angles equal, and arcs equal.
In this chapter
you will study relations
between line segments that are unequal,
between unequal angles,
and between unequal arcs.
The study of inequalities
is very interesting and instructive.

517. Orders of Inequalities

The symbol for "is unequal to" is $\neq$; for "is greater than" is $>$; and for "is less than" is $<$. The inequalities $a > x$ and $b > y$ are said to be in the *same order*, since the same symbol is used in each inequality. The inequalities $a > x$ and $b < y$ are said to be in *reverse order*, since the symbol in one inequality is the reverse of the symbol in the other.

EXERCISES Oral Exercises

A

1. Read the following: $5 < x$; $x > 5$; $AB + BC > AC$.
2. State which of the following pairs of inequalities are in reverse order:

a. $\angle x > \angle y$ and $\angle m > \angle n$.
b. $7 < 9$ and $8 > 4$.
c. $6 < 8$ and $7 < 9$.
d. $c > d$ and $e < f$.

518. Assumptions on Inequalities

In addition to the assumptions on inequalities listed elsewhere in the book we shall need the following:

Assumption 53. If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.

Thus if $a > b$ and $b > c$, then $a > c$.

Assumption 54.* If unequals are increased by, diminished by, multiplied by, or divided by positive equals, the results are unequal in the same order.

Example 1. If $6 < 8$
and $\frac{4}{10} = \frac{4}{12}$
then $\frac{4}{10} < \frac{4}{12}$

Example 2. If $6 < 8$
and $\frac{4}{2} = \frac{4}{4}$
then $\frac{4}{2} < \frac{4}{4}$

Example 3. If $6 < 8$
and $\frac{4}{24} = \frac{4}{32}$
then $\frac{4}{24} < \frac{4}{32}$

Example 4. If $6 < 8$
and $\frac{4}{1\frac{1}{2}} = \frac{4}{2}$
then $\frac{4}{1\frac{1}{2}} < \frac{4}{2}$

*Also, if unequals are increased by or diminished by negative equals, the results are unequal in the same order.

Assumption 55. If unequals are subtracted from equals, the remainders are unequal in the reverse order.

Example 1. If $12 = 12$
and $7 > 4$
then $5 < 8$

Example 2. If $2x + 6 = 2x + 6$
and $x + 2 < x + 3$
then $x + 4 > x + 3$

Assumption 56. If unequals are added to unequals in the same order, the sums are unequal in the same order.

Example 1. If $5 < 8$
and $3 < 4$
then $8 < 12$

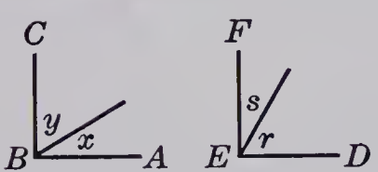
Example 2. If $3x + 5 > x + 6$
and $x + 7 > 5$
then $4x + 12 > x + 11$

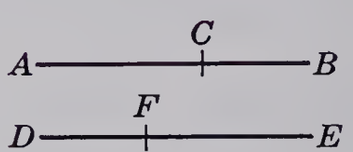
Assumption 57. Like powers and like positive roots of positive unequals are unequal in the same order.

Example. If $x^2 > 9$, then $x > 3$.

EXERCISES — Oral Exercises

A





1. If $\triangle ABC$ and DEF are rt $\triangle$, and if $\angle x < \angle r$, what is true of $\triangle y$ and s? Why?

2. If $AB = DE$ and $CB < FE$, what is true of AC and DF? Why?

3. If $AD + DC > AC$, why is $AD > AC - DC$?

4. If $\angle BAC > \angle y$ and $\angle y = \angle x$, why is $\angle BAC > \angle x$?

5. If $x^2 - r^2 = y^2 - s^2$ and $r^2 > s^2$, why is $x^2 > y^2$?

6. Are complements of unequal angles unequal in the same order, or in reverse order? Prove your statement.

7. If $c < d$, what do you know of m when $cm < dm$?

8. In $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$ and $\angle B < \angle B'$. Show that $\angle C > \angle C'$.

9. Complete: If $a < b$ and if both a and b are positive, then a^2 __?__ b^2.

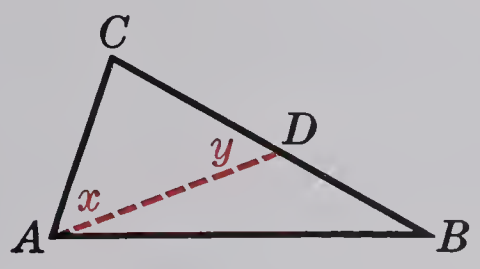
Theorem 89

519. Each side of a triangle is less than the sum of the other two sides.

(The proof is left to the student. Use Asmt. 14.)

Theorem 90

520. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.



Given $\triangle ABC$ with $BC > AC$.
To prove that $\angle A > \angle B$.

Planning the Proof: 1. We can prove $\measuredangle$ unequal by Asmts. 9, 10, 53, 54, 55, 56, 57, and § 93.
2. We shall use Asmts. 9, 53, and § 93.

Proof

STATEMENTS	REASONS
1. On CB construct $CD = AC$.	1. Why possible?
2. Draw AD.	2. Why possible?
3. $\angle x = \angle y$.	3. Why?
4. $\angle BAC > \angle x$.	4. Asmt. 9.
5. $\therefore \angle BAC > \angle y$.	5. Asmt. 6.
6. $\angle y > \angle B$.	6. § 93.
7. $\therefore \angle BAC > \angle B$.	7. Asmt. 53.

EXERCISES

A

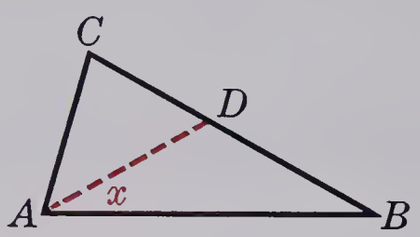
1. In $\triangle ABC$, $AB = 6$, $BC = 5$, and $AC = 7$. Which angle of the triangle is the smallest? the largest?

2. What does the contrapositive of Th. 12 tell us?

3. What is the inverse of Th. 90?

Theorem 91

521. If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle.



Given $\triangle ABC$ with $\angle A > \angle B$.
To prove that $BC > AC$.

Planning the Proof: 1. We can prove line segments unequal by Asmts. 9, 10, 14, 21, 53–57, and § 519.
2. We shall use § 519.

Proof

STATEMENTS	REASONS
1. Draw AD, making $\angle x = \angle B$.	1. Why possible?
2. $\therefore AD = DB$.	2. Why?
3. $AD + DC > AC$.	3. § 519.
4. $\therefore BD + DC > AC$.	4. Why?
5. $\therefore BC > AC$.	5. Why?

522. Corollary I. The perpendicular from a point to a line is the shortest line segment from the point to the line.

NOTE. The proof of this corollary proves Asmt. 21. The proof of this corollary is not based on any theorems that depend on Asmt. 21.

523. Corollary II. The hypotenuse of a right triangle is greater than either leg.

EXERCISES

A

1. In $\triangle MNP$, $\angle M = 80°$ and $\angle P = 60°$. Which side is the shortest? the longest?

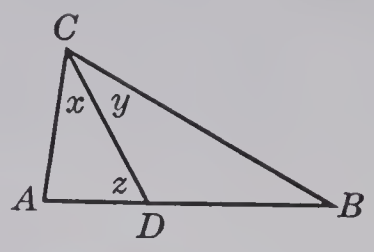
2. In $\triangle ABC$, $\angle A = 59°$ and $\angle C = 60°$. Which side is the longest? the shortest?

3. What does the contrapositive of Theorem 3 tell us?

B

4. The difference between two sides of a triangle is less than the third side.

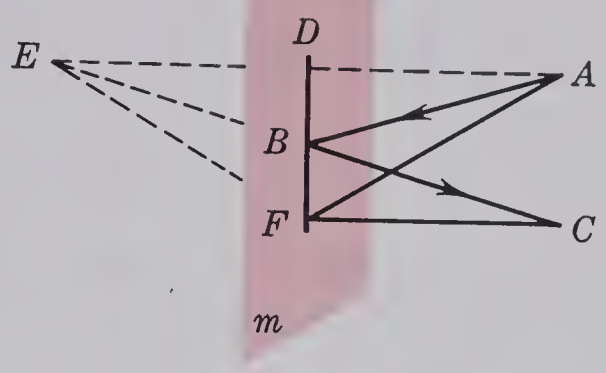
5. *Given* $\triangle ABC$ with CD bisecting $\angle ACB$.
Prove that $AC > AD$.



SUGGESTION. $\angle z > \angle y$. Why?

6. In a 30°-60° right triangle which is the shortest side?

7. ABC is the path of a ray of light reflected by a plane mirror m from A to C. Prove that the path ABC is shorter than the path AFC, F being any other point in the line BD of the mirror.



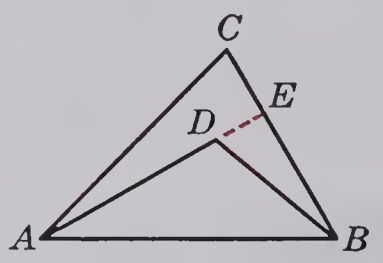
SUGGESTION. See Ex. 1, p. 168.

$$EBC < EF + FC.$$

8. *Given* $\triangle ABC$ with D a point within the triangle.

a. *Prove* that $AC + CB > AD + DB$.

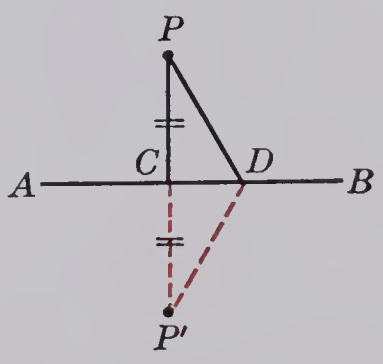
b. *Prove* that $\angle ADB > \angle C$.



C

9. The perimeter of a quadrilateral is greater than the sum of its diagonals.

10. Prove Cor. I, § 522, using the figure shown at the right.



11. The sum of the line segments drawn from any point within a triangle to the vertices is greater than one half the perimeter of the triangle.

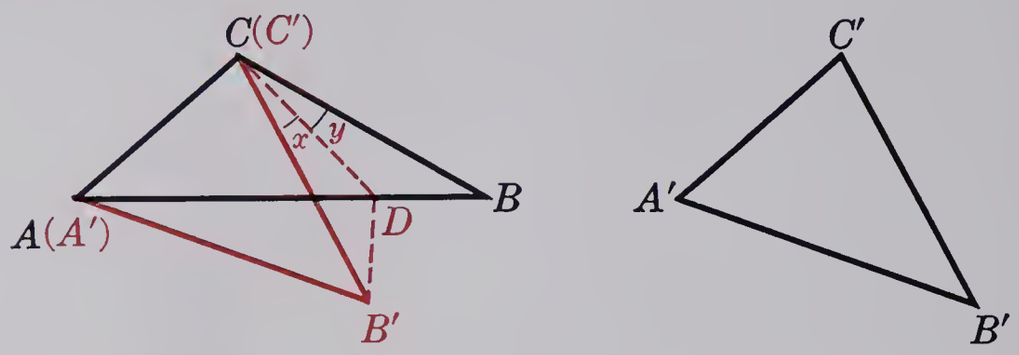
12. If two angles of a triangle are unequal, the bisector of the third angle is oblique to the opposite side.

13. The median of a triangle is less than half the sum of the two adjacent sides.

SUGGESTIONS. Let CM be the median of $\triangle ABC$. Extend CM to C' so that $MC' = MC$. Draw AC' and BC'. Compare CC' with $CB + BC'$. Prove that $AC = BC'$.

Theorem 92

524. If two triangles have two sides of one equal respectively to two sides of the other and the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.



Given $\triangle ABC$ and $A'B'C'$ with $AC = A'C'$, $BC = B'C'$, and $\angle ACB > \angle A'C'B'$.

To prove that $AB > A'B'$.

Planning the Proof: 1. We can prove line segments unequal by §§ 519, 521, 522, 523, and assumptions.
2. We shall use § 519, Asmt. 6.

Proof

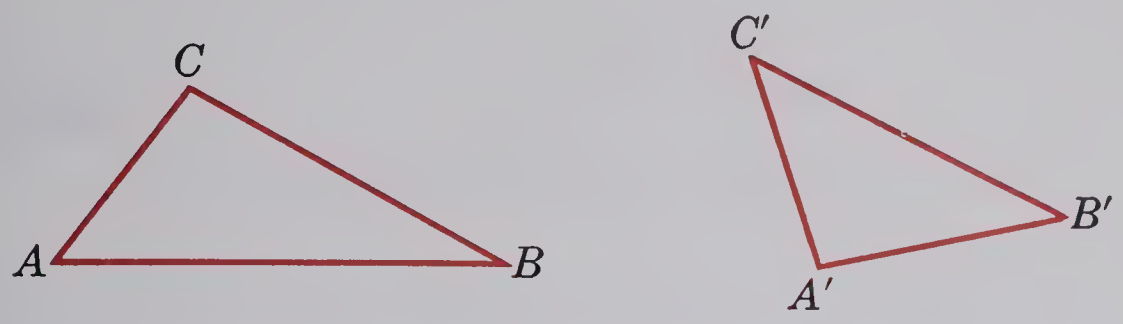
STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $A'C'$ coincides with its equal, AC, and $C'B'$ falls within $\angle ACB$.	1. Why possible?
2. Draw CD bisecting $\angle B'CB$. Draw $B'D$.	2. Why possible?
3. In $\triangle B'CD$ and BCD, $CB' = CB$,	3. Why?
4. $\angle x = \angle y$, and	4. Why?
5. $CD = CD$.	5. Why?
6. $\therefore \triangle B'CD \cong \triangle BCD$.	6. Why?
7. $\therefore DB' = DB$.	7. Why?
8. $AD + DB' > AB'$.	8. Why?
9. $\therefore AD + DB > A'B'$.	9. Why?
10. $\therefore AB > A'B'$.	10. Why?

EXERCISES

In $\triangle ABC$ median CM makes $\angle AMC > \angle BMC$. Prove $AC > BC$.

Theorem 93

525. If two triangles have two sides of one equal respectively to two sides of the other and the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.



Given $\triangle ABC$ and $A'B'C'$ with $AC = A'C'$, $BC = B'C'$, and $AB > A'B'$.
To prove that $\angle C > \angle C'$.

Planning the Proof: 1. We can prove $\measuredangle$ unequal by §§ 93, 520, and assumptions.
2. We shall use Asmt. 10.

Proof

STATEMENTS	REASONS
1. $\angle C < \angle C'$, $\angle C = \angle C'$, or $\angle C > \angle C'$.	1. Asmt. 10.
2. If $\angle C < \angle C'$, then $AB < A'B'$.	2. § 524.
3. But $AB > A'B'$.	3. Given.
4. $\therefore \angle C$ is not $< \angle C'$.	4. Why?
5. If $\angle C = \angle C'$, $\triangle ABC \cong \triangle A'B'C'$.	5. Why?
6. Then $AB = A'B'$.	6. Why?
7. But $AB > A'B'$.	7. Why?
8. $\therefore \angle C \neq \angle C'$.	8. Why?
9. $\therefore \angle C > \angle C'$.	9. Asmt. 10.

EXERCISES

A

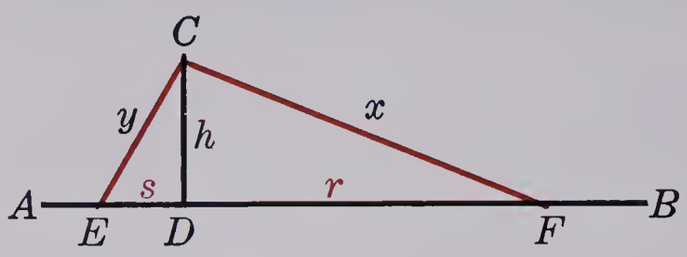
1. Compare the length of AB of $\triangle ABC$ with that of DF of $\triangle DEF$, if $AC = 10$, $BC = 12$, $DE = 10$, $EF = 12$, $\angle C = 70°$, and $\angle E = 65°$.

2. The diagonals of a rhombus that is not a square are unequal.

3. Given P a point on AB the base of isosceles $\triangle ABC$ and $AP < PB$. Prove that $\angle ACP < \angle PCB$.

Theorem 94

526. If two oblique line segments are drawn to a line from a point in a perpendicular to that line, the one having the greater projection upon the line is the greater.



Given $CD \perp AB$, DE and DF the respective projections of CE and CF on AB, and $DF > DE$.

To prove that $CF > CE$.

Planning the Proof: 1. We can prove line segments unequal by §§ 519, 521, 522, 523, 524, and assumptions.
2. We shall use Asmts. 54, 57.

Proof

STATEMENTS	REASONS
1. $h^2 + r^2 = x^2$ and $h^2 + s^2 = y^2$.	1. Why?
2. Then $h^2 = x^2 - r^2$ and $h^2 = y^2 - s^2$.	2. Why?
3. $\therefore x^2 - r^2 = y^2 - s^2$.	3. Why?
4. But $r > s$, and	4. Why?
5. $r^2 > s^2$.	5. Asmt. 57.
6. From (3) and (5), $x^2 > y^2$.	6. Asmt. 54.
7. $\therefore x > y$.	7. Asmt. 57.

EXERCISES

A

1. In $\triangle DEF$, H is the foot of the perpendicular from F to DE. What is the projection of DF on DE? Compare DF and EF if $DH = 8$ and $HE = 10$.

2. Prove that the sum of the altitudes of a triangle is less than its perimeter.

3. Prove that either diagonal of a quadrilateral is less than half of its perimeter.

Theorem 95

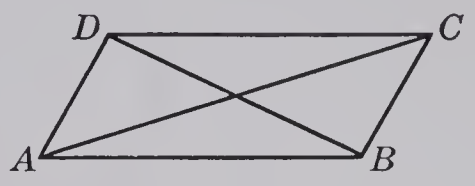
527. If two unequal oblique line segments are drawn to a line from a point in a perpendicular to that line, the greater line segment has the greater projection upon the line.

(The demonstration is left to the student.)

EXERCISES

C

1. If the median of a triangle is oblique to the base, the adjacent sides are unequal.

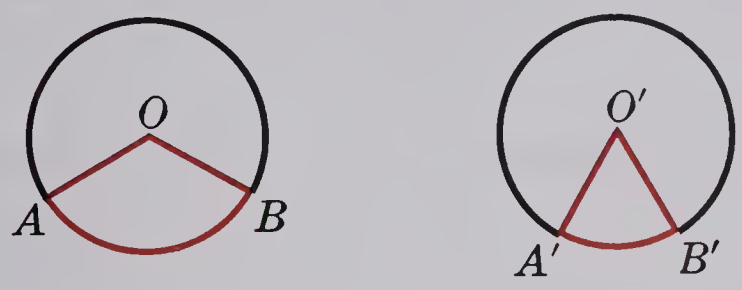


2. The diagonals of an oblique parallelogram are unequal.

3. If a triangle is not isosceles, the median to any side is greater than the altitude to that side.

Theorem 96

528. In a circle or in equal circles the greater of two unequal central angles has the greater arc.



Given the $= ⓢ$ O and O' with $\angle O > \angle O'$.

To prove that $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.*

Planning the Proof: 1. We can prove arcs unequal by assumptions.
2. We shall use § 315 and Asmt. 6.

Proof

STATEMENTS	REASONS
1. $\odot O = \odot O'$.	1. Why?
2. $\angle O \doteq \overset{\frown}{AB}$, $\angle O' \doteq \overset{\frown}{A'B'}$.	2. § 315.
3. $\angle O > \angle O'$.	3. Why?
4. $\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$.	4. Asmt. 6.

*The arcs referred to on this and on the following pages are minor arcs.

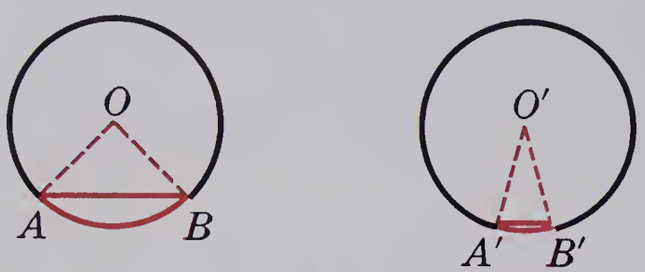
Theorem 97

529. In a circle or in equal circles the greater of two unequal arcs has the greater central angle.

(The demonstration is left to the student.)

Theorem 98

530. In a circle or in equal circles the greater of two unequal chords has the greater arc.



Given the $= \odot O$ and O' with chord $AB >$ chord $A'B'$.

To prove that $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

Planning the Proof: 1. We can prove arcs unequal by § 528, assumptions.
2. We shall use § 528.

Proof

STATEMENTS	REASONS
1. Draw radii OA, OB, $O'A'$, and $O'B'$.	1. Why possible?
In $\triangle AOB$ and $A'O'B'$,	
2. $AO = A'O'$ and $BO = B'O'$.	2. Why?
3. $AB > A'B'$.	3. Why?
4. $\therefore \angle O > \angle O'$.	4. § 525.
5. $\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$.	5. § 528.

Theorem 99

531. In a circle or in equal circles the greater of two unequal arcs has the greater chord.

(The demonstration is left to the student.)

EXERCISES

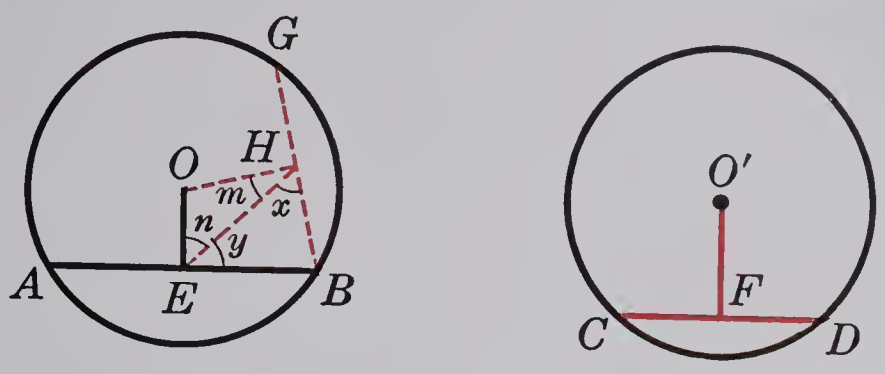
1. In the same circle, do arcs have the same ratio as their central angles?

2. Prove that in a circle the chord of an arc of 180° is twice as long as the chord of an arc of 60°.

3. In a given circle, if $\overset{\frown}{AB} = 2\,\overset{\frown}{CD}$, is chord AB twice, more than twice, or less than twice chord CD? Prove that your answer is correct.

Theorem 100

532. In a circle or in equal circles the greater of two unequal chords is nearer the center.



Given the $= \odot O$ and O' with chord $AB >$ chord CD, $OE \perp AB$, and $O'F \perp CD$.

To prove that $OE < O'F$.

Planning the Proof: 1. We can prove line segments unequal by §§ 519, 521–524, 526, 527, 531, and assumptions.
2. We shall use § 521.

Proof

STATEMENTS	REASONS
1. Draw $BG = CD$.	1. Why possible?
2. Draw $OH \perp BG$.	2. Why possible?
3. Draw EH.	3. Why possible?
4. $AB > CD$.	4. Why?
5. $\therefore AB > BG$.	5. Why?
6. $EB = \frac{1}{2}\,AB$ and $BH = \frac{1}{2}\,BG$.	6. § 290.
7. $\therefore EB > BH$.	7. Why?
8. $\angle OHB = \angle OEB$.	8. Why?
9. From (7), $\angle x > \angle y$.	9. § 520.
10. $\therefore \angle m < \angle n$.	10. Asmt. 55.
11. $\therefore OE < OH$.	11. § 521.
12. $GB = CD$.	12. Why?
13. Then $OH = O'F$.	13. § 293.
14. $\therefore OE < O'F$.	14. Why?

Theorem 101

533. In a circle or in equal circles if two chords are unequally distant from the center, the chord nearer the center is the greater.

(The demonstration is left to the student. Prove by reversing the order of steps in Th. 100.)

534. Corollary. A diameter of a circle is greater than any other chord.

EXERCISES

B

1. If $\triangle ABC$ is inscribed in a circle and if $\angle B > \angle A$, prove that $\overset{\frown}{AC} > \overset{\frown}{BC}$.

2. A $\triangle ABC$ is inscribed in a circle and $\overset{\frown}{AC} = 70°$ and $\overset{\frown}{BC} = 150°$. Name the angles of the triangle in order of size.

3. If a square and an equilateral triangle are inscribed in the same circle, prove that the apothem of the square is greater than the apothem of the triangle.

4. If two unequal chords intersect on a circle, the greater chord makes the smaller angle with the diameter through the point of intersection.

C

5. The shortest chord through a point inside a circle is perpendicular to the radius through this point.

6. If AB is a diameter of a circle and CD is an intersecting diameter of a smaller concentric circle, prove that $ACBD$ is an oblique parallelogram.

TEST 40 — True-False Statements

[6 Minutes]

Copy the numbers of these statements on your paper. If a statement is *always* true, write T after its number. If a statement is *not always* true, write F after its number. Do not guess.

1. If unequals are subtracted from equals, the remainders are unequal in the same order.

2. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.

3. In a circle or in equal circles the greater of two unequal central angles has the greater arc.

4. If two oblique line segments are drawn to a line from a point in a perpendicular to that line, the one having the smaller projection upon the line is the greater.

5. In a circle or in equal circles the smaller of two unequal chords is nearer the center.

6. The bisector of an angle of a triangle is less than the median drawn from the vertex of the angle.

7. A median of a triangle is less than half the sum of the two adjacent sides.

8. Any side of a triangle is greater than half the perimeter.

TEST 41 — Completing Statements

[9 Minutes]

On your paper write one word, and only one, for each blank to make the following statements true:

1. Like powers and like roots of unequals are _ _?_ _.

2. If one angle of a triangle is greater than a second angle, the side opposite the first angle is _ _?_ _ _ _?_ _ the side opposite the second angle.

3. A diameter of a circle is greater than _ _?_ _ _ _?_ _ _ _?_ _.

4. In a circle, the greater of two unequal _ _?_ _ has the greater arc.

5. If a central angle is increased, its arc is _ _?_ _.

6. The greater of two arcs of a circle has the _ _?_ _ chord.

7. The difference between two sides of a triangle is _ _?_ _ _ _?_ _ the third side.

8. The _ _?_ _ of a right triangle is greater than either leg.

9. The shortest chord through a point within a circle is _ _?_ _ to the radius through that point.

10. The area of the largest triangle inscribed in a semicircle with a radius of 20 inches is _ _?_ _ square inches.

CHAPTER

17

Coordinate Geometry

IN THIS CHAPTER
you will study geometry
by the methods of algebra.

Most of the geometry given in Chapters 1–14 and in Chapter 16 was developed by the early Greeks during the height of their culture.

Algebra was developed much later and it was not until the Middle Ages that the symbols and notations that we now use in mathematics were commonly employed.

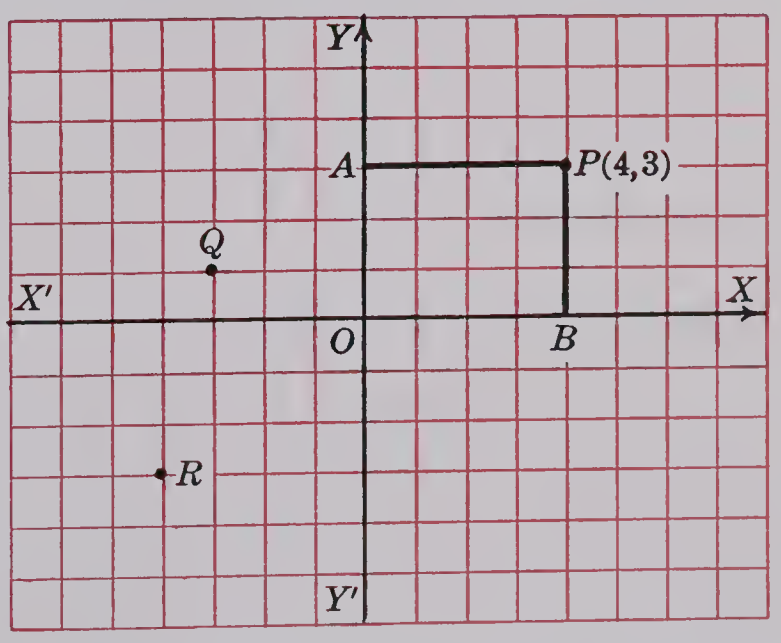
In 1637 René Descartes, a great French mathematician, showed in his *Géométrie* how the methods of algebra could be applied to geometry. This kind of geometry is known as *coordinate geometry*, or *analytic geometry*. Before you complete the study of this chapter, you will understand why it is called coordinate geometry. In coordinate geometry numbers are associated with points and equations with geometric figures.

Coordinate geometry is not entirely new to you if you have plotted points on coordinate paper (graph paper) and have solved pairs of equations graphically.

535. Cartesian, or Rectangular, Coordinates

As you know, we can locate any point in a city by giving its distances from two streets which are perpendicular to each other. In like manner we can locate a point on graph paper by giving its distances from two perpendicular lines. Let us see how this is done.

On graph paper we draw two lines perpendicular to each other. We label these lines $X'X$ and YY' and label their intersection O, as shown in the figure. The line $X'X$ is called the *x-axis* (plural *axes*) and the line YY' is called the *y-axis*. The intersection O of the axes is called the *origin*. Values to the right of the origin on the horizontal axis, or x-axis, are positive, and to the left of the origin are negative; values above the origin on the vertical axis, or y-axis, are positive and below the origin are negative. PA, the distance of a point from the y-axis, is called the *abscissa* of P; PB, the distance of P from the x-axis, is called the *ordinate* of P. Together the values of the abscissa and the ordinate are called the *coordinates* of P.



These coordinates are called *cartesian* coordinates after the Latin name of Descartes. Do you know why they are also called rectangular coordinates? The point P is called the point (4, 3) because its abscissa is 4 and its ordinate is 3. Note that we write the x-distance first and the y-distance second. Similarly, the point Q is called the point $(-3, 1)$ and the point R is called the point $(-4, -3)$.

EXERCISES

A

1. Plot each group of points, using separate co-ordinate axes:
   a. $(3, 4)$, $(-2, 3)$, $(5, -1)$, $(-2, -4)$.
   b. $(-3, 5)$, $(0, 2)$, $(-3, 0)$.
   c. $(5, 0)$, $(0, -3)$, $(7, -4)$, $(-1, -1)$.

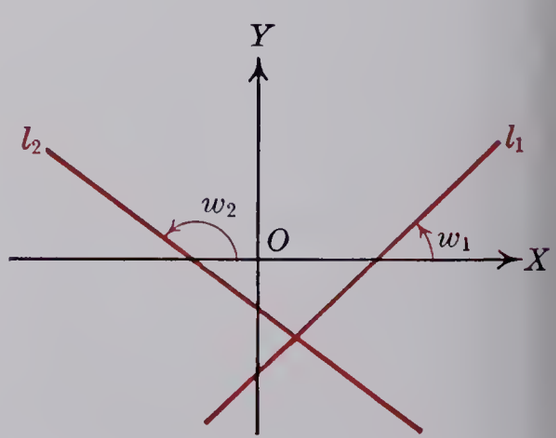
2. On graph paper, using the same set of axes, draw the lines which pass through the following pairs of points:
   a. $(2, 1)$ and $(0, 5)$.
   b. $(-3, -4)$ and $(4, 5)$.
   c. $(-2, 5)$ and $(4, 0)$.
   d. $(0, 0)$ and $(2, -3)$.

3. Draw the triangles whose vertices are as follows:
   a. $(3, -1)$, $(-2, 6)$, $(-8, -4)$.
   b. $(0, 0)$, $(-7, 7)$, $(4, 6)$.
   c. $(4, -5)$, $(-4, -5)$, $(0, 5)$.

4. Draw the quadrilaterals whose vertices are as follows:
   a. $(-3, 0)$, $(2, 4)$, $(6, 0)$, $(3, -5)$.
   b. $(0, 0)$, $(5, 3)$, $(0, 8)$, $(5, 11)$.
   c. $(1, -6)$, $(8, 2)$, $(1, 6)$, $(-8, 2)$.

536. Inclination of a Line

The *inclination* of a line is the smaller of the two angles that the line makes with the positive end of the x-axis measured counterclockwise from the x-axis to the line. In the figure, w_1 is the inclination of the line l_1 and w_2 is the inclination of the line l_2.



537. The Slope of a Line

If you understand what is meant by the slope of a roof and the slope of a hill, you should have no difficulty in learning the meaning of the slope of a line. We can describe the inclination of a line (its steepness) by giving its slope.

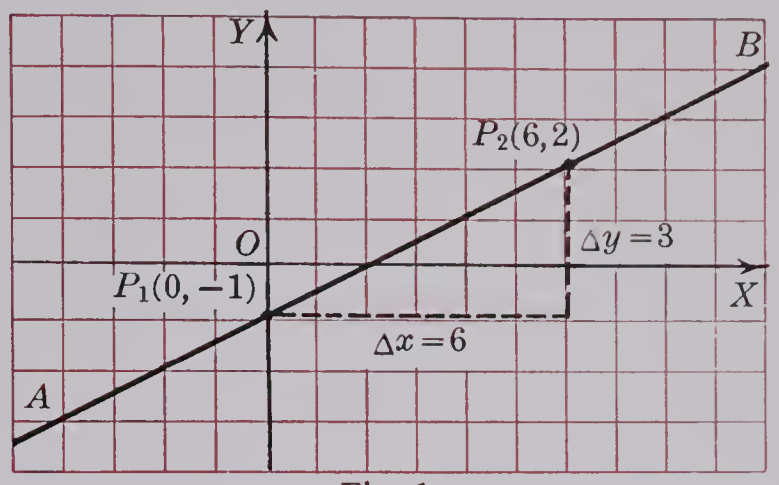


Fig. 1

In Fig. 1, let a point P move along the line AB from the point P_1 $(0, -1)$ to the point P_2 $(6, 2)$. Note that we let the point P move so that its x value increases. When P moves from P_1 to P_2, its x value *increases* from 0 to $+6$ and its y value *increases* from -1 to $+2$. In this case y increases as x increases. The x change in value $= (6) - (0) = +6$ and the y change in value $= (2) - (-1) = +3$.

The slope of a line is the ratio of the change in y to the change in x.*

We designate the change in y by Δy (delta y) and the change in x by Δx, as shown in the figure. Then the slope of $AB = \dfrac{\Delta y}{\Delta x} = \dfrac{3}{6}$, or $\dfrac{1}{2}$.

In Fig. 2, as a point P moves along the line CD from the point P_1 $(-3, 1)$ to the point P_2 $(3, -2)$, its x value *increases* from -3 to $+3$ and its y value *decreases* from $+1$ to -2. Note that y decreases when x increases. The x change in value $= 3 - (-3) = +6$ and the y change in value $= (-2) - (+1) = -3$. Since $\Delta y = -3$ and $\Delta x = 6$, the slope of $CD = \dfrac{\Delta y}{\Delta x} = -\dfrac{3}{6}$, or $-\dfrac{1}{2}$. The slope of a line is negative when its inclination is greater than 90°.

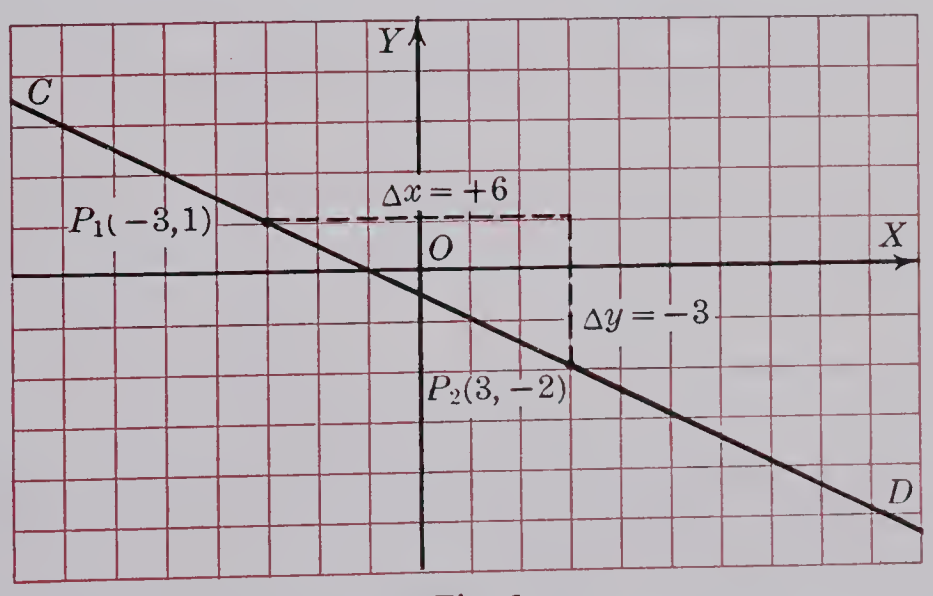


Fig. 2

*This definition applies when y is a function of x.

Any line which is not parallel to either axis has either a positive or a negative slope. If a line, such as l_1 in Fig. 3, is parallel to the x-axis, it

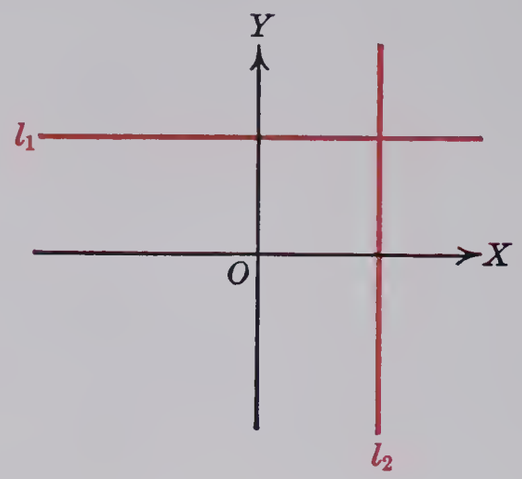


Fig. 3

has a zero slope. If any line, such as l_2 in Fig. 3, is parallel to the y-axis, it has no slope. Do you know why each of these statements is true? The converses of these statements are also true.

We shall now derive a formula which can be used to find the slope of a line without making the graph.

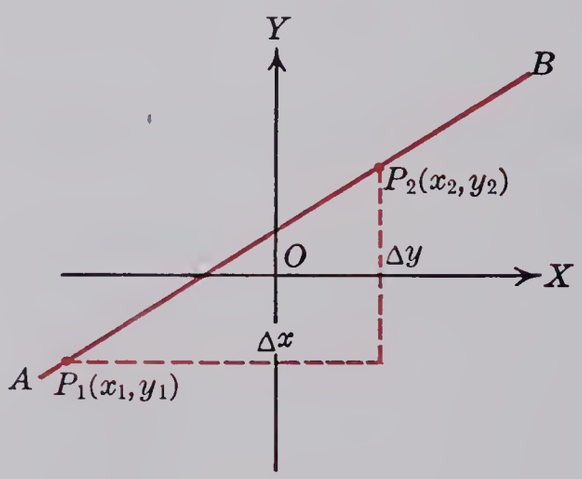


Fig. 4

Let a point P (Fig. 4) move along the line AB from P_1 to P_2. Then $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$. Let m denote the slope of AB. Then

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Also $m = \dfrac{y_1 - y_2}{x_1 - x_2}$. Why?

Example. Find the slope m of the line which passes through the points $(-4, 1)$ and $(3, 5)$.

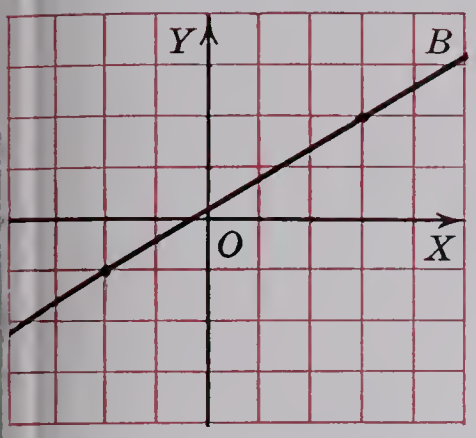
Solution. Let $(-4, 1)$ be (x_1, y_1) and $(3, 5)$ be (x_2, y_2). Then

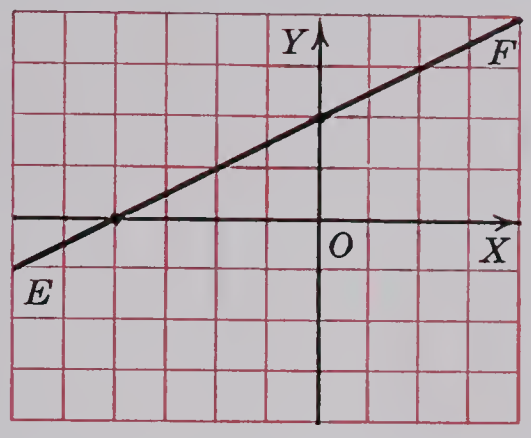
$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1 - 5}{-4 - 3} = \frac{-4}{-7} = \frac{4}{7}.$$

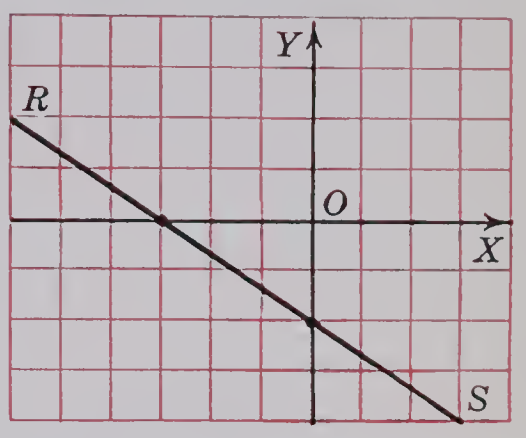
EXERCISES

A

1. In the figures, find the slope of AB; of EF; of RS.







2. Use the formula to find the slopes of the lines that pass through the following pairs of points:

a. $(0, 0)$ and $(4, 4)$.
b. $(0, 5)$ and $(5, 0)$.
c. $(3, -2)$ and $(-2, 3)$.
d. $(-5, -3)$ and $(2, 1)$.
e. $(4, 0)$ and $(-4, 0)$.
f. $(7, 7)$ and $(1, 5)$.

3. On coordinate paper draw the x- and y-axes and plot several points each of which has its abscissa equal to its ordinate. Draw the locus of points satisfying the condition $x = y$.

4. On coordinate paper plot several points each of which has an abscissa 4. Draw the locus of points whose abscissa is 4. What equation does this locus of points satisfy?

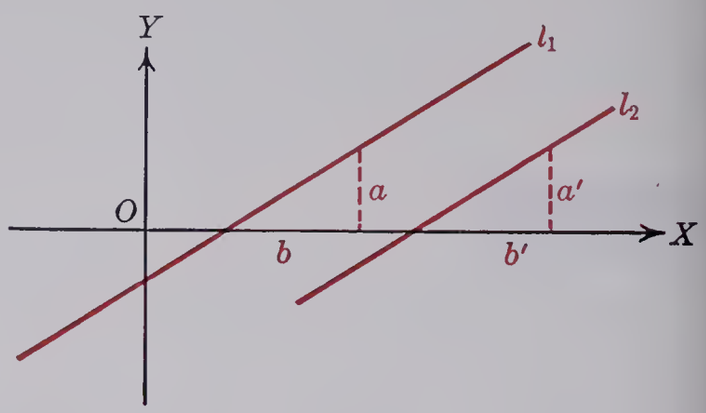
5. On coordinate paper plot several points each of which has an ordinate 5. Draw the locus of points satisfying the condition $y = 5$.

6. On coordinate paper draw the x- and y-axes and plot the points $A(-1, 1)$, B $(6, 3)$, C $(4, 6)$, and D $(-3, 4)$. Draw the quadrilateral $ABCD$. Find the slope of each side of $ABCD$. Which sides have the same slope?

7. What special quadrilateral is formed in exercise 6?

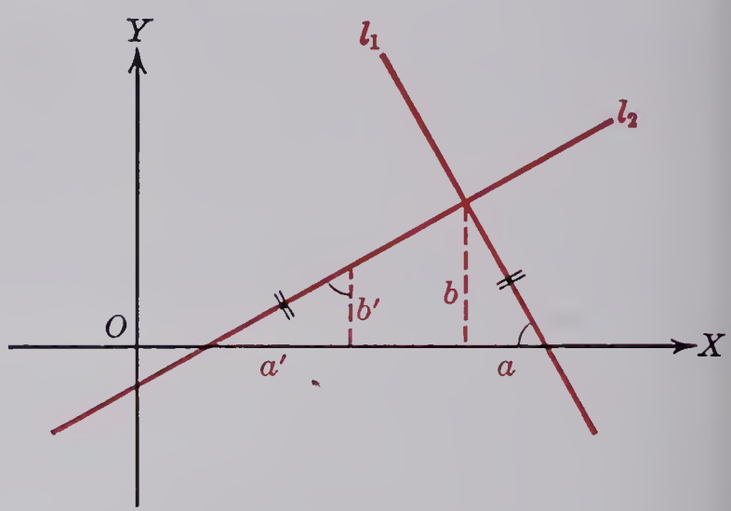
538. Parallel and Perpendicular Lines

If two lines l_1 and l_2 have the same slope, they are parallel to each other; and conversely, if two lines are parallel to each other, they have the same slope. Can you prove these two statements?



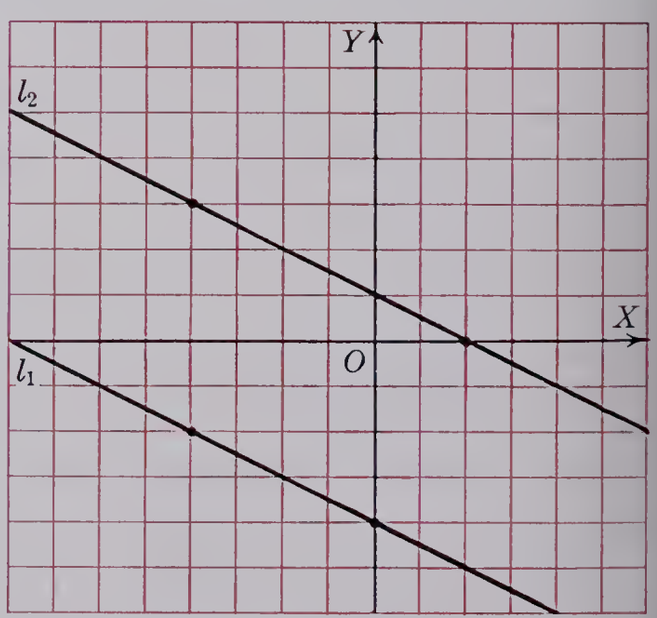
SUGGESTIONS. To prove $l_1 \parallel l_2$, prove the two triangles congruent (§ 95a). Conversely, to prove $m_1 = m_2$, prove the two triangles congruent (§ 95b).

If two lines l_1 and l_2 are perpendicular to each other, the product of their slopes is -1, that is, $m_1 m_2 = -1$; and conversely, if the product of the slopes of two lines is -1, the lines are perpendicular to each other. Can you prove these two statements?



SUGGESTIONS. To prove $m_1 m_2 = -1$, find m_1 and m_2 in terms of a', b', a, and b. To prove $l_1 \perp l_2$, prove the triangles congruent (§ 156a).

Example 1. Show that the line l_1 passing through the points $(-4, -2)$ and $(0, -4)$ is parallel to the line l_2 passing through the points $(-4, 3)$ and $(2, 0)$.



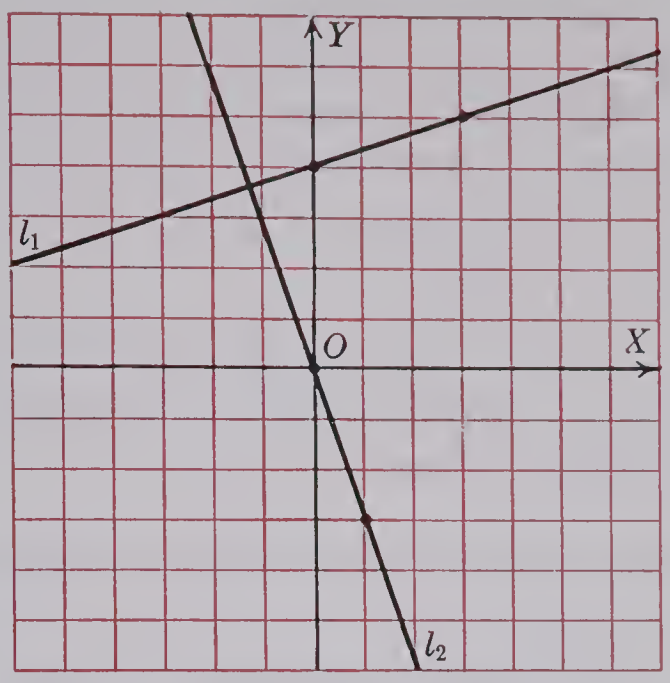
Solution. Let m_1 denote the slope of l_1 and m_2 denote the slope of l_2.

Then $m_1 = \dfrac{-2+4}{-4-0} = -\dfrac{1}{2}$

and $m_2 = \dfrac{3-0}{-4-2} = -\dfrac{1}{2}$,

Since $m_1 = m_2$, $l_1 \parallel l_2$.

Example 2. Show that the line l_1 passing through the points (0, 4) and (3, 5) is perpendicular to the line l_2 passing through the points (1, − 3) and (0, 0).



Solution. Let m_1 denote the slope of l_1 and m_2 the slope of l_2. Then

$$m_1 = \frac{4-5}{0-3} = \frac{1}{3}$$

and $$m_2 = \frac{-3-0}{1-0} = -3.$$

$$m_1 m_2 = (\tfrac{1}{3})(-3) = -1.$$

Then $$l_1 \perp l_2.$$

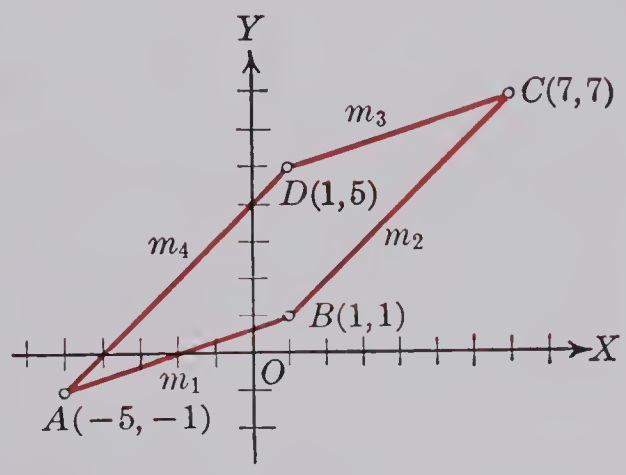
Example 3. Show that (− 5, − 1), (1, 1), (7, 7), and (1, 5) are the vertices of a parallelogram.

Solution.

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

$$m_1 = \frac{-1-1}{-5-1} = \frac{-2}{-6} = \frac{1}{3}.$$

$$m_3 = \frac{5-7}{1-7} = \frac{-2}{-6} = \frac{1}{3}.$$



Since AB and DC have the same slope, they are parallel.

$$m_2 = \frac{7-1}{7-1} = \frac{6}{6} = 1, \quad \text{and} \quad m_4 = \frac{5+1}{1+5} = \frac{6}{6} = 1.$$

Since AD and BC have the same slope, they are parallel.

$\therefore$ $ABCD$ is a $\square$. Why?

EXERCISES

A

1. Prove that (2, 8), (6, 4), (3, − 2), and (− 1, 2) are the vertices of a parallelogram.

2. Prove by means of slopes that the following points lie on a straight line in each case:

a. (0, 0), (2, 3), (4, 6).

b. (0, 0), (2, 1), (− 4, − 2).

c. (3, 1), (3, 4), (3, − 2).

d. (1, 4), (2, 3), (4, 1).

3. Show that the polygon having the vertices (2, 3), (4, 2), (1, − 3), and (− 1, − 2) taken in order is a parallelogram.

4. Prove by means of slopes that (5, 7), (6, 2), and (3, 5) are the vertices of a right triangle. (Use $m_1 m_2 = -1$.)

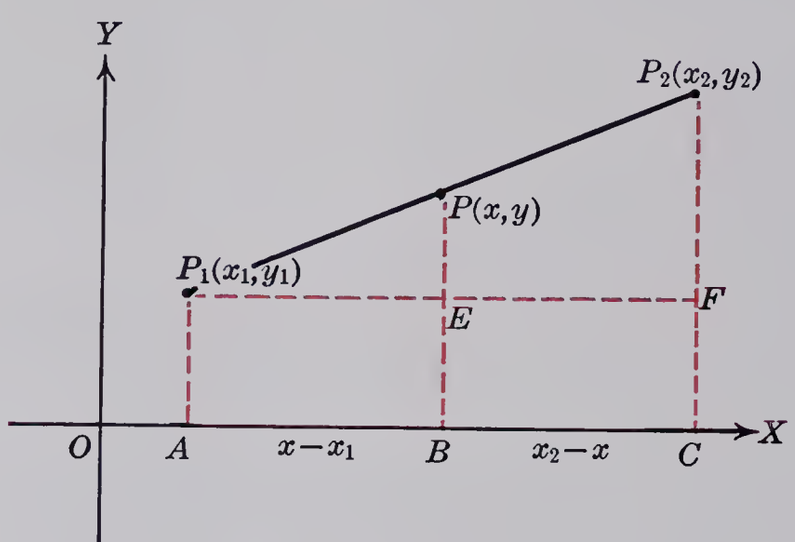
5. Prove that (− 1, 2), (1, 3), (− 5, 10), and (− 3, 11) are the vertices of a rectangle.

6. Show that the polygon with vertices (0, 0), (2, 1), (4, − 3), and (2, − 4) taken in order is a rectangle.

539. The Midpoint of a Line Segment

Theorem 102. The coordinates of the midpoint of a line segment are one-half the sums of the coordinates of the end points, or

$$x = \frac{x_1 + x_2}{2} \text{ and } y = \frac{y_1 + y_2}{2}.$$



Given P the midpoint of line segment P_1P_2.

To prove that $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$.

Proof

STATEMENTS	REASONS
1. Draw P_1A, PB, and $P_2C \perp$ the x-axis.	1. Why possible?
2. $P_1A \parallel PB \parallel P_2C$.	2. Why?
3. $P_1P = PP_2$.	3. Why?
4. $x - x_1 = x_2 - x$.	4. Why?
5. $2x = x_2 + x_1$.	5. Why?
6. $\therefore x = \frac{x_2 + x_1}{2}$.	6. Why?

In like manner by drawing lines through the points P_1, P, and P_2 perpendicular to the y-axis we can prove that $y = \frac{y_1 + y_2}{2}$. (The proof is left to the student.)

EXERCISES

A

1. Find the coordinates of the midpoints of the sides of these triangles:
a. $(-1, 2)$, $(5, -2)$, $(5, 6)$. **b.** $(-2, 3)$, $(-4, 3)$, $(6, 1)$.

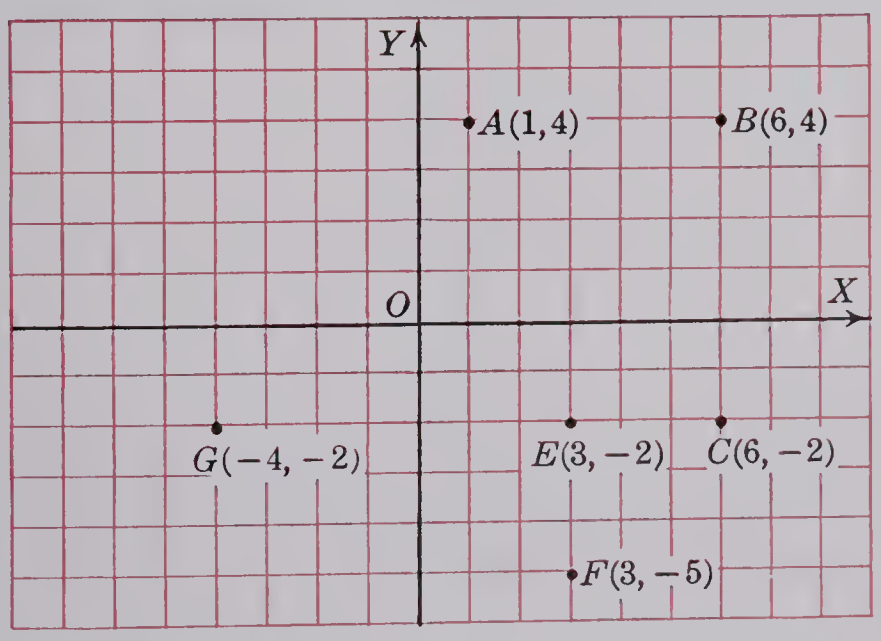
2. The midpoint of a line segment is $(2, 1)$ and one end point is $(8, 10)$. Find the other end point.

3. The points $A(2, 0)$, $B(6, 0)$, and $C(4, 8)$ are the vertices of a triangle. Find the area of the triangle.

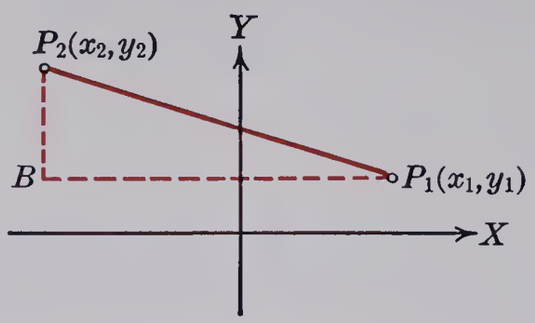
540. The Distance between Two Points

It is readily apparent from the figure that the distance between the two points A and B, having the same ordinate, is the difference of their abscissas, or $6 - 1 = 5$. Also, the distance between the two points B and C, having the same abscissa, is the difference of their ordinates, or $4 - (-2) = 6$. In like manner, we see that the distance $EF = -2 - (-5) = 3$ and that the distance $GE = 3 - (-4) = 7$.

We shall now prove a theorem which can be used to find the distance between two points which may have unequal coordinates.



Theorem 103. The distance between two points is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Given points P_1 and P_2 with the coordinates (x_1, y_1) and (x_2, y_2) respectively. $P_1P_2 = d$.

To prove that $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Proof

STATEMENTS	REASONS
1. Draw $P_1B \parallel$ to the x-axis and $P_2B \parallel$ to the y-axis.	1. Why possible?
2. $\angle P_2BP_1$ is a right angle.	2. Why?
3. $BP_1 = x_2 - x_1$ and $P_2B = y_2 - y_1$.	3. Why?
4. $\overline{P_2P_1}^2 = \overline{BP_1}^2 + \overline{P_2B}^2$.	4. Why?
5. $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.	5. Why?
6. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.	6. Why?

Example 1. Show that $(3, 6)$, $(-6, 4)$, $(1, -2)$ are the vertices of an isosceles triangle.

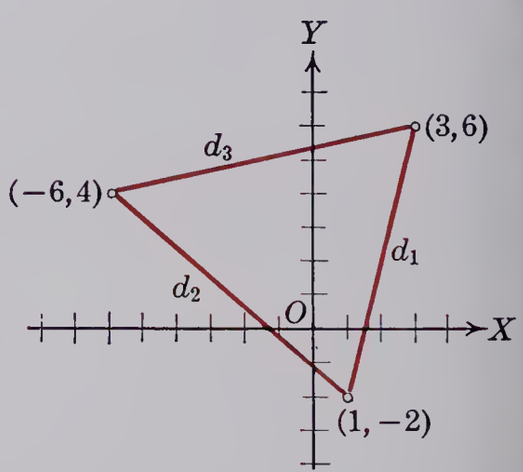
Solution.

$d_1 = \sqrt{(3 - 1)^2 + (6 + 2)^2} = \sqrt{68}$.

$d_2 = \sqrt{(-6 - 1)^2 + (4 + 2)^2} = \sqrt{85}$.

$d_3 = \sqrt{(3 + 6)^2 + (6 - 4)^2} = \sqrt{85}$.

Since $d_2 = d_3$, the triangle is isosceles.



Example 2. Prove that $(-2, -3)$, $(4, 5)$, $(-4, 1)$ are the vertices of a right triangle.

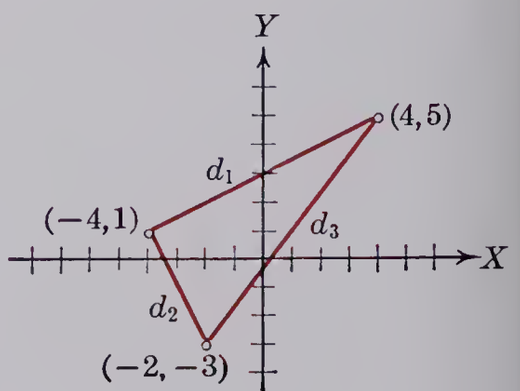
Solution.

$d_1^2 = (4 + 4)^2 + (5 - 1)^2 = 80$.

$d_2^2 = (-4 + 2)^2 + (1 + 3)^2 = 20$.

$d_3^2 = (4 + 2)^2 + (5 + 3)^2 = 100$.

Since $d_1^2 + d_2^2 = d_3^2$, the triangle is a right triangle.



EXERCISES

1. Find the lengths of the sides of a triangle whose vertices are

a. $(1, -2)$, $(5, 6)$, $(-2, 3)$. **b.** $(3, 4)$, $(-2, 2)$, $(5, -3)$.

2. Show that the following triangles are isosceles:

a. $(9, 5)$, $(5, -1)$, $(-1, 3)$. **b.** $(4, 5)$, $(2, -1)$, $(-3, 4)$.

3. Show that the following are right triangles:

a. $(7, 4)$, $(4, 5)$, $(1, -4)$. **b.** $(1, 3)$, $(5, 6)$, $(4, -1)$.
c. $(-6, 2)$, $(5, -1)$, $(4, 4)$. **d.** $(13, -1)$, $(-9, 3)$, $(-3, -9)$.

4. Show (**a**) that $(-5, -3)$, $(1, -11)$, $(7, -6)$, and $(1, 2)$ are the vertices of a parallelogram; (**b**) that the quadrilateral $(3, 2)$, $(0, 5)$, $(-3, 2)$, $(0, -1)$ is a rhombus.

5. What kind of quadrilateral has vertices at $(6, 4)$, $(-1, 2)$, $(3, -2)$, $(2, 8)$?

6. Find the lengths of the diagonals of a quadrilateral whose vertices are $(1, 3)$, $(-2, -1)$, $(5, 0)$, and $(-4, 2)$.

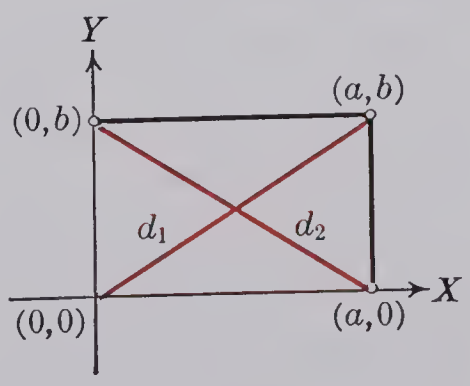
7. Prove that $(0, -2)$, $(4, 2)$, $(0, 6)$, and $(-4, 2)$ are the vertices of a square.

541. Proving Theorems by Coordinate Geometry

Many of the theorems of plane geometry can be proved more easily by coordinate geometry than by plane geometry. Thus far in this chapter the proofs have related to special figures. Now we shall prove theorems which apply to general figures. When studying the three examples which follow, you should observe how the coordinates of the given points are represented.

Example 1. *Prove:* The diagonals of a rectangle are equal.

Solution. Place the given rectangle with one vertex at the origin and two adjacent sides along the x- and y-axes respectively. Label the vertices $(0, 0)$, $(a, 0)$, (a, b), and $(0, b)$.



$$d_1 = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}.$$

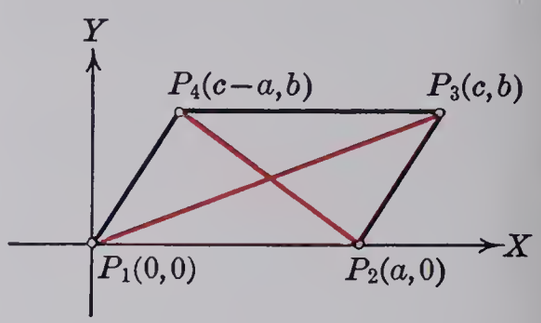
$$d_2 = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}.$$

Therefore the diagonals of the rectangle are equal.

Example 1 could have been proved by placing the rectangle in any position in reference to the axes, but the algebraic computation is simplified when a suitable position is chosen. The properties of a geometric figure depend upon the relation of its parts and not upon the position in which the figure is placed in reference to the axes.

Example 2. The diagonals of a parallelogram bisect each other.

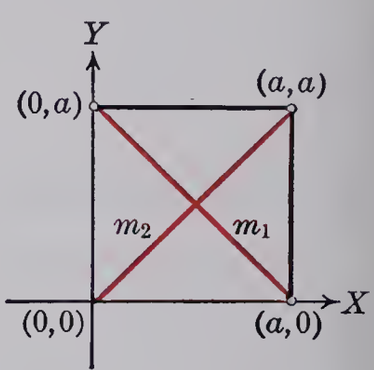
Solution. Represent the $\square$ by $P_1P_2P_3P_4$ and place one vertex at the origin and one side along the x-axis. Represent P_2 by $(a, 0)$ and P_3 by (c, b). Then the fourth vertex will be $(c - a, b)$.



By means of the midpoint formula, we find that the midpoint of P_1P_3 is $\left(\frac{c}{2}, \frac{b}{2}\right)$ and the midpoint of P_2P_4 is $\left(\frac{a+c-a}{2}, \frac{b}{2}\right)$, or $\left(\frac{c}{2}, \frac{b}{2}\right)$. Since the diagonals have the same midpoint, they bisect each other.

Example 3. Prove that the diagonals of a square are perpendicular.

Solution. Place the square as shown.



Use $$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

$$m_1 = \frac{a-0}{0-a} = \frac{a}{-a} = -1.$$

$$m_2 = \frac{a-0}{a-0} = \frac{a}{a} = 1.$$

$$m_1 m_2 = (-1)(1) = -1.$$

Therefore the diagonals are $\perp$ (§ 538).

EXERCISES

A

Prove the following theorems by coordinate geometry:

1. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices.

2. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal in length to one-half the third side. [Place the triangle with one vertex at the origin and the base along the x-axis. Label the vertices $(0, 0)$, $(a, 0)$, and (b, c).]

3. The line segments joining the midpoints of the sides of a square taken in order form a square.

4. The line segments joining the midpoints of the sides of a rectangle taken in order form a rhombus.

5. The diagonals of an isosceles trapezoid are equal. [Place the figure so that the vertices are $(0, 0)$, $(a, 0)$, (c, b), and $(a - c, b)$.]

6. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

7. The median of a trapezoid is one-half the sum of its bases.

542. The Graph of a Linear Equation

The graph of a linear equation (a first-degree equation) is a straight line; and conversely, the equation of a straight line is linear.

If you have forgotten how to graph a linear equation, study the following example:

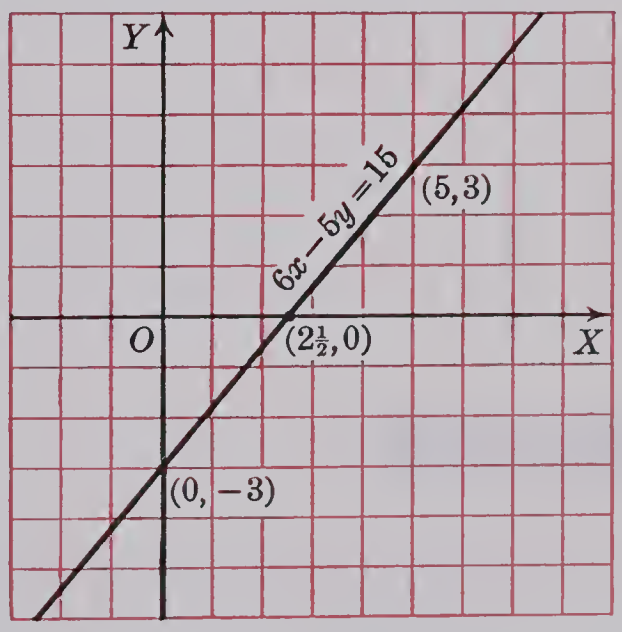
Example. Graph the equation $6x - 5y = 15$.

Solution. Since the equation is linear, its graph is a straight line. Two points are needed to determine the line and a third point is needed to check it.

To graph the equation we draw the x- and y-axes on coordinate paper. Next we prepare a table of values of x and y, using either of the following forms:

x	0	$2\frac{1}{2}$	5
y	-3	0	3

x	y
0	-3
$2\frac{1}{2}$	0
5	3



In preparing the table, we let $x = 0$, substitute 0 for x in the equation, and solve for y, getting $y = -3$.

We let $y = 0$, substitute 0 for y in the equation, and solve for x, getting $x = 2\frac{1}{2}$.

Next we plot the points $(0, -3)$ and $(2\frac{1}{2}, 0)$ and draw the straight line through the two points. To check the correctness of the graph, we let $x = 5$ in the equation and solve for y, getting $y = 3$. We plot the point $(5, 3)$ and find that it is on the line. The *x-intercept* of the graph is $2\frac{1}{2}$ and the *y-intercept* is -3. Why is it usually convenient in graphing to find the intercepts?

EXERCISES

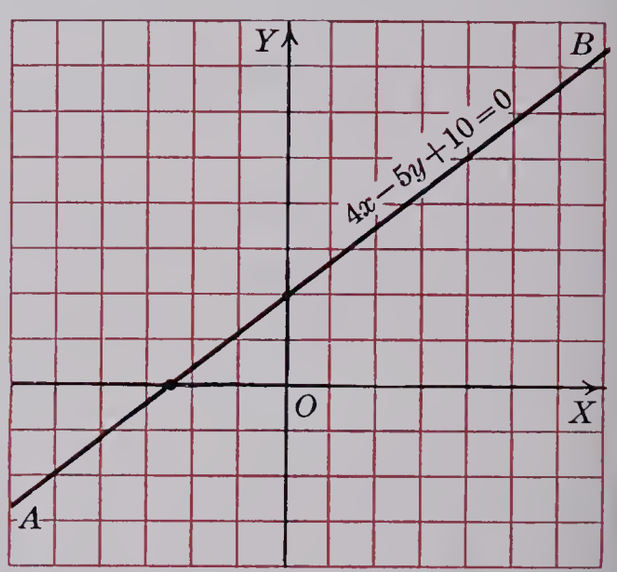
A

Graph each of the following equations:

1. $x = y + 6$.
2. $y = 3x$.
3. $2x + 3y = 12$.
4. $3x - 2y = 6$.
5. $x + 2y = 0$.
6. $x - y = 0$.

543. Linear Loci

The line AB shown here is the graph of the equation $4x - 5y + 10 = 0$. The coordinates of every point of AB satisfy the equation $4x - 5y + 10 = 0$; and conversely, any pair of values satisfying the equation $4x - 5y + 10 = 0$ are the coordinates of a point on AB. Therefore, AB is the locus of points satisfying the condition $4x - 5y + 10 = 0$. In general, we can say that the locus of points satisfying the linear (first-degree) equation $ax + by + c = 0$ is a straight line.



EXERCISES

A

Draw the locus of points satisfying each of the following conditions:

1. $3x - y = 9$.
2. $4x + 5y = 20$.
3. $2x - 7y = 14$.

4. Plot the points $(0, 2)$, $(5, 2)$, and $(-7, 2)$. Do the coordinates of these points satisfy the equation $0y + x = 2$? the equation $y + 0x = 2$?

5. What geometric figure is the locus of points satisfying each of the following conditions?

a. $x = 3$. **c.** $y = 2$. **e.** $x - c = 0$.
b. $x = -4$. **d.** $y = -5$. **f.** $y + k = 0$.

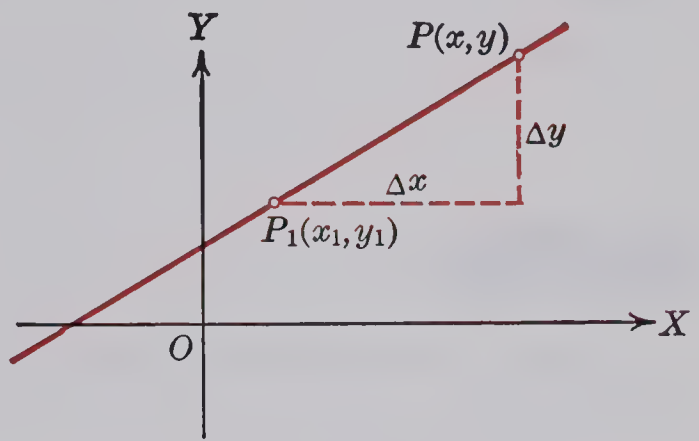
6. Graph the locus of a point the sum of whose coordinates is 8.

7. Graph the locus of a point whose abscissa exceeds its ordinate by 4.

544. The Point-Slope Form, $y - y_1 = m(x - x_1)$

You know how to graph a linear equation and you know how to find the slope of a line passing through two given points. We shall learn how to form the equation of a line when we are given the slope of the line and the coordinates of one point on the line.

Given m the slope of the line and one point on the line, as $P_1(x_1, y_1)$. Represent any other point on the line as $P(x, y)$.



Then by § 537 $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y - y_1}{x - x_1}$,

or $$y - y_1 = m(x - x_1).$$

Example. Find the equation of the line through (3, 4) whose slope is $\frac{1}{2}$.

Solution.
$$y - y_1 = m(x - x_1).$$
$$y - 4 = \tfrac{1}{2}(x - 3).$$
$$2y - 8 = x - 3, \text{ or } x - 2y = -5.$$

EXERCISES

A

1. Find the equations of the lines through the following points and having the given slopes:

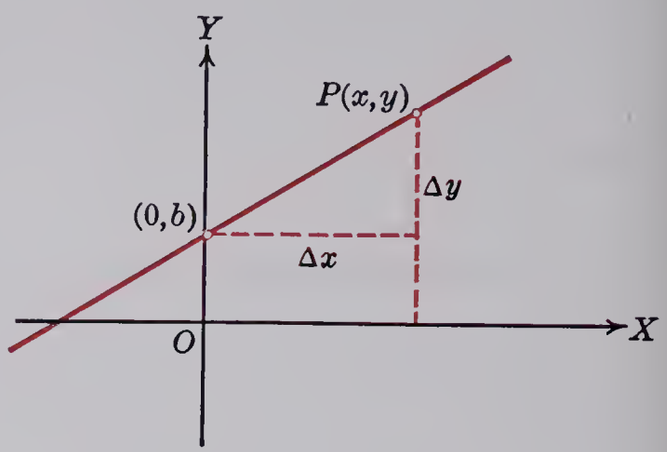
a. (2, 3), $m = \frac{5}{2}$. **c.** (2, − 4), $m = -\frac{1}{2}$. **e.** (− 5, 2), $m = \frac{1}{2}$.
b. (3, 0), $m = 1$. **d.** (2, 7), $m = 0$. **f.** (0, 6), $m = -2$.

2. Use the point-slope formula to graph the equation $\dfrac{y - 3}{x - 2} = \dfrac{1}{2}$.

545. The Slope-Intercept Form, $y = mx + b$

When the point (x_1, y_1) in the formula $y - y_1 = m(x - x_1)$ becomes $(0, b)$, the point-slope formula becomes the slope-intercept formula.

Given the slope m and the y-intercept equal to b. Represent any other point on the line by $P(x, y)$.



Then $$m = \frac{\Delta y}{\Delta x} = \frac{y - b}{x - 0}.$$

$$mx = y - b.$$

$$y = mx + b.$$

Example. Find the equation of a line whose slope is 2 and y-intercept 5.

Solution. $y = mx + b.$

$y = 2x + 5$, or $2x - y = -5.$

EXERCISES

A

1. Find the equations of the lines having the following slopes and y-intercepts:

a. $m = 2, b = 5.$ c. $m = 5, b = -2.$

b. $m = \frac{1}{3}, b = -3.$ d. $m = \frac{1}{2}, b = 3.$

2. Change each of the following equations to the form $y = mx + b$ and give the slope and y-intercept of its graph:

a. $2x + 3y = 12.$ b. $5x - y = 8.$ c. $x = y + 6.$

3. Tell which of the following equations have graphs that are parallel and which have graphs that are perpendicular:

a. $x - 3y = 9.$ c. $x + 2y = 6.$ e. $3x - 9y = 18.$

b. $2x + y = 10.$ d. $y - 3x = 12.$ f. $4x - 2y = 4.$

SUGGESTION. Change each equation to the form $y = mx + b$.

546. The Two-Point Form, $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$.

The two-point form equation enables us to form the equation of a line when we know the coordinates of two points on the line.

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Take any other point $P(x, y)$ on the line.

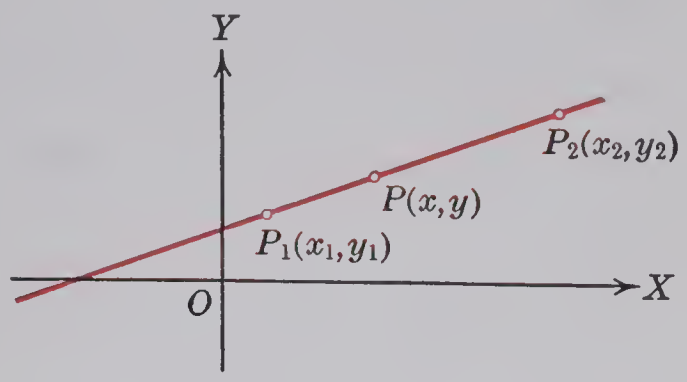
For PP_1,

$$m = \frac{y - y_1}{x - x_1}.$$

For P_2P_1,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$



Example. Find the equation of the line determined by (2, 3) and (− 3, 6).

Solution.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$\frac{y - 3}{x - 2} = \frac{6 - 3}{-3 - 2}.$$

$$\frac{y - 3}{x - 2} = \frac{3}{-5}.$$

$$-5y + 15 = 3x - 6, \text{ or } 3x + 5y = 21.$$

EXERCISES

A

1. Find the equations of the lines through the points
   - **a.** (4, 3), (− 2, 7).
   - **b.** (3, − 1), (5, 3).
   - **c.** (6, − 4), (2, − 6).
   - **d.** (0, 3), (− 7, 5).
   - **e.** (4, 2), (− 5, − 3).
   - **f.** (6, 0), (− 1, − 2).

2. The vertices of a triangle are (1, − 4), (5, 4), and (− 5, 6). Find the equations of its sides.

3. The vertices of a triangle are (0, 0), (6, 0), and (8, 6). Find the equations of the medians.

4. The vertices of a triangle are (0, 8), (4, 0), and (− 4, 0). Find the equations of the medians.

C

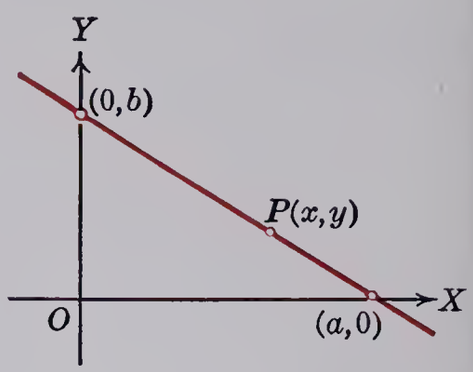
5. The vertices of a triangle are (1, 5), (7, − 1), and (7, 7).
   - **a.** Find the altitude of the triangle from the vertex (7, 7).
   - **b.** Find the area of the triangle.

547. The Intercept Form, $\frac{x}{a}+\frac{y}{b}=1$

The intercept form of equation enables us to form the equation of a line when we know both the x- and y-intercepts of the line.

Given the x-intercept a and the y-intercept b.

The line intersects the x-axis in the point $(a, 0)$ and the y-axis in the point $(0, b)$.



From § 546, $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.

Substituting $(a, 0)$ for (x_1, y_1) and $(0, b)$ for (x_2, y_2) in the equation, we get

$$\frac{y-0}{x-a}=\frac{b-0}{0-a}.$$

Simplifying, $\quad bx+ay=ab.$

Dividing by ab, $\quad \frac{x}{a}+\frac{y}{b}=1.$

Example. Find the equation of a line whose x-intercept is 4 and y-intercept -5.

Solution.

$$\frac{x}{a}+\frac{y}{b}=1.$$

$$\frac{x}{4}+\frac{y}{-5}=1.$$

$$\frac{x}{4}-\frac{y}{5}=1.$$

$$5x-4y=20.$$

EXERCISES

A

1. Write the equations of the lines having the following x- and y-intercepts:

a. $a=3,\ b=4.$ c. $a=-5,\ b=2.$

b. $a=5,\ b=-1.$ d. $a=3,\ b=-7.$

2. Write the equations of the following lines in intercept form:

a. $3x+5y=15.$ b. $3x+4y=12.$ c. $x-5y=5.$

548. Equation of a Circle, $(x-h)^2+(y-k)^2=r^2$

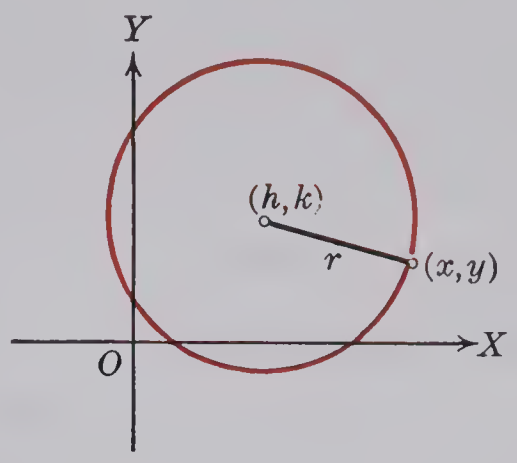
We know that the locus of points having a given distance from a given point is a circle having its radius equal to the given distance and its center at the given point (Asmt. 47).

Given a circle with radius r and center (h, k). Take (x, y) any point on the circle.

Then, by the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2}=r,$$

or $$(x-h)^2+(y-k)^2=r^2.$$



If h and k are both equal to 0, the center is at the origin and the equation becomes

$$x^2+y^2=r^2.$$

Example 1. Find the equation of a circle whose center is $(3, -4)$ and whose radius is equal to 6.

Solution.

$$(x-h)^2+(y-k)^2=r^2.$$
$$(x-3)^2+(y+4)^2=6^2.$$
$$x^2-6x+9+y^2+8y+16=36.$$
$$x^2+y^2-6x+8y=11.$$

Example 2. Find the center and radius of the circle

$$x^2+y^2-6x+10y-2=0.$$

Solution. To complete the squares in the left-hand member, we first write the equation in the form

$$(x^2-6x+\quad)+(y^2+10y+\quad)=\quad 2.$$

Since the coefficient of x^2 and of y^2 is unity, we can complete the squares by adding the squares of half the coefficient of x and of y respectively. Making these additions, we have

$$(x^2-6x+9)+(y^2+10y+25)=9+25+2,$$

or $$(x-3)^2+(y+5)^2=36.$$

Then the center is $(3, -5)$ and the radius is 6.

EXERCISES

A

1. Write the equations of the following circles and draw their graphs, using compasses:

a. Center (0, 0), radius 5.
b. Center (0, 2), radius 3.
c. Center (− 3, 0), radius 4.
d. Center (− 1, 3), radius 2.
e. Center (1, − 4), radius 8.
f. Center (2, 5), radius 6.

2. Write the equation of the circle with center $(-4, 3)$ and passing through the origin.

3. Write the equation of a circle for which $(5, 2)$ and $(-1, 8)$ are the ends of a diameter.

4. Find the center and radius of each of the following circles and draw their graphs with compasses:

a. $x^2 + y^2 = 100$.

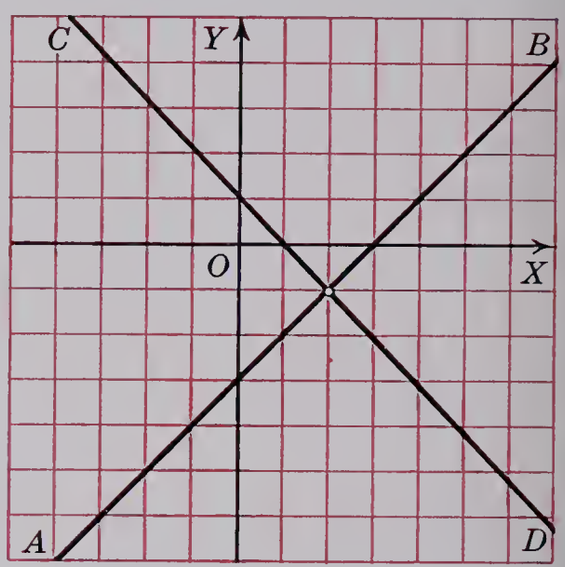
b. $x^2 + y^2 + 10x + 8y + 25 = 0$.

c. $x^2 + y^2 + 10x - 24y = 0$.

d. $x^2 + y^2 - 6x - 7 = 0$.

549. Compound Loci

In the figure at the right, every point on the line AB satisfies the condition $x - y = 3$; and conversely, every point that satisfies the condition $x - y = 3$ is on the line AB. Therefore we can say that AB is the locus of points satisfying the condition $x - y = 3$. In like manner, CD is the locus of points satisfying the condition $x + y = 1$. AB and CD intersect at point $(2, -1)$, the locus of points satisfying the two conditions $x - y = 3$ and $x + y = 1$.

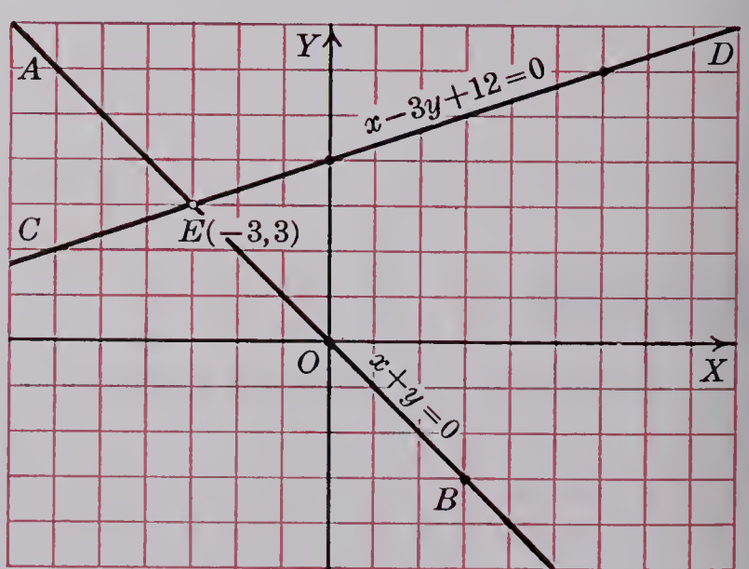


Example. Find the locus of points satisfying the conditions

(1) $x + y = 0$

and (2) $x - 3y + 12 = 0$.

Solution. We graph both equations using the same set of axes. The line AB is the locus of points satisfying the condition $x + y = 0$ and the line CD is the locus of points satisfying the condition $x - 3y + 12 = 0$. The intersection $E(-3, 3)$ of AB and CD is the locus of points satisfying both conditions.



EXERCISES

A

Find the locus of points satisfying the following sets of conditions:

1. $x + y = 12$
   $x - y = 6$

2. $x + y = 14$
   $x - y = 6$

3. $4x + y = 14$
   $3x - y = 1$

4. $2x + y = 2$
   $x - 2y = 11$

5. $x + y = -6$
   $3x - y = 2$

6. $x - 3 = 0$
   $x + y = 5$

B

7. $x^2 + y^2 = 36$
   $x - y = 0$

8. $x^2 + y^2 = 64$
   $x + 2y = 8$

9. $(x - 3)^2 + y^2 = 9$
   $x - 3 = 0$

TEST 42 — Applications

[40 Minutes]

1. Find the lengths of the sides of the triangle whose vertices are $(2, 3)$, $(-5, 6)$, and $(-4, -3)$.

2. Find the midpoints of the sides of the triangle whose vertices are $(-3, 6)$, $(5, 4)$, and $(7, -6)$.

3. Find the slope of the line segment joining $(2, 1)$ and $(5, 4)$.

4. Prove that $(4, 0)$, $(7, 8)$, $(0, 10)$, and $(-3, 2)$ are the vertices of a parallelogram.

5. Prove that $(-1, -3)$, $(-3, 3)$, $(6, 6)$, and $(8, 0)$ are the vertices of a rectangle.

6. Find the equation of the line through $(2, 5)$ whose slope is $\frac{2}{3}$.

7. Find the equation of the line whose slope is 2 and y-intercept 5.

8. Write the equation of the line through $(3, 5)$ and $(7, -1)$.

9. Write the equation of the line whose x-intercept is 3 and y-intercept -4.

10. Find the equation of the circle whose center is at the origin and which has a radius equal to 6.

11. Write the equation of the circle which passes through the origin and has its center at $(4, -3)$.

12. Find the equation of the circle whose center is $(6, 2)$ and radius 4.

CHAPTER 18

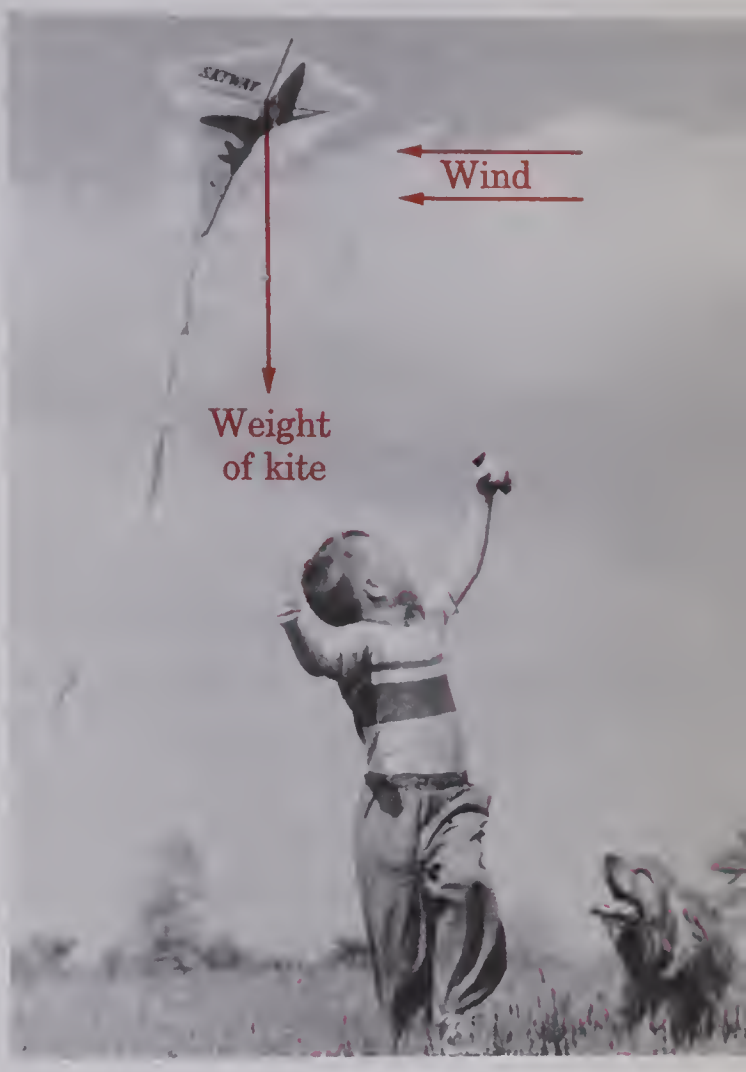
Additional Topics

AERONAUTICS, or air navigation,
is the science or art of sailing in the air
by means of a balloon,
airship, glider, or airplane.
Four of the most needed subjects
in the study of aeronautics
are mathematics, physics, physical
geography, and meteorology.
The second topic in this chapter,
Geometric Relations and Orthographic
Projection, will test your power in judging
relationships among geometric figures.

1. Aeronautics

550. Why a Kite Flies

When a boy flies his kite, he keeps it stationary as the wind flows past it. When a pilot flies a plane, the engine of the plane pulls it through the air. Since the kite and plane are kept aloft in equilibrium by the action of similar forces, we shall first make a short study of flying the simple kite.



John Gajda F.P.G.

You probably know that three forces are acting upon the kite to keep it in the air. One force is the wind which blows against the surface of the kite. Another force is the pull of gravity. This force is called the weight of the kite. The third force is the one applied by the boy through the string.

In Fig. 1 below, AG shows the direction of the force of gravity, AP shows the direction of the pull of the boy through the string, and AB shows the direction of the effective force of the wind WA.

When two nonparallel forces act at a point on a body, their *resultant* (the single force which has the same effect on the motion of the body as the two forces acting together) can be represented by the diagonal of a parallelogram

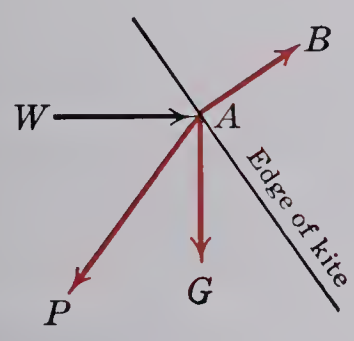


Fig. 1

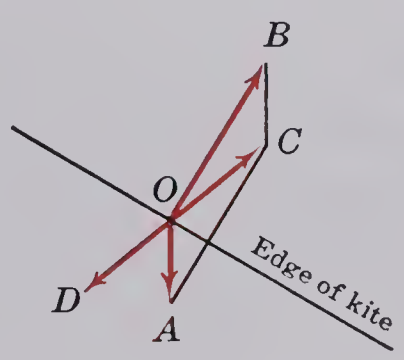


Fig. 2

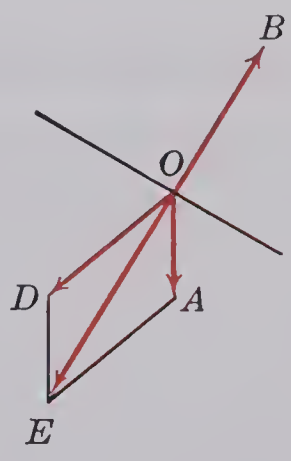


Fig. 3

which has two adjacent sides representing the two nonparallel forces. (See Ex. 3, p. 437.)

In Fig. 2 (page 541), OA represents the magnitude and direction of the force of gravity, or the weight of the kite; OB represents the magnitude and direction of the effective force of the wind (part of the wind's force is directed along the face of the kite); and OC represents the resultant of these two forces both in magnitude and in direction. To keep the kite in equilibrium, the force OD, which is equal to the force OC but opposite in direction, must be applied at O. Then OD represents the pull of the string.

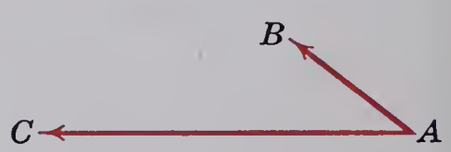
In the same manner one can find the resultant OE of the two forces OA and OD. Then OB (Fig. 3) represents the effective force of the wind.

EXERCISES

1. Two forces act at a point upon a body at an angle of 60°. One of the forces is 40 pounds and the other is 80 pounds. Construct their resultant, letting 1 cm. stand for 10 pounds. Measure their resultant and find what single force will have the same effect on the body as the two given forces.

2. Force AB is 30 pounds and force AC is 25 pounds. What is their resultant if they act in opposite directions?

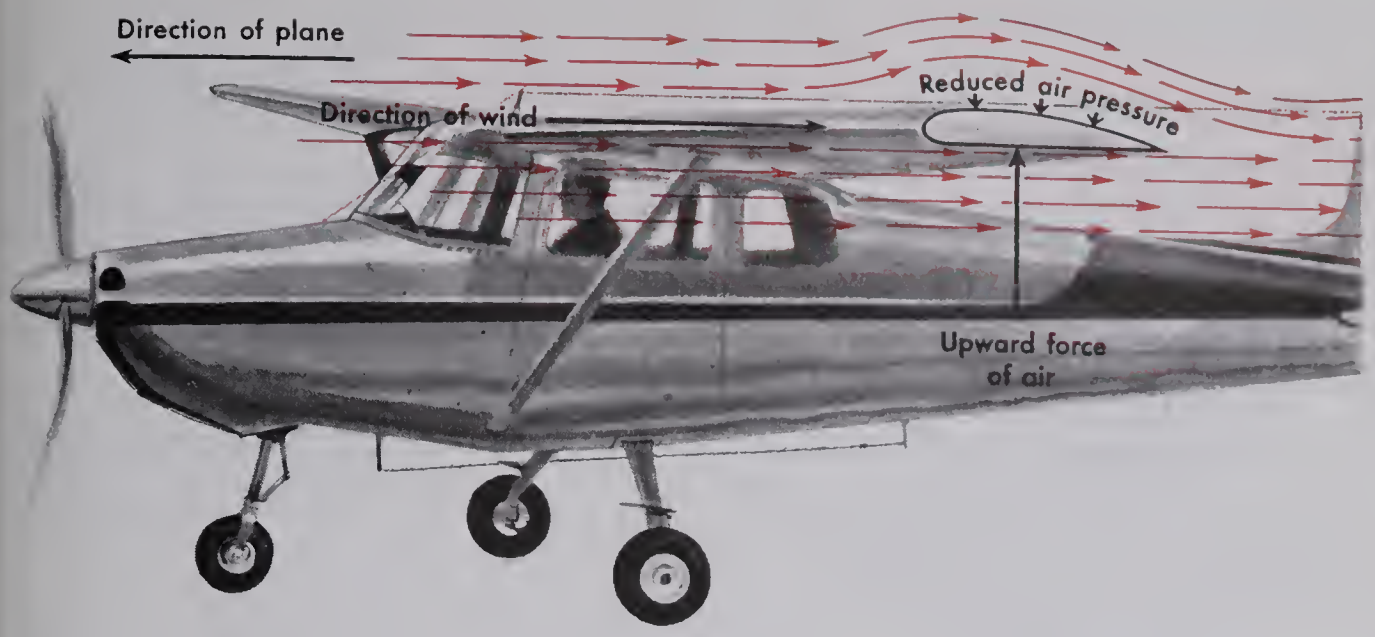
3. Two forces, AB and AC, act on a body at A. If the force AB is 30 pounds, find the resultant of the two forces. What single force AD must be applied at A to maintain equilibrium? Find the number of degrees in $\angle BAD$.



551. Why a Plane Flies

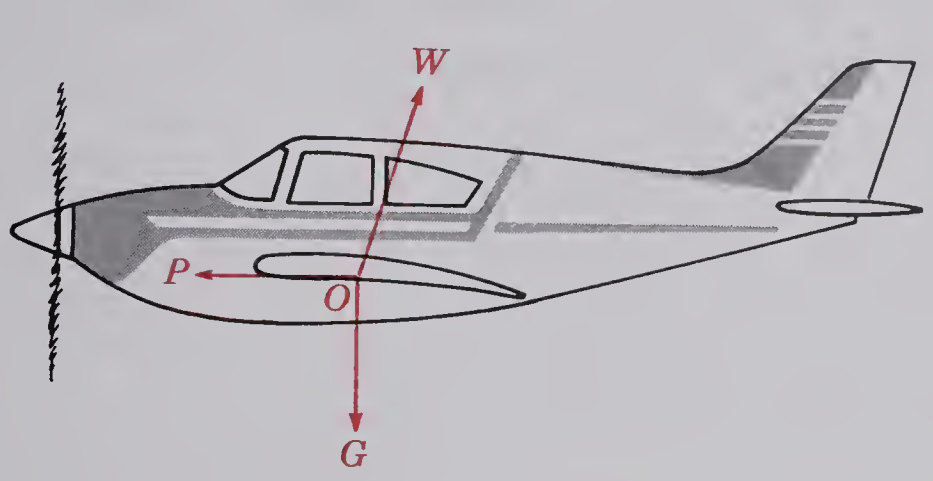
All the forces which keep a kite in the air are present in the flight of a plane, but not all of them act on the plane in the same directions as they do on the kite. For example, in the level flight of a plane the propeller force is almost horizontal, while the string pull on the kite is toward a point on the ground.

One force which helps to keep the modern plane aloft is due to the streamlining of the plane. Let us see why the wings of a plane are streamlined. When an *airfoil* (any part of a plane which is used to keep the plane in the air), having a cross section like the one shown here, is pulled rapidly through the air toward the left, there is a decrease in the atmos-



pheric pressure on the upper side of the airfoil and an increase in the atmospheric pressure on the underside of the airfoil. The difference in pressure between these two pressures is called the *lift* of the airfoil.

When a plane is in a straight and level flight, the air forces acting against the plane may be resolved into one single force, OW, which tends



to lift the plane. The weight of the plane, OG, is directed toward the center of the earth. The engine of the plane through the propeller acts on the plane by the force OP. When the plane is in equilibrium, the forces OW, OP, and OG can be represented by the parallelogram of forces.

EXERCISES

1. The lifting surface of the wings of a plane is 262 square feet. If the atmospheric pressure on the underside of the wings is 12.8 pounds per square inch and the atmospheric pressure above the wings is 12.1 pounds per square inch, what is the total lift?

2. The proportion $\frac{V_1}{V_2} = \sqrt{\frac{W_1}{W_2}}$ shows the relation of the velocities and loads of a plane. In this proportion V_1 and V_2 are the velocities when the respective loads are W_1 and W_2.

a. Show that the load of a plane varies directly as the square of its velocity.

b. If a plane weighing 2400 pounds can fly 100 m.p.h., what velocity must it have if its load is increased 600 pounds?

3. An airplane has an air speed of 100 m.p.h. If the wind is blowing east at 20 m.p.h., how far east can the plane go and return in $7\frac{1}{2}$ hours?

552. How Navigators Express Time

The navigator counts the time of day from midnight. Each day begins at midnight and ends at the following midnight. Instead of writing "4:37 A.M." the navigator writes "0437"; instead of writing "8:12 P.M." he writes "2012"; and instead of writing "11 A.M." he writes "1100."

Frequently it is necessary to add and subtract measures of time.

Example 1. Add: 2 hr. 32 min. 18 sec.
14 hr. 29 min. 45 sec.
The sum is 16 hr. 61 min. 63 sec. = 17 hr. 2 min. 3 sec.

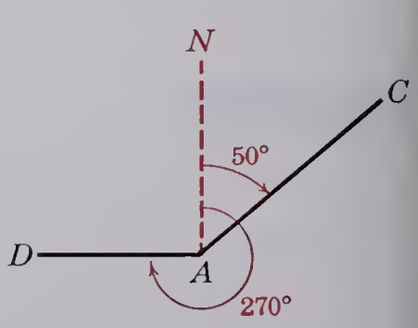
Example 2. If the difference in sun time between two places is 2 hr. 48 min., find the slow time when the fast time is 1823.

Solution. 1823 = 18 hr. 23 min. = 17 hr. 83 min.
2 hr. 48 min.
The difference is 15 hr. 35 min.

The slow time is 1535, or 3:35 P.M.

553. Meaning of Direction

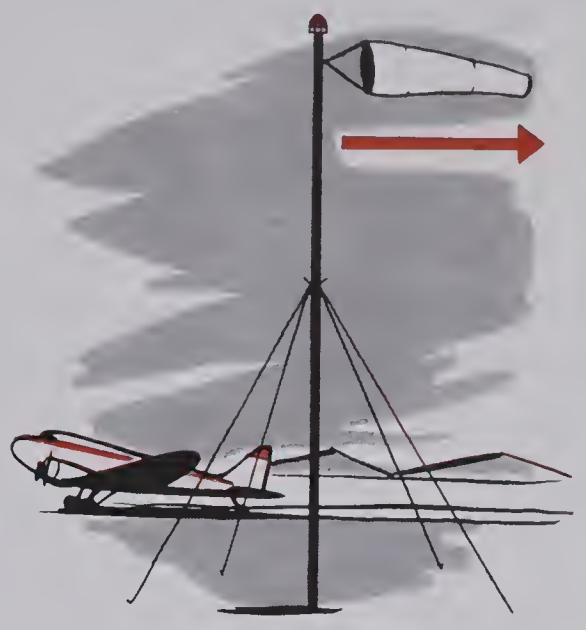
The direction of any ray from a point A is the angle that the ray makes with the ray running north from A. The direction of any point on the ray is the same as that of the ray. The angle is measured clockwise. Thus the direction of C from A is 50° and the direction of D from A is 270°.



554. Direction of the Wind

When indicating the direction of the wind, it is customary to give the direction *from* which the wind is blowing. For example, a west wind means a wind blowing from the west. Remember that a west wind blows from the west toward the east and that a south wind blows from the south toward the north.

The direction of the wind at an airport is usually shown by an *air-sock*, which is attached to a pole and flies in the wind.



555. Bearing of a Point

The *bearing* of one point from another is the direction of the first point from the second. In Fig. 1 the bearing of C from P is 45°, the bearing of D from P is 120°, and the bearing of E from P is 200°.

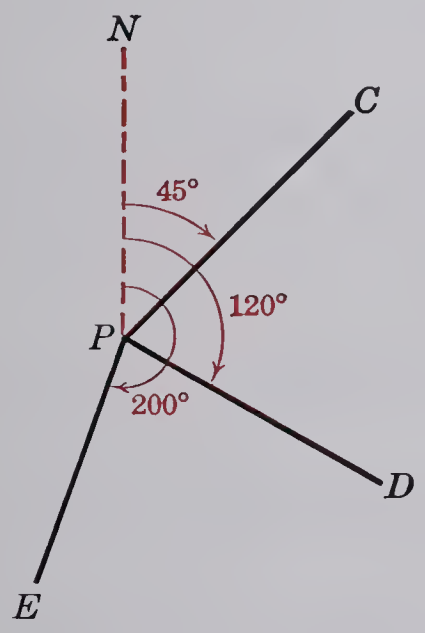


Fig. 1

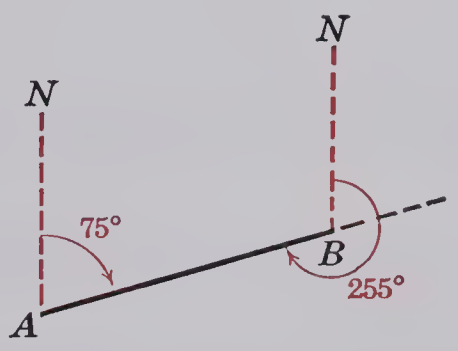


Fig. 2

If the bearing of a point B from point A is known (Fig. 2), the bearing of point A from point B can be determined easily. For example, if the bearing of B from A is 75°, the bearing of A from $B = 75° + 180° = 255°$. Can you prove it? What would be the bearing of B from A if the bearing of A from B were 180°?

EXERCISES

1. Change the following times to the form used by navigators:

a. 4:18 A.M.	**c.** 6:15 P.M.	**e.** 7 A.M.
b. 12 noon.	**d.** 11:32 P.M.	**f.** midnight.

2. When it is 0815 Wednesday, what is the time 17 hr. 4 min. later?

3. What was the time 16 hr. 50 min. before 1345 Tuesday?

4. $\angle x = 27° 18' 34''$ and $\angle y = 18° 37' 52''$.

a. Find $\angle m$ if $\angle m = \angle x + \angle y$. **b.** Find $\angle n$ if $\angle n = \angle x - \angle y$.

5. If the bearing of a ship from a lighthouse is 52° 18′, what is the bearing of the lighthouse from the ship?

6. A pilot finds that the bearing of an airport from his plane is 137°. What is the bearing of his plane from the airport?

7. What is the approximate bearing of the North Star?

8. Columbus, Ohio, is about 170 miles east of Indianapolis, Indiana. The pilot of a plane observed that the bearing of Columbus was 70° and the bearing of Indianapolis was 330°. Make a scale drawing to find his distance from Columbus.

556. Methods of Navigation

There are four general ways of navigating a plane—piloting, celestial navigating, radio navigating, and navigating by dead reckoning. *Aviga-tion* is a word often used for air navigation because "navigation" has for many years been applied to steering ships at sea.

Piloting is finding one's way by the use of maps and charts and by the observation of certain landmarks such as railways, roads, rivers, and towns.

Celestial navigating is done by observing the position of the sun, moon, planets, and stars.

Radio navigating means finding one's way by radio communication with airports, planes, or ships.

Dead reckoning is flying by the use of a chart and compass. Before a pilot or navigator starts on a flight by dead reckoning, he must chart his course. To do this he uses his chart to find the bearing of his destination from the beginning of his flight. He determines the direction the plane must be *headed* (pointed) to make allowance for the *wind drift*. Since the

compass needle at most places does not point true north, the pilot must add or subtract the magnetic variation to know what compass reading should be used in the flight. The pilot should be able to tell exactly where he is at any time in the flight.

557. How to Establish a Fix

Establishing a fix is finding one's position. A pilot may be sure that he is flying in the right direction toward his destination, but uncertain of the remaining distance to be covered. The following example illustrates a method of finding his position:

Example 1. While flying a straight course from airport A to airport B, the pilot wishes to find how far he is from the airport B. Hearing the radio signal from airport C, he finds the bearing of it from his position to be 125°. Find his distance to B.

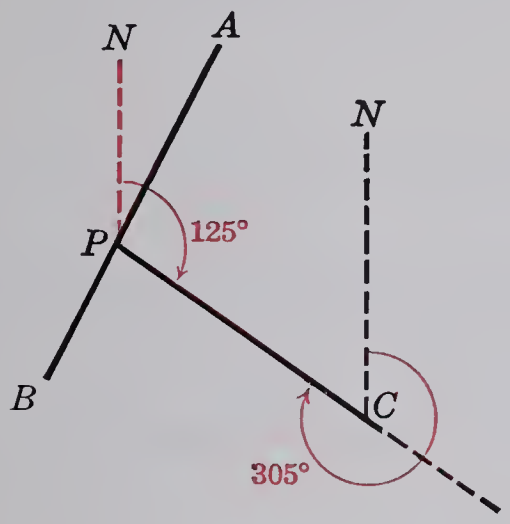


Fig. 1

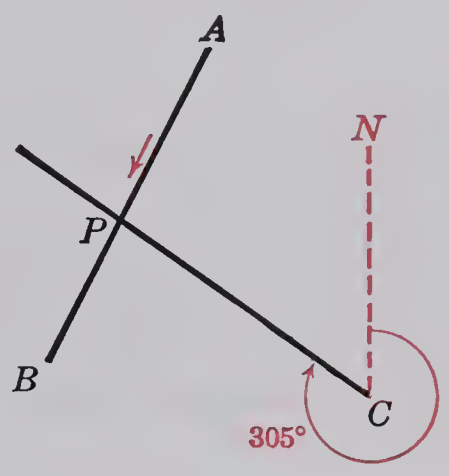


Fig. 2

Analysis. (Fig. 1) 1. Suppose the problem solved and that P is his position in AB. Draw PC.

2. Draw the lines running north from P and C.

3. Since the bearing of C from P is 125°, the bearing of P from C is $(125° + 180°)$, or 305°.

4. Then P is the intersection of AB and the line through C with direction 305°.

Solution. (Fig. 2) 1. On the chart draw AB.

2. Draw the line through C in direction 305°.

3. This line intersects AB in point P, which is the fix.

4. By using the chart, the distance PB can be calculated.

Suppose a pilot is not flying a straight course and wishes to check his location by radio from two airports. The following example illustrates a method he may use:

Example 2. A pilot at a point P recognized the radio signals from two airports A and B. Using his direction-finder, he observed that the bearing of A was 78° and the bearing of B was 300°. Find his position.

Analysis. (Fig. 1) 1. Suppose the problem solved.

2. Since the bearing of A from P is 78°, the bearing of P from A is $(180° + 78°)$, or 258°.

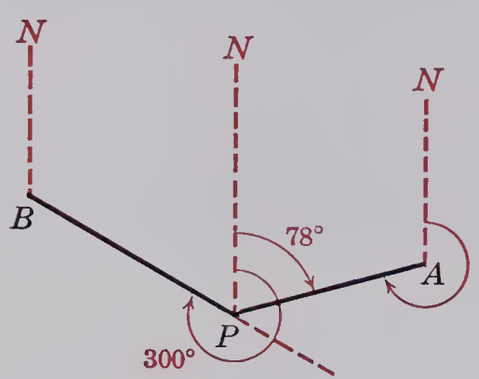


Fig. 1

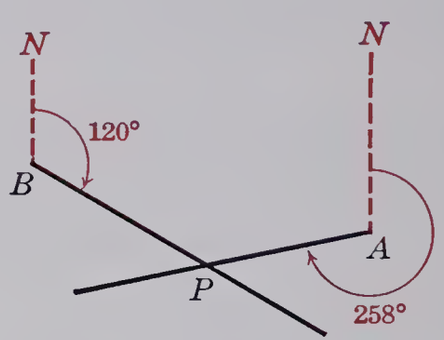


Fig. 2

3. Since the bearing of B from P is 300°, the bearing of P from B is $(300° - 180°)$, or 120°.

Solution. (Fig. 2) 1. On the map draw the line from A in direction 258°.

2. From B draw the line in direction 120°.

3. These two lines intersect in point P, which is the position or fix of the pilot.

EXERCISES

1. Station B is 60 miles due east from station A. The bearing of a plane from station A is 40° and from B is 290°. How far is the plane from station A?

2. Airport S is 240 miles from airport R, and its bearing from R is 138°. Airport T has a bearing of 115° from R and a bearing of 10° from S. Find the distance of T from R and from S.

3. A pilot is flying course 225° from airport X. Determine his fix when the bearing of station Y is 76° from the plane, if station Y is in direction 115° from X and the distance of Y from X is 150 miles.

558. How the Wind Affects the Direction and Speed of a Plane

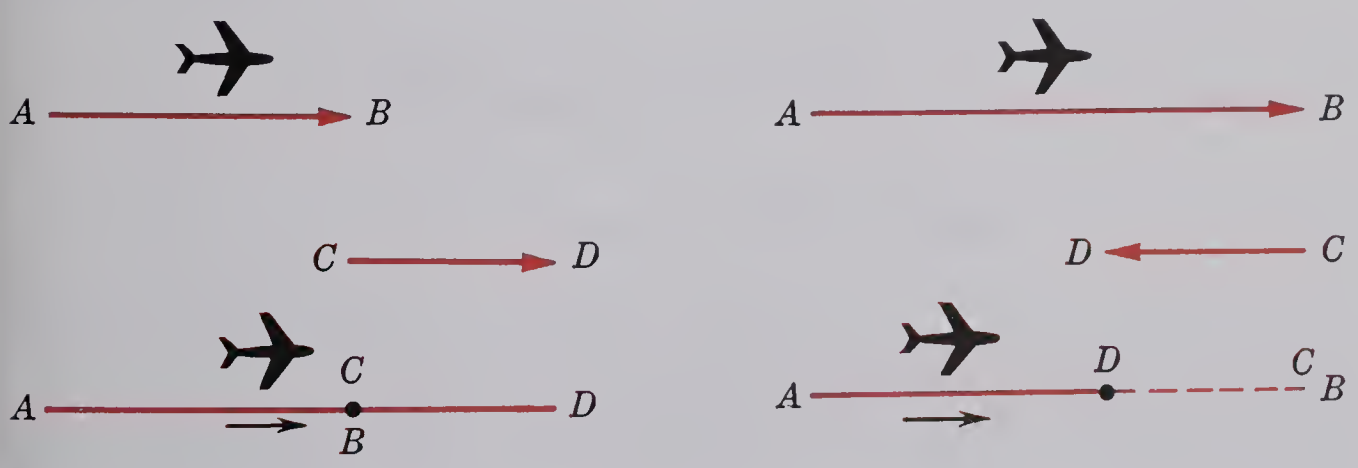
When an airplane is flying in still air toward a given destination, its nose is pointed in the direction it is going. A wind blowing in the same direction as the plane is flying is called a *tail wind*; and a wind blowing in the opposite direction is called a *head wind*. Neither a tail wind nor a head wind changes the direction of a plane's flight, but either one affects the speed of the plane relative to the ground.

559. Speed, Track, and Course

The *air speed* of a plane is the speed of the plane in still air. The *ground speed* of a plane is the speed of the plane relative to the ground over which the plane flies. The *heading* of a plane is the direction in which the nose of the plane is pointed. The *track* of a plane is the path along which the plane flies. The *course* of a plane is the direction of the track. The meanings of these terms will become clearer as we study the next article.

560. How to Find the Course, Ground Speed, and Drift Angle

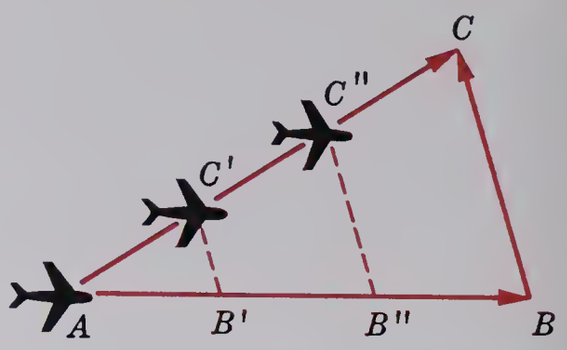
When a plane is flying with a tail wind, the ground speed of the plane is equal to the sum of the air speed and the speed of the wind. Thus if AB represents the air speed of the plane and CD the speed of the wind, then AD represents the ground speed of the plane.



When a plane is flying against a head wind, the ground speed is equal to the air speed of the plane decreased by the speed of the wind.

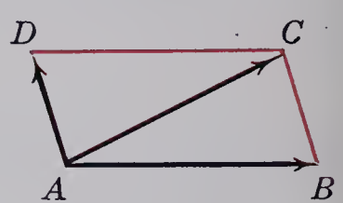
When a plane is flying in a wind that is blowing obliquely to the heading of the plane, both the speed and the direction of the plane are affected.

Suppose that points A and B are 100 miles apart and that a plane at A is headed toward B at 100 m.p.h. Also suppose that the wind is blowing in the direction BC at 40 m.p.h. In 20 minutes the plane has traveled from A to B', but the wind has carried the plane from B' to C'; in 40 minutes the plane has traveled from A to B'', but the wind has carried the plane from B'' to C''; and in 1 hour the plane has traveled from A to B, while the wind has caused it to arrive at C. If AB represents the air speed of the plane and BC represents the velocity of the air, then AC represents the course of the plane and its ground speed.

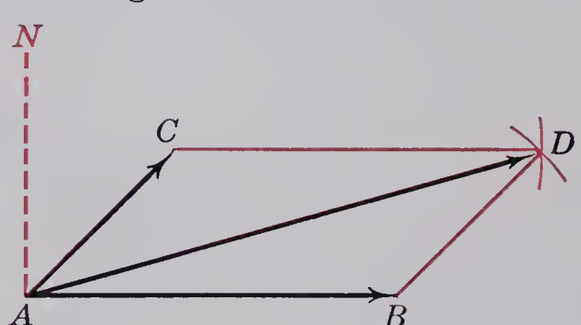


The parallelogram of velocities can be used to find the ground velocity of the plane.

AC is the diagonal of the parallelogram $ABCD$ drawn from A. In either figure, the plane is headed along AB, its track is along AC, and its course is in direction AC. Due to the wind the plane's direction of flight (course) is changed from AB to AC. The $\angle BAC$ is called the *drift angle*.



Example 1. A plane is headed east with an air speed of 100 m.p.h. The wind is blowing from direction 225° at 40 m.p.h. Find the course, ground speed, and drift angle.



Solution.* 1. If 1 cm. represents 20 m.p.h., 5 cm. will represent 100 m.p.h. Draw AB in direction 90° and 5 cm. long.

2. $40 \div 20 = 2$. Draw AC in direction $(225° - 180°)$, or 45°, and 2 cm. long.

3. Construct $\square\, ABDC$ and draw diagonal AD.

4. Measure AD. It is 6.5 cm. long approximately.

*The drawings are to a reduced scale. The student will use the measurements of the problems.

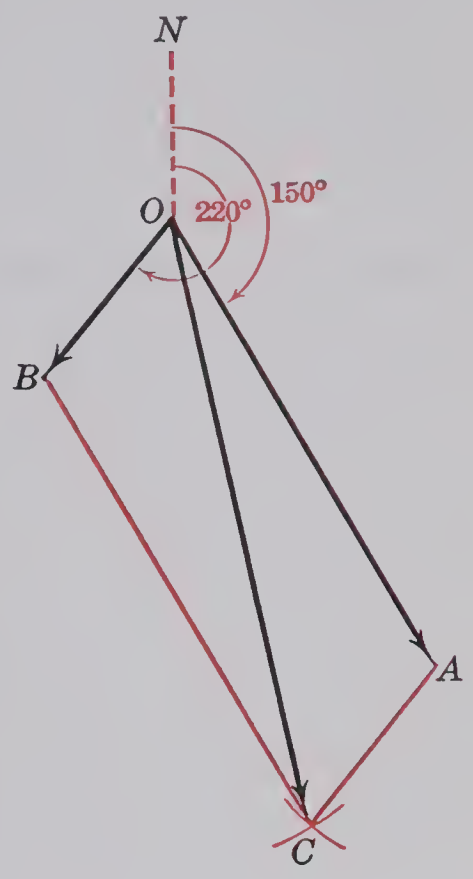
5. $6.5 \times 20 = 130$. The ground speed is 130 m.p.h.

6. Measure $\angle BAD$, the drift angle. It is 16° approximately.

7. 16° is the drift angle. The course is 130 m.p.h. in direction AD, 74°.

In Example 1 above, the line segments AB, AC, and AD that represent the velocities are called *vectors*.

Example 2. Find the course, ground speed, and drift angle of an airplane with an air speed of 100 m.p.h. and headed at 150° if there is a 40 m.p.h. wind from direction 40°.



Solution. 1. Let 1 cm. represent 20 m.p.h. Then 5 cm. will represent 100 m.p.h. and 2 cm. will represent 40 m.p.h.

2. Draw $OA = 5$ cm. and in direction 150°.

3. Draw $OB = 2$ cm. and in direction 220°.

4. Construct $\square OACB$ and draw diagonal OC.

5. $OC = 6$ cm. $6 \times 20 = 120$. The ground speed is 120 m.p.h.

6. Measure $\angle NOC$. $\angle NOC = 168°$. The course is 168° from O.

7. The drift angle $= \angle AOC = 18°$.

EXERCISES

1. An airplane is headed 30° with an air speed of 120 m.p.h. The wind is blowing 45 m.p.h. from due west. Find the course, ground speed, and drift angle.

2. Solve Ex. 1, assuming that the wind is blowing from 135° instead of from the west.

3. Solve Ex. 1, assuming that the wind is blowing from 340°.

4. An airplane is headed in direction 260° with an air speed of 280 m.p.h. The wind is blowing 20 m.p.h. in direction 280°. Find the ground speed of the plane.

5. An airplane is headed in direction 140° with an air speed of 300 m.p.h. The wind is blowing north 20 m.p.h. Find the ground speed.

561. How to Find the Ground Speed and the Wind-Correction Angle

On pages 549–550 it was shown how a pilot can determine the course and ground speed when the wind is blowing. More often a pilot must determine in what direction the plane must be headed so that he can fly the required course.

Study the following example and its solution:

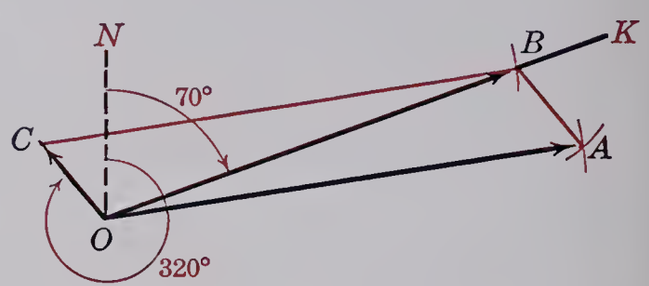
Example. A pilot is to fly on course 70° when the wind is blowing 30 m.p.h. from direction 140°. If he plans on an air speed of 150 m.p.h., what wind-correction angle is needed and what will be his ground speed?

Solution. 1. Draw line OK in direction 70°.

2. Using the scale 1 cm. = 30 m.p.h., draw OC in direction (140° + 180°), or 320°, and 1 cm. long.

3. With C as the center and radius = 5 cm. draw an arc intersecting OK in B.

4. Complete the parallelogram $OABC$.



5. $OA = CB$. Why? The plane should be headed along OA. Measure $\angle NOA$. The heading is about 81° and the wind-correction angle AOB is about 11°.

6. The ground speed represented by vector OB is about 137 m.p.h.

EXERCISES

1. A pilot must fly a course of 20° when the wind has a velocity of 20 m.p.h. from direction 135°. Find his heading and the ground speed if his air speed is 100 m.p.h.

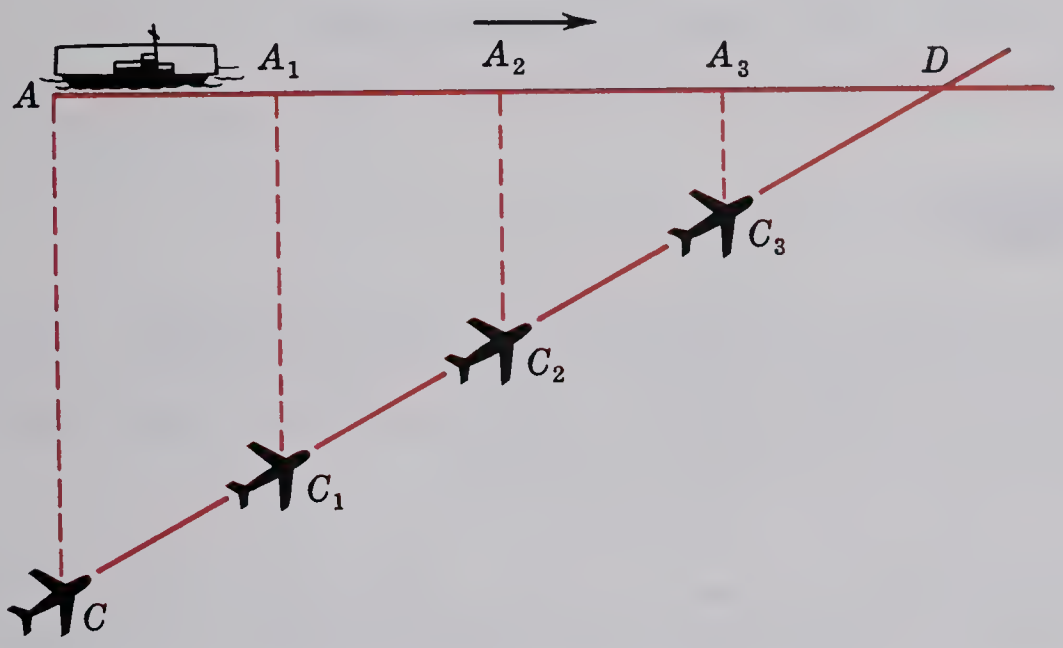
2. The required course is 280°, the wind is blowing from 200° at 40 m.p.h., and the air speed is 240 m.p.h. Find the heading and ground speed.

3. A pilot wishes to take his plane on course 280° with a ground speed of 160 m.p.h. The wind is blowing due east at 40 m.p.h. Find the air speed and heading he must use. (Draw a sketch of the parallelogram and use analysis.)

562. Interception

Any boy or girl who has played basketball knows that in passing a ball to a running teammate the ball is aimed in the direction ahead of the intended receiver so that it will reach a certain place at the same time the receiver arrives. The antiaircraft gunner tries to aim his gun so that the shell and target will be in the same position at the same time. Likewise when a pilot wishes to intercept a moving plane or ship in the shortest time, he must fly a straight course to the spot where they will meet.

Suppose that a pilot at point C wishes to intercept a ship at A sailing east at constant speed. If he wishes to reach the ship in the shortest time possible, he must head his plane toward the point D where the plane and ship will meet. Suppose that they will meet in 60 minutes. If their veloc-



ities are uniform, at the end of 15 minutes the ship will be at A_1 and the plane at C_1; in 30 minutes their respective positions will be at A_2 and C_2; and at the end of 45 minutes their respective positions will be at A_3 and C_3. From the figure it can be seen that when the plane and the ship intercept each other under the above conditions, the bearing of one from the other is constant. The converse of the last statement has long been known among seamen and is stated "constant bearing means collision." This means that if two objects are moving in straight courses and the bearing of one from the other remains constant, they will meet.

Example. A plane at point C starts to meet a ship which starts to take course AB from port A at 20 m.p.h. If there is no wind, how long will it take the plane to intercept the ship at a speed of 100 m.p.h.?

Solution. 1. Draw AC.

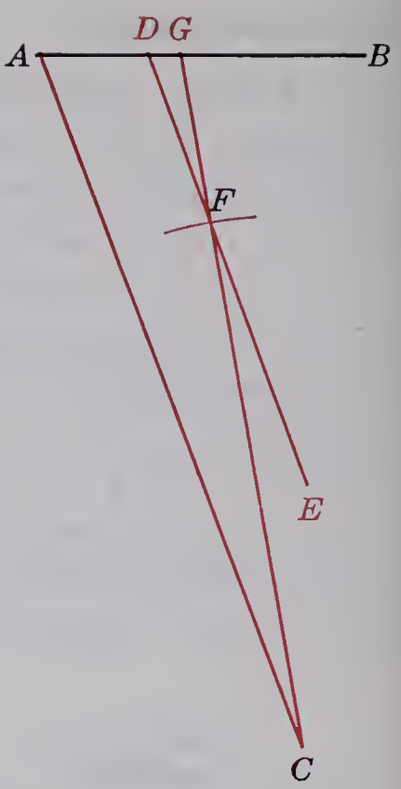
2. On AB mark off $AD = 1$ cm. to represent the distance the ship will go in 1 hour.

3. Draw $DE \parallel AC$. When the ship is at D, the plane is somewhere in line DE. Why?

4. In 1 hour the plane is 100 miles from C. So with C as a center and a radius of 5 cm. draw an arc intersecting DE in F.

5. Draw CF intersecting AB in G, which is the place of interception.

6. If CF represents 1 hour of flying time, CG represents about 1 hr. 24 min. (1.35 hr.), the time required for interception. AG is about 1.35 cm. and CG is about 6.65 cm. $1.35 \times 20 = 27$ and $6.65 \times 20 = 133$. Then the ship has gone about 27 miles, and the plane has gone about 133 miles.



Scale: 1 cm.= 20 miles

EXERCISES

1. A pursuit plane starts to intercept a transport plane which is 60 miles due north and flying at 320 m.p.h. due east. Find the course of the pursuit plane if it flies 450 m.p.h. How soon after it starts will the pursuit plane intercept the transport plane?

2. A jet plane starts to intercept a transport plane which is 80 miles due east and flying in direction 120° at 280 m.p.h. How soon after it starts will the jet plane intercept the transport plane if its ground speed is 650 m.p.h.?

3. At 6 A.M. a troopship sails on course 79° from a coastal town A at 25 m.p.h. At 8 A.M. an airplane leaves an airport 150 miles due south from A with a speed of 180 m.p.h. to intercept the ship. How far will the plane have to fly to intercept the ship? How far from A is the point of interception? How long does it take the plane to intercept the ship?

4. A bomber is flying due south at 220 m.p.h. A pursuit plane starts to intercept the bomber when the bearing of the bomber from the pursuit plane is 280° and the distance between the planes is 80 miles. Find the course of the pursuit plane and the time needed for the interception, if the pursuit plane flies 300 m.p.h.

5. At the hour 1600 a battleship starts on course 260° at 16 m.p.h. from island X. At the hour 2000 an airplane leaves an airport 340 miles south of X with a speed of 280 m.p.h. to intercept the battleship. Find the course of the airplane. How far from X will the point of interception be? How long will it take the airplane to make the interception?

6. A hunter fires a rifle at a hare when it is 120 feet due north of him and running due east at 40 feet a second. The rifle the hunter uses will give the bullet an average velocity of 2000 feet a second. In what direction must he sight the gun so that his shot will hit the hare? How far will the hare go from the time the hunter fires the gun until it is shot?

563. Radius of Action

One of a pilot's problems is finding how far he can safely fly a plane and return on a given amount of gasoline. After a plane is in the air and has attained its required speed, the consumption of gasoline is independent of the velocity of the air.

The distance from an airport that a plane can safely fly and return is called the *radius of action* for the plane at this port.

Example. The tank of a plane holds 56 gallons, and the plane consumes 10 gallons an hour. When no wind is blowing, what distance can the plane safely fly and return if 6 gallons are allowed for safety and getting into the air and the plane's speed is 100 m.p.h.?

Solution. 1. $56 - 6 = 50$.

2. $50 \div 10 = 5$, the number of hours of safe flying. The time going $= 2\frac{1}{2}$ hours.

3. $2\frac{1}{2} \times 100 = 250$, the distance in miles that the plane can go, or the radius of action.

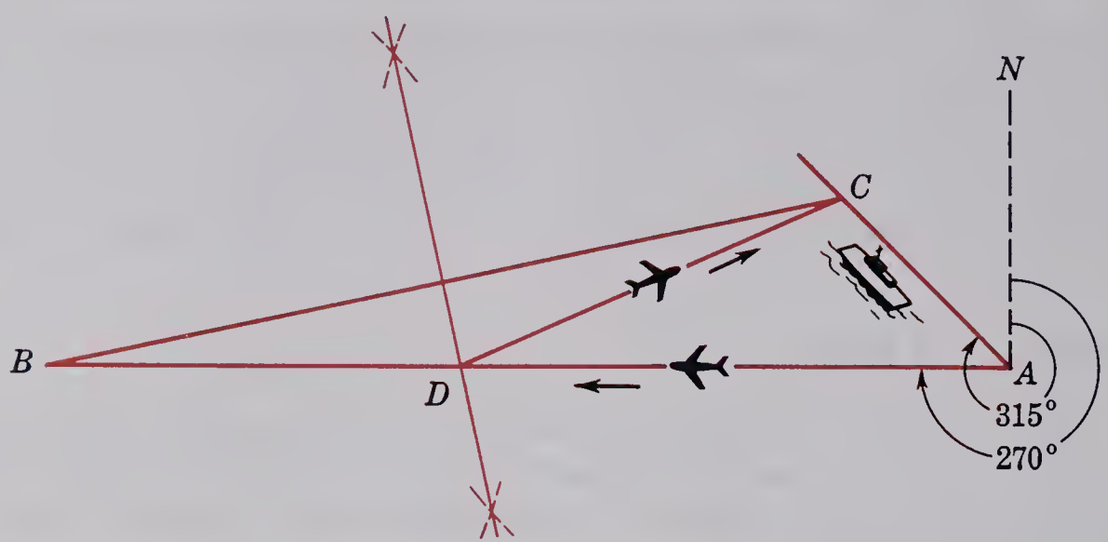
When a pilot flies a straight course from one point as far as he can and returns on a straight course to a second point, the distance between the first point and the turning point is the radius of action for the plane at the first point.

Example. An airplane carrier is steaming on course 315° at 25 m.p.h. A scouting plane with a speed of 100 m.p.h. leaves the ship on course 270° with sufficient gasoline for 4 hours of flight. How far can the plane keep its course before starting its return to the carrier? What heading should it take in returning?

Solution. 1. Draw AB the distance the plane can go on course 270° at 100 m.p.h. in 4 hours.

2. Draw AC the distance the carrier will go in 4 hours.

3. Draw BC and construct its $\perp$ bisector, which intersects AB in D.



4. Draw DC. Prove $DC = DB$.

5. D is the point where the plane must start on its return trip. AD, the radius of action, is about 228 miles.

6. The course returning, DC, is about 65°.

EXERCISES

1. A pilot is flying a plane whose safe flying time is 6 hours. How far southeast can he fly and return if no wind is blowing and his air speed is 120 m.p.h.?

2. An airplane carrier is steaming on course 240° at 20 m.p.h. If a plane with an air speed of 160 m.p.h. leaves the carrier to scout on course 280° and no wind is blowing, how far can it go on its course before starting its return to the carrier, allowing $1\frac{1}{2}$ hours flying time? What heading should it take on its returning course?

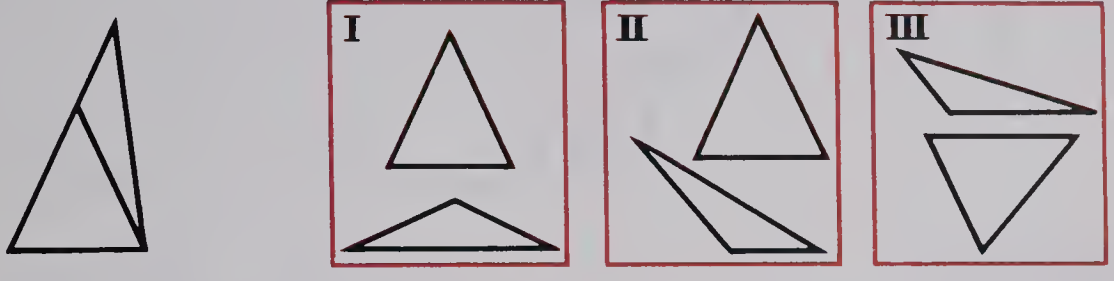
3. The wind is blowing 20 m.p.h. and the air speed of the plane is 110 m.p.h. How far can the pilot fly with the wind and return against the wind in 4 hours?

2. Geometric Relations; Orthographic Projection

564. Geometric Relations

The following exercises are designed to test your powers of observation with simple geometric figures.

Example. The figure on the left below is formed from the two parts in one of the large rectangles on the right. We may think of the parts being moved about in the plane to form the figure, but it is not permissible to take the parts out of the plane. Which rectangle contains the correct parts?



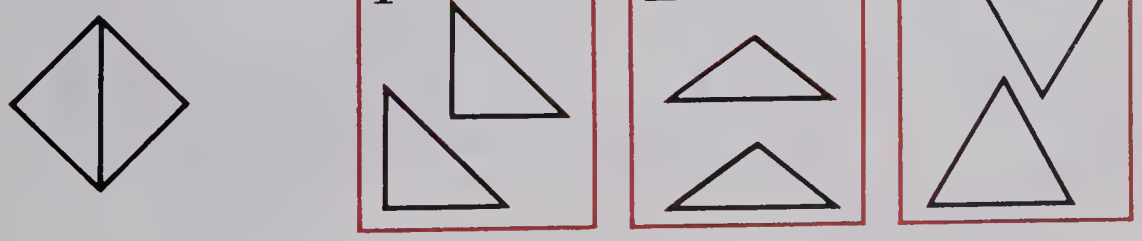
Rectangle III has the correct parts to form the given figure.

EXERCISES

A

In each exercise decide which rectangle contains the parts that can be moved about in the plane to form the given figure.

1.



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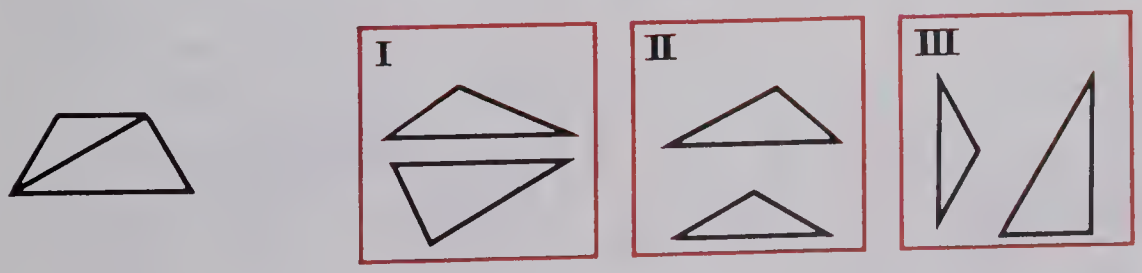

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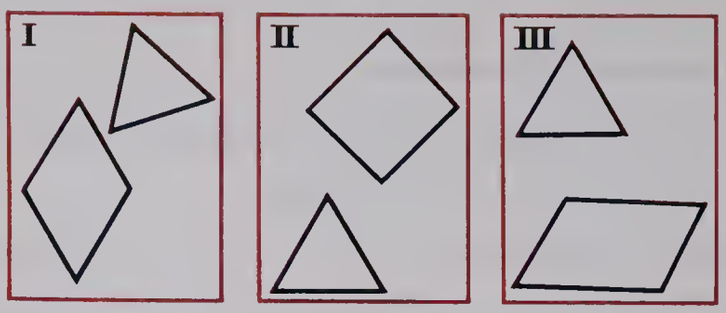

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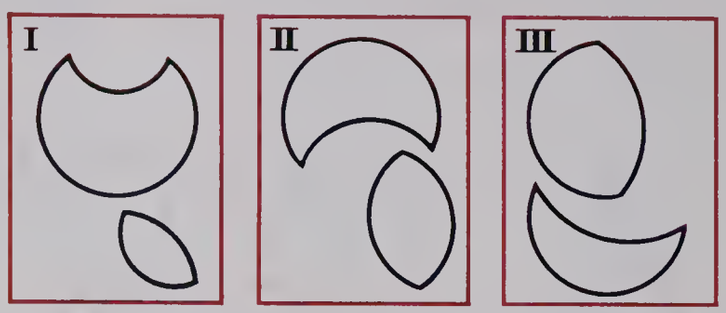

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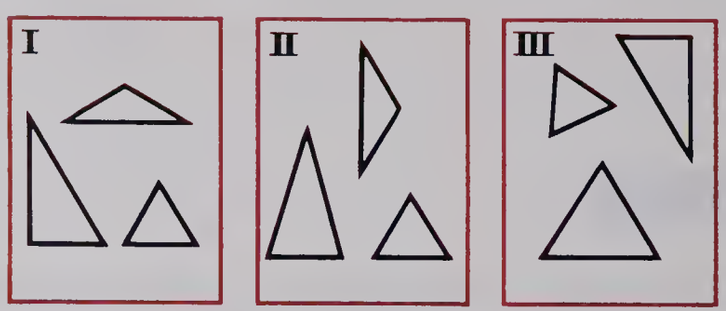

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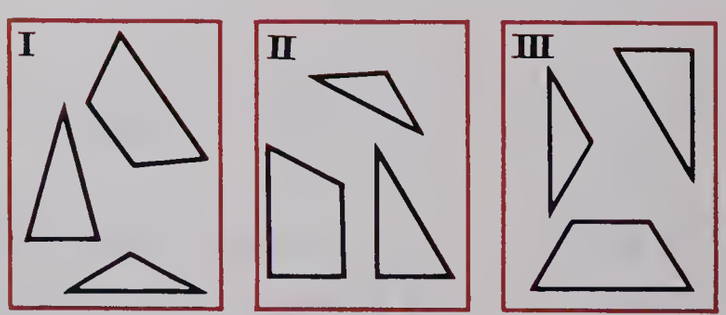

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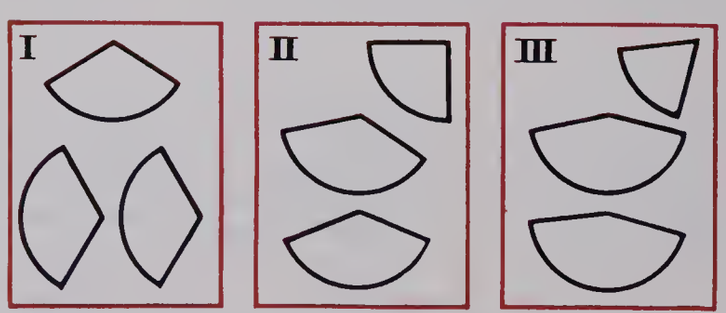


EXERCISES

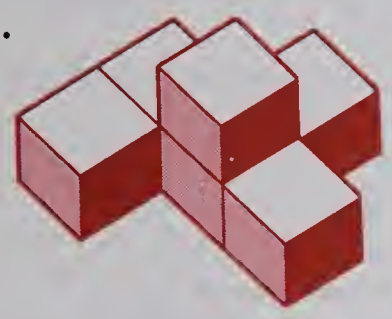
A

In the following exercises each figure was first constructed with cubical blocks and then the coinciding faces were glued together. In counting the cubes in each figure be sure to count all hidden cubes which are necessary to hold up the structure.

Answer the following questions for each figure:

a. How many cubes are there in the figure?
b. How many cubes have one and only one face glued?
c. How many cubes have two and only two faces glued?
d. How many cubes have three and only three faces glued?
e. How many cubes have four and only four faces glued?
f. How many cubes have five and only five faces glued?

1.



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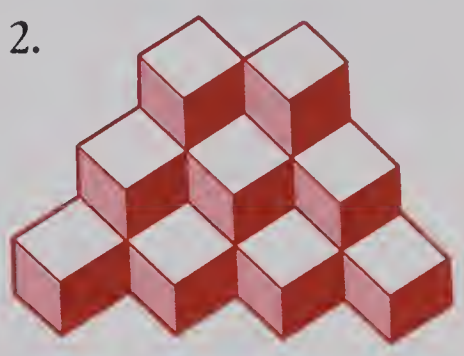

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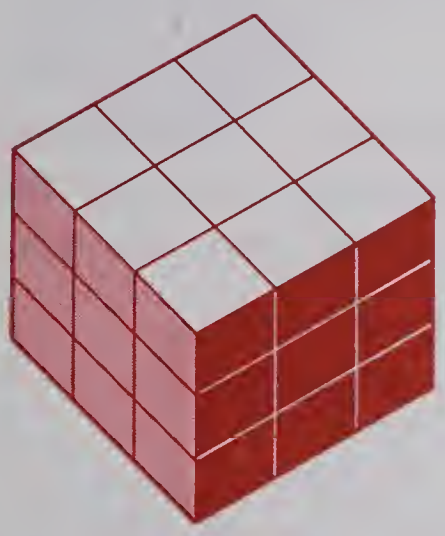

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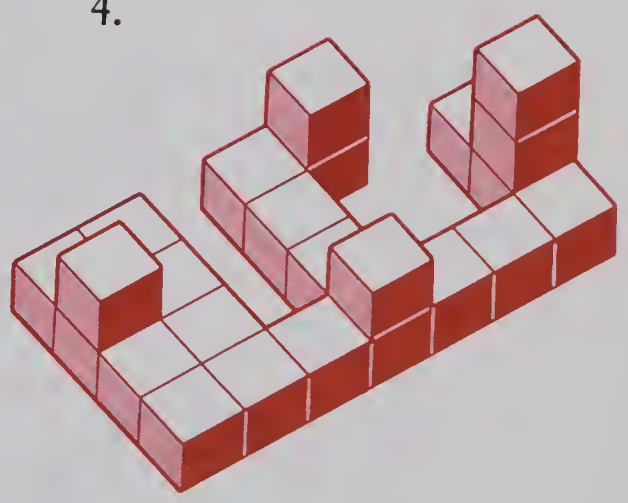

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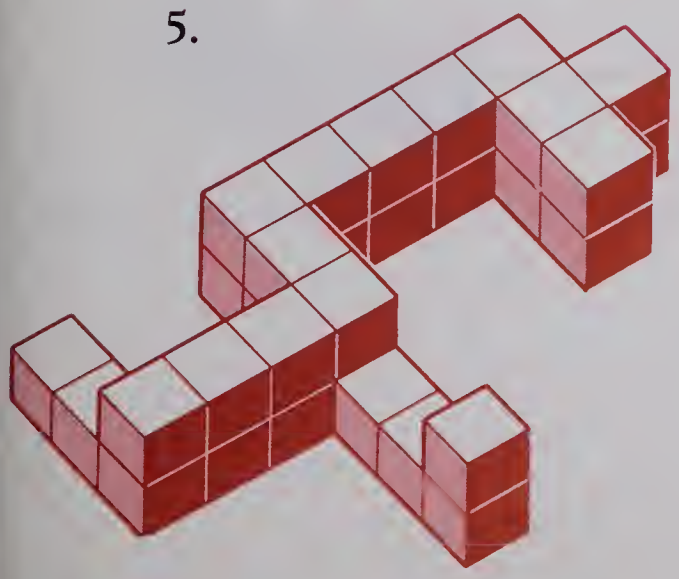

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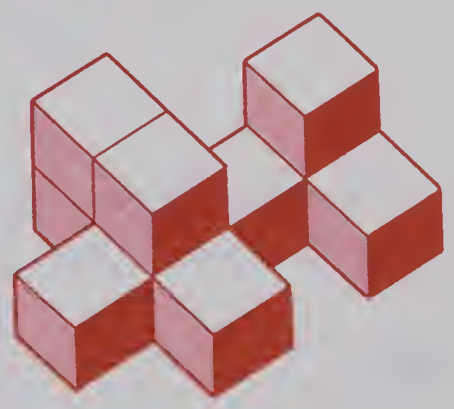


565. Orthographic Projection

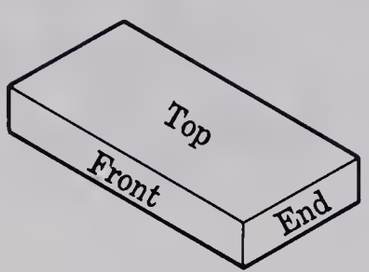
To represent material objects of three dimensions on the two-dimensional surface of a sheet of paper the system of orthographic projection is used. The drawing consists of a set of separate views made by the observer from different positions and arranged in a definite order. The usual combination is the top, front, and end views.

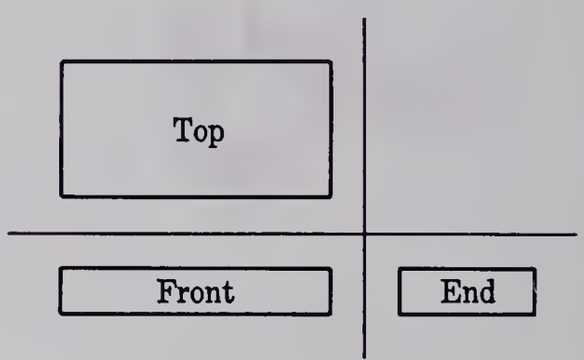
The *top view* is the view of the object as seen by an observer directly above it.

The *front view* of the object is the view observed from a position directly in front of it.

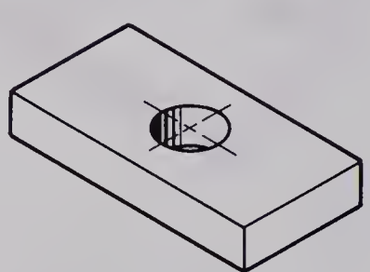
The *end view* is the view of the object as seen from the right side.

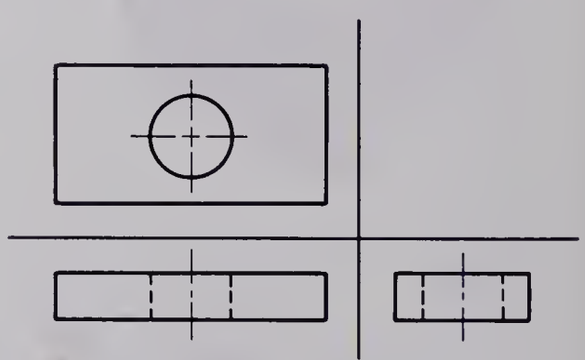
In the illustrations below, the three views of the rectangular solid at the left are shown at the right. The views usually occupy the positions shown in reference to the vertical and horizontal lines. In the example and exercises that follow, you can tell from their positions what views are shown.



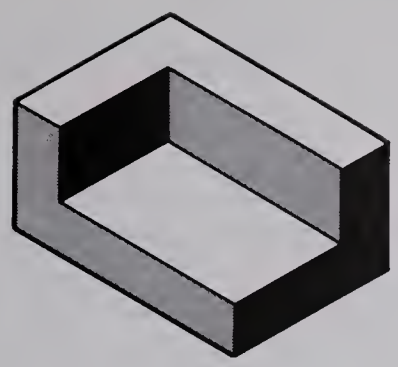


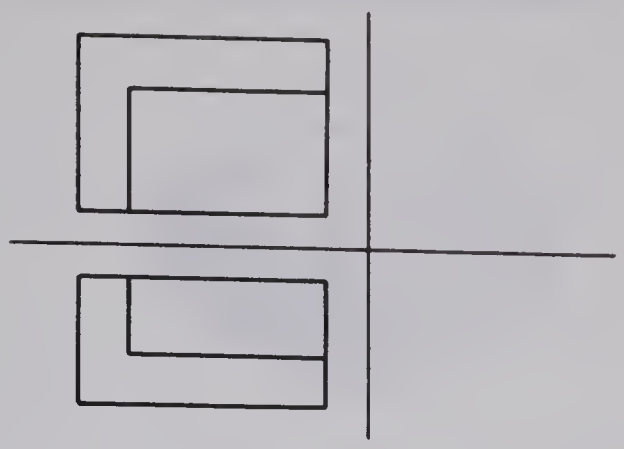
In the figure at the left below, the rectangular solid pictured has a circular hole in it. The three views are shown at the right. Since the circular hole cannot be seen in the front and end views, it is represented by dotted lines.



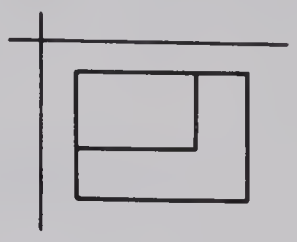


Example. In the diagram which follows, the top and front views of the solid are given. Draw the end view.





Solution. The correct end view is shown at the right.

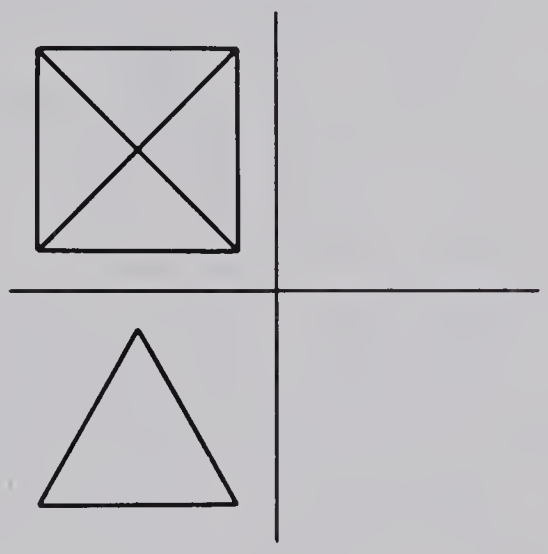


EXERCISES

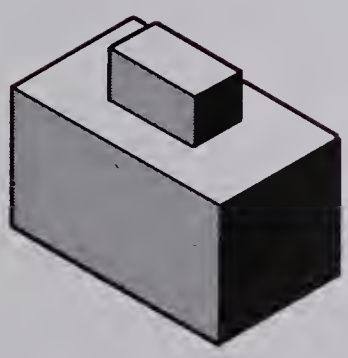
A

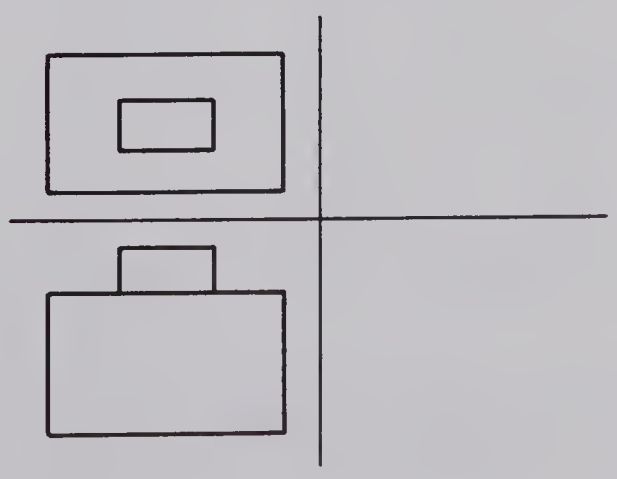
In each exercise two views of the solid are shown. Draw the third view.

1.

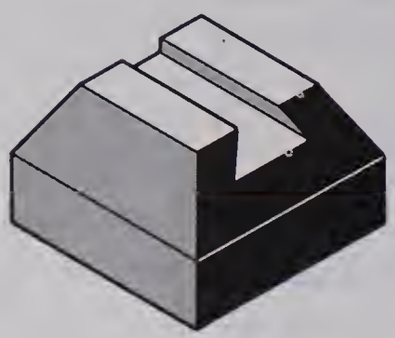


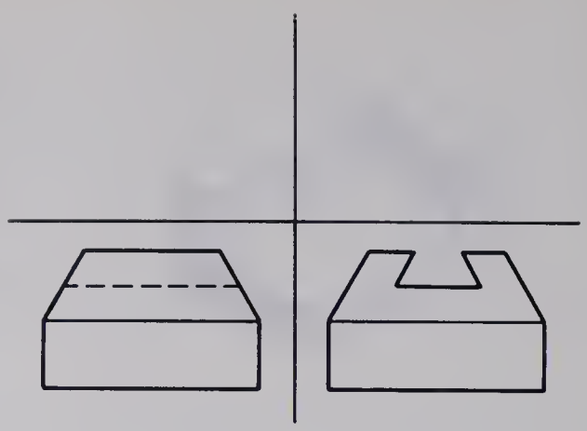
2.





3.

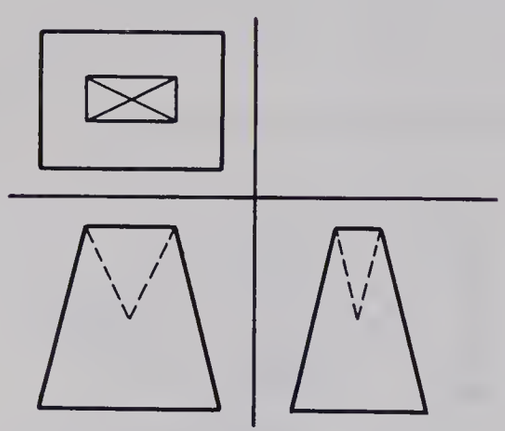




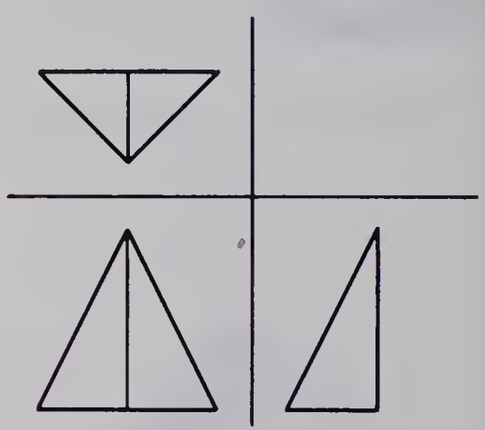
B

In each exercise the three views of a solid are given. Sketch the solid.

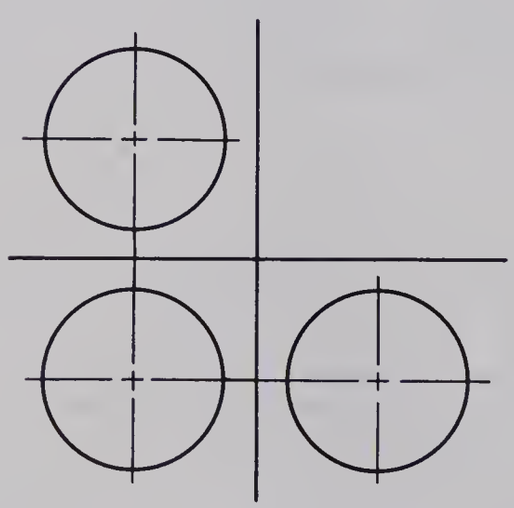
4.



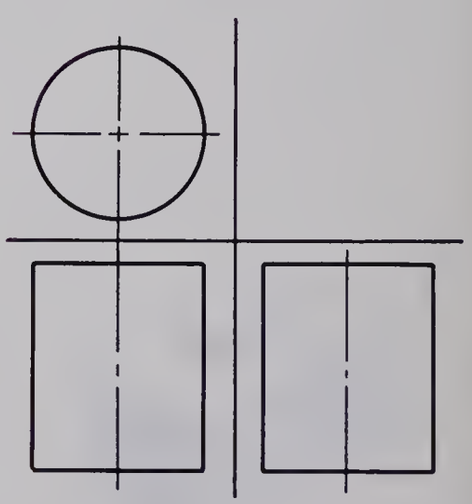
5.



6.



7.



SUMMARY OF PRINCIPAL BASES FOR PROOF

566. Triangles congruent

a. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent (§ 65).

b. If two right triangles have the two legs of one equal respectively to the two legs of the other, the triangles are congruent (§ 71).

c. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent (§ 73).

d. If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent (§ 85).

e. If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, the triangles are congruent (§ 125) .

f. If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent (§ 135).

g. A diagonal of a parallelogram divides it into two congruent triangles (§ 187).

567. Line segments equal

a. All radii of a circle or of equal circles are equal (§ 36, Asmt. 26).

b. Corresponding sides of congruent polygons are equal (§ 78).

c. Line segments are equal if they join any point in the perpendicular bisector of a line segment to the end-points of the segment (§ 79).

d. Line segments are equal if they are perpendiculars drawn from any point in the bisector of an angle to the sides of the angle (§ 126).

e. If two angles of a triangle are equal, the sides opposite these angles are equal (§ 128).

f. An equiangular triangle is equilateral (§ 129).

g. If one acute angle of a right triangle is 30°, the side opposite this angle is one half the hypotenuse (§ 136).

h. The opposite sides of a parallelogram are equal (§ 188).

i. Segments of parallels included between parallels are equal (§ 192).

j. The diagonals of a parallelogram bisect each other (§ 195).

k. If parallels intercept equal segments on one transversal, they intercept equal segments on any transversal (§ 202).

l. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side (§ 203).

m. The line segment joining the midpoints of two sides of a triangle is equal to one half the third side (§ 204).

n. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle (§ 205).

o. The median for a trapezoid is equal to one half the sum of the bases (§ 207).

p. In a circle or in equal circles equal arcs have equal chords (§ 287).

q. If a line through the center of a circle is perpendicular to a chord, it bisects the chord (§ 290).

r. In a circle or in equal circles equal chords are equidistant from the center (§ 293).

s. In a circle or in equal circles chords equidistant from the center are equal (§ 294).

t. The tangents to a circle from an external point are equal (§ 300).

u. The locus of points within an angle equidistant from the sides is the bisector of the angle (§ 343).

v. The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points (§ 345).

568. Angles equal

a. All right angles are equal (§ 36, Asmt. 19).

b. Complements of the same angle or of equal angles are equal (§ 36, Asmt. 28, 29).

c. Supplements of the same angle or of equal angles are equal (§ 36, Asmt. 30, 31).

d. If two lines intersect, the vertical angles are equal (§ 49).

e. Corresponding angles of congruent polygons are equal (§ 78).

f. If two sides of a triangle are equal, the angles opposite these sides are equal (§ 83).

g. An equilateral triangle is equiangular (§ 84).

h. If two parallels are cut by a transversal, the alternate interior angles are equal (§ 111).

i. If two parallels are cut by a transversal, the corresponding angles are equal (§ 112).

j. If two angles have their sides parallel, right side to right side and left side to left side, the angles are equal (§ 117).

k. If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal (§ 121).

l. If two angles have their sides perpendicular, right side to right side and left side to left side, the angles are equal (§ 127).

m. The opposite angles of a parallelogram are equal (§ 189).

n. In a circle or in equal circles equal arcs have equal central angles (§ 284).

o. Inscribed angles which intercept the same arc are equal (§ 320).

p. Corresponding angles of similar polygons are equal (§ 413).

569. Angles supplementary

a. Two angles are supplementary when their sum is a straight angle (§ 21).

b. If two adjacent angles have their exterior sides in a straight line, they are supplementary (§ 36, Asmt. 24).

c. If two parallels are cut by a transversal, the two interior angles on the same side of the transversal are supplementary (§ 113).

d. Any two consecutive angles of a parallelogram are supplementary (§ 190).

e. If a quadrilateral is inscribed in a circle, the opposite angles are supplementary (§ 321).

570. Angles complementary

a. Two angles are complementary when their sum is a right angle (§ 20).

b. The acute angles of a right triangle are complementary (§ 124).

571. Lines perpendicular; angle a right angle

a. Two lines are perpendicular to each other if one of them forms two equal adjacent angles with the other; or if the lines form a right angle (§ 14).

b. The perpendicular is the shortest line segment that can be drawn from a given point to a given line (§ 36, Asmt. 21 and § 522).

c. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other (§ 114).

d. A line passing through two points each equidistant from the end points of a line segment is the perpendicular bisector of the segment (§ 140).

e. All the angles of a rectangle are right angles (§ 191).
f. The diagonals of a rhombus are perpendicular (§ 196).
g. If a line through the center of a circle bisects a chord that is not a diameter, it is perpendicular to the chord (§ 291).
h. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact (§ 295).
i. If two circles intersect, the line joining their centers is the perpendicular bisector of the common chord (§ 301).
j. An angle inscribed in a semicircle is a right angle (§ 319).
k. The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points (§ 345).

572. Lines parallel

a. Two lines are parallel if they lie in the same plane and do not intersect even if extended (§ 99).
b. Two straight lines in the same plane are either parallel lines or intersecting lines (§ 100).
c. If two lines form equal alternate interior angles with a transversal, the lines are parallel (§ 104).
d. If two lines form equal corresponding angles with a transversal, the lines are parallel (§ 105).
e. If two lines form supplementary interior angles on the same side of a transversal, the lines are parallel (§ 106).
f. Two lines perpendicular to a third line are parallel (§ 107).
g. Two lines parallel to a third line are parallel (§ 109).
h. A parallelogram is a quadrilateral having two pairs of parallel sides (§ 185).
i. The line segment joining the midpoints of two sides of a triangle is parallel to the third side (§ 204).
j. The median of a trapezoid is parallel to the bases (§ 207).
k. If a line divides two sides of a triangle proportionally, it is parallel to the third side (§ 402).
l. If a line divides two sides of a triangle so that either side is to one of its segments as the other side is to its corresponding segment, the line is parallel to the third side (§ 403).

573. A quadrilateral a parallelogram

a. A parallelogram is a quadrilateral having two pairs of parallel sides (§ 185).

b. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram (§ 197).

c. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram (§ 198).

d. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram (§ 199).

e. If the opposite angles of a quadrilateral are equal the figure is a parallelogram (§ 200).

574. Angles and sums of angles

a. The sum of all the angles about a point in a plane is two straight angles (§ 36, Asmt. 22).

b. The sum of the angles of a triangle is a straight angle (§ 120).

c. An exterior angle of a triangle equals the sum of the two nonadjacent interior angles (§ 123).

d. The sum of the exterior angles of a polygon is two straight angles (§ 208).

e. Each exterior angle of a regular polygon of n sides contains $\frac{360°}{n}$ (§ 209).

f. The sum of the interior angles of a polygon having n sides is $(n-2)$ straight angles (§ 210).

g. Each interior angle of a regular polygon of n sides contains $\frac{180(n-2)°}{n}$ (§ 211).

h. The central angle of a regular polygon of n sides contains $\frac{360°}{n}$ (§ 462).

575. Arcs equal

a. In a circle or in equal circles equal central angles have equal arcs (§ 282).

b. A diameter of a circle bisects the circle (§ 283).

c. In a circle or in equal circles equal chords have equal arcs (§ 286).

d. If a line through the center of a circle is perpendicular to a chord, it bisects its arc (§ 290).

e. In a circle or in equal circles two arcs which have the same number of arc degrees are equal (§ 317).

f. Parallel lines intercept equal arcs on a circle (§ 327).

576. Comparison of angles and their arcs

a. A central angle is equal in degrees to its intercepted arc (§ 315).

b. An inscribed angle is equal in degrees to one half its intercepted arc (§ 318).

c. An angle formed by a tangent and a chord is equal in degrees to one half its intercepted arc (§ 322).

d. An angle formed by two chords intersecting within a circle is equal in degrees to one half the sum of the arcs intercepted by it and its vertical angle (§ 323).

e. An angle formed by two tangents, a tangent and a secant, or two secants intersecting outside a circle, is equal in degrees to one half the difference of the intercepted arcs (§ 324).

577. Lines passing through a point

a. The perpendicular bisector of a chord passes through the center of the circle (§ 292).

b. If a line is perpendicular to a tangent at the point of contact, it passes through the center of the circle (§ 296).

c. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices (§ 356).

d. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides (§ 358).

e. The altitudes of a triangle are concurrent (§ 360).

f. The medians of a triangle are concurrent in a point which lies two thirds of the distance from each vertex to the midpoint of the opposite side (§ 362).

578. Line segments in proportion

a. A line parallel to one side of a triangle divides the other two sides proportionally (§ 397).

b. A line parallel to one side of a triangle divides the other two sides so that either side is to one of its segments as the other side is to its corresponding segment (§ 398).

c. Parallel lines intercept proportional segments on two transversals (§ 399).

d. The bisector of an interior (or exterior) angle of a triangle divides the opposite side internally (or externally) into segments which are proportional to the adjacent sides (§§ 405, 406).

e. Any two corresponding sides of two similar polygons have the same ratio as two other corresponding sides (§ 413).

f. The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides (§ 417).

g. The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse (§ 425).

h. Either leg of a right triangle is the mean proportional between the hypotenuse and its projection on the hypotenuse (§ 427).

i. The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the segments of the diameter (§ 428).

j. If two chords intersect within a circle, the product of the segments of one chord is equal to the product of the segments of the other (§ 431).

k. If a tangent and a secant are drawn to a circle from the same point, the tangent is the mean proportional between the secant and its external segment (§ 433).

579. Proportions involving polygons and circles

a. Two rectangles, parallelograms, or triangles having equal bases have the same ratio as their altitudes (§§ 231, 237, 239).

b. Two rectangles, parallelograms, or triangles having equal altitudes have the same ratio as their bases (§§ 232, 237, 240).

c. The areas of two triangles have the same ratio as the products of their bases and altitudes (§ 241).

d. The perimeters of two similar polygons have the same ratio as any two corresponding sides (§ 434).

e. The areas of two similar triangles have the same ratio as the squares of any two corresponding sides or as the squares of any two corresponding altitudes (§§ 438, 439).

f. The areas of two similar polygons have the same ratio as the squares of any two corresponding sides (§ 440).

g. The areas of two regular polygons of the same number of sides have the same ratio as the squares of any two corresponding sides or as the squares of their radii or as the squares of their apothems (§§ 469, 471).

h. The circumferences of two circles have the same ratio as their radii or as their diameters (§§ 479, 480).

i. The areas of two circles have the same ratio as the squares of their radii or as the squares of their diameters (§ 489).

j. The area of a sector of a circle is to the area of the circle as the angle of the sector is to 360° (§ 490).

580. Similar polygons

a. Similar polygons are polygons whose corresponding angles are equal and whose corresponding sides are proportional (§ 412).

b. If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar (§ 415).

c. If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar (§ 416).

d. If two triangles are similar to a third triangle, they are similar to each other (§ 418).

e. If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar (§ 422).

f. If two triangles have their sides respectively proportional, they are similar (§ 423).

g. The altitude on the hypotenuse of a right triangle forms two right triangles which are similar to the given triangle and to each other (§ 424).

h. If two polygons are similar, they can be separated into the same number of triangles which are similar each to each (§ 435).

i. If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar (§ 436).

j. Two regular polygons of the same number of sides are similar (§ 468).

581. Areas

a. The area of a rectangle is equal to the product of its base and altitude (§ 228).

b. The area of a square is equal to the square of one of its sides (§ 233).

c. The area of a parallelogram is equal to the product of its base and altitude (§ 235).

d. The area of a triangle is equal to one half the product of its base and altitude (§ 238).

e. The area of a rhombus is equal to one half the product of its diagonals (§ 242).

f. The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases (§ 244).

g. The area of an equilateral triangle having a side s is given by the formula $A = \frac{s^2}{4}\sqrt{3}$ (§ 254).

h. The area of a triangle whose sides are a, b, and c is given by the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ (§ 257).

i. The area of a regular polygon is equal to one half the product of its apothem and its perimeter (§ 467).

j. The area of a circle is equal to one half the product of its radius and its circumference (§ 487).

k. The area of a circle is given by the formula $S = \pi r^2$ (§ 488).

582. Figures equal in area

a. Parallelograms having equal bases and equal altitudes are equal (§ 236).

b. If a triangle and a parallelogram have equal bases and equal altitudes, the area of the triangle is half the area of the parallelogram (§ 243).

c. The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the legs (§ 250).

583. Miscellaneous numerical relations

a. The diagonal of a square is equal to one side multiplied by $\sqrt{2}$ (§ 253).

b. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs (§ 430).

c. The circumference of a circle is expressed by the formula $c = \pi d$, or $c = 2\,\pi r$ (§ 482).

d. $\pi = \dfrac{c}{d} = 3.1416$, approximately (§ 486).

584. Polygons regular

a. A regular polygon is a polygon which is both equilateral and equiangular (§ 184).

b. If a circle is divided into three or more equal arcs, the chords of these arcs form a regular inscribed polygon and the tangents at the points of division form a regular circumscribed polygon (§§ 463, 466).

c. An equilateral polygon inscribed in a circle is a regular polygon (§ 464).

d. If the midpoints of the arcs of a regular inscribed polygon are joined to the extremities of the respective sides, a regular inscribed polygon of double the number of sides is formed (§ 465).

585. Inequalities

a. An exterior angle of a triangle is greater than either nonadjacent interior angle (§ 93).

b. The perpendicular is the shortest line segment that can be drawn from a given point to a given line (§ 36, Asmt. 21, and § 522).

c. Each side of a triangle is less than the sum of the other two sides (§ 519).

d. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side (§ 520).

e. If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle (§ 521).

f. The hypotenuse of a right triangle is greater than either leg (§ 523).

g. If two triangles have two sides of one equal respectively to two sides of the other and the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second (§ 524).

h. If two triangles have two sides of one equal respectively to two sides of the other and the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second (§ 525).

i. If two oblique line segments are drawn to a line from a point in a perpendicular to that line, the one having the greater projection upon the line is the greater (§ 526); and conversely (§ 527).

j. In a circle or in equal circles the greater of two unequal central angles has the greater arc (§ 528); and conversely (§ 529).

k. In a circle or in equal circles the greater of two unequal chords has the greater arc (§ 530); and conversely (§ 531).

l. In a circle or in equal circles the greater of two unequal chords is nearer the center (§ 532); and conversely (§ 533).

m. A diameter of a circle is greater than any other chord (§ 534).

586. Constructions

a. To construct a line segment equal to a given line segment (§ 9).

b. To construct a triangle, given the three sides (§ 165).

c. To bisect a given angle (§ 166).

d. To construct an angle equal to a given angle (§ 167).

e. To construct a triangle, given two sides and the included angle (§ 169).

f. To construct a triangle, given two angles and the included side (§ 170).

g. To construct the perpendicular bisector of a given line segment (§ 171).

h. To construct a perpendicular to a given line from a given point not on the line (§ 172).

i. To construct a perpendicular to a given line at a given point in the line (§ 173).

j. Through a given point to construct a line parallel to a given line (§ 174).

k. To divide a given line segment into any number of equal parts (§ 212).

l. To construct a square equal to the sum of two given squares (§ 262).

m. To construct a triangle equal to a given polygon (§ 263).

n. To bisect a given arc of a circle (§ 328).

o. To construct the tangent to a given circle at a given point on the circle (§ 329).

p. To construct the tangents to a given circle from a given external point (§ 330).

q. To inscribe a square in a given circle (§ 331).

r. To inscribe a regular hexagon in a given circle (§ 332).

s. To circumscribe a circle about a given triangle (§ 364).

t. To inscribe a circle in a given triangle (§ 369).

u. Upon a given line segment as a chord to construct an arc of a circle in which a given angle can be inscribed (§ 372).

v. To construct the fourth proportional to three given line segments (§ 400).

w. To divide a given line segment into n parts which are proportional to n given line segments (§ 401).

x. To construct the mean proportional between two given line segments (§ 429).

y. Upon a given line segment corresponding to one side of a given polygon to construct a polygon similar to the given polygon (§ 437).

z. To construct a square equal to a given parallelogram (§ 441).

TABLES FOR REFERENCE

LINEAR MEASURE

12 in. (12″) = 1 ft. (1′)
3 ft. = 1 yd.
16.5 ft. = 1 rd.
320 rd. = 1 mi.
5280 ft. = 1 mi.

SQUARE MEASURE

144 sq. in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
272.25 sq. ft. = 1 sq. rd
160 sq. rd. = 1 acre
100 sq. ft. = 1 square*

FORMULAS OF PLANE GEOMETRY

Notation

a, apothem of a regular polygon
a, b, c, sides of $\triangle ABC$
A, area
b, base
c, circumference
c, the hypotenuse of rt. $\triangle ABC$
d, diagonal of a square
d, d', diagonals of a rhombus
h, altitude
p, perimeter
r, radius
s, semiperimeter, side

Formulas

Circle: $A = \pi r^2$, $c = 2\pi r$
Equilateral triangle: $A = \frac{1}{4} s^2\sqrt{3}$, $h = \frac{1}{2} s\sqrt{3}$
Parallelogram: $A = bh$
Regular polygon: $A = \frac{1}{2} ap$
Rhombus: $A = \frac{1}{2} dd'$
Right triangle: $A = \frac{1}{2} ab$, $c^2 = a^2 + b^2$
Sector of a circle: $A = \frac{\text{central } \angle}{360^\circ} \pi r^2$
Segment of a circle: $A = \text{sector} - \triangle$
Square: $A = \frac{1}{2} d^2$, $A = s^2$
Trapezoid: $A = \frac{1}{2} h(b + b')$
Triangle: $A = \frac{1}{2} bh$, $A = \sqrt{s(s-a)(s-b)(s-c)}$

*A square is a unit of area used by roofers.

Table I. Square Roots

N	$\sqrt{N}$	N	$\sqrt{N}$	N	$\sqrt{N}$	N	$\sqrt{N}$	N	$\sqrt{N}$	N	$\sqrt{N}$
1	1.000	51	7.141	101	10.050	151	12.288	201	14.177	251	15.843
2	1.414	52	7.211	102	10.100	152	12.329	202	14.213	252	15.875
3	1.732	53	7.280	103	10.149	153	12.369	203	14.248	253	15.906
4	2.000	54	7.348	104	10.198	154	12.410	204	14.283	254	15.937
5	2.236	55	7.416	105	10.247	155	12.450	205	14.318	255	15.969
6	2.449	56	7.483	106	10.296	156	12.490	206	14.353	256	16.000
7	2.646	57	7.550	107	10.344	157	12.530	207	14.388	257	16.031
8	2.828	58	7.616	108	10.392	158	12.570	208	14.422	258	16.062
9	3.000	59	7.681	109	10.440	159	12.610	209	14.457	259	16.093
10	3.162	60	7.746	110	10.488	160	12.649	210	14.491	260	16.124
11	3.317	61	7.810	111	10.536	161	12.689	211	14.526	261	16.155
12	3.464	62	7.874	112	10.583	162	12.728	212	14.560	262	16.186
13	3.606	63	7.937	113	10.630	163	12.767	213	14.595	263	16.217
14	3.742	64	8.000	114	10.677	164	12.806	214	14.629	264	16.248
15	3.873	65	8.062	115	10.724	165	12.845	215	14.663	265	16.279
16	4.000	66	8.124	116	10.770	166	12.884	216	14.697	266	16.310
17	4.123	67	8.185	117	10.817	167	12.923	217	14.731	267	16.340
18	4.243	68	8.246	118	10.863	168	12.962	218	14.765	268	16.371
19	4.359	69	8.307	119	10.909	169	13.000	219	14.799	269	16.401
20	4.472	70	8.367	120	10.955	170	13.038	220	14.832	270	16.432
21	4.583	71	8.426	121	11.000	171	13.077	221	14.866	271	16.462
22	4.690	72	8.485	122	11.045	172	13.115	222	14.900	272	16.492
23	4.796	73	8.544	123	11.091	173	13.153	223	14.933	273	16.523
24	4.899	74	8.602	124	11.136	174	13.191	224	14.967	274	16.553
25	5.000	75	8.660	125	11.180	175	13.229	225	15.000	275	16.583
26	5.099	76	8.718	126	11.225	176	13.267	226	15.033	276	16.613
27	5.196	77	8.775	127	11.269	177	13.304	227	15.067	277	16.643
28	5.292	78	8.832	128	11.314	178	13.342	228	15.100	278	16.673
29	5.385	79	8.888	129	11.358	179	13.379	229	15.133	279	16.703
30	5.477	80	8.944	130	11.402	180	13.416	230	15.166	280	16.733
31	5.568	81	9.000	131	11.446	181	13.454	231	15.199	281	16.763
32	5.657	82	9.055	132	11.489	182	13.491	232	15.232	282	16.793
33	5.745	83	9.110	133	11.533	183	13.528	233	15.264	283	16.823
34	5.831	84	9.165	134	11.576	184	13.565	234	15.297	284	16.852
35	5.916	85	9.220	135	11.619	185	13.602	235	15.330	285	16.882
36	6.000	86	9.274	136	11.662	186	13.638	236	15.362	286	16.912
37	6.083	87	9.327	137	11.705	187	13.675	237	15.395	287	16.941
38	6.164	88	9.381	138	11.747	188	13.711	238	15.427	288	16.971
39	6.245	89	9.434	139	11.790	189	13.748	239	15.460	289	17.000
40	6.325	90	9.487	140	11.832	190	13.784	240	15.492	290	17.029
41	6.403	91	9.539	141	11.874	191	13.820	241	15.524	291	17.059
42	6.481	92	9.592	142	11.916	192	13.856	242	15.556	292	17.088
43	6.557	93	9.644	143	11.958	193	13.892	243	15.588	293	17.117
44	6.633	94	9.695	144	12.000	194	13.928	244	15.620	294	17.146
45	6.708	95	9.741	145	12.042	195	13.964	245	15.652	295	17.176
46	6.782	96	9.798	146	12.083	196	14.000	246	15.684	296	17.205
47	6.856	97	9.849	147	12.124	197	14.036	247	15.716	297	17.234
48	6.928	98	9.899	148	12.166	198	14.071	248	15.748	298	17.263
49	7.000	99	9.950	149	12.207	199	14.107	249	15.780	299	17.292
50	7.071	100	10.000	150	12.247	200	14.142	250	15.811	300	17.321

Table II. Trigonometric Ratios

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	.0000	1.0000	.0000	45°	.7071	.7071	1.0000
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	∞

INDEX

PAGE

A

Abbreviations viii
Abscissa 519
Ahmes (ä′měz) 38
Algebra
  solutions by . 63–65, 220, 248–251, 254, 328, 331, 397, 414, 477
Altitude
  of an equilateral triangle 256
  of a parallelogram 202
  of a trapezoid 215
  of a triangle 118
Analysis of a problem 369
Analytic geometry 519
Angle 16
  acute 21
  bisector of an 25
  central 285
  common 80
  dihedral 228
  exterior 119
  formed by rays 16
  generating an 16
  inscribed in arc 286
  measurement of an 22, 320
  naming an 16
  oblique 21
  obtuse 21
  of depression 501
  of elevation 501
  right 20
  sides of an 16, 138
  size of an 17
  vertex 78
  vertex of an 16
  wind-correction 552
Angles
  addition of 17
  adjacent 20
  alternate exterior 129
  alternate interior 129
  base 78
  complementary 24
  corresponding, of lines and transversals 129
  corresponding, of polygons . 98, 403

PAGE

  equal 17
  opposite sides of a triangle . . . 77
  subtraction of 17
  supplementary 24
  vertical 24
Antenna, parabolic 361
Antiphon (ăn′tĭ·fŏn) 40
Any, meaning of word 103
Apothem 454
Applications of geometry relating to
  architecture 133, 203, 293, 312, 437, 453
  arts 292–294
  aviation 541–546
  botany 265
  carpentry 114, 163, 164, 225, 273, 339
  city planning 19
  debating 11
  designs 338, 483
  dressmaking 83
  engineering . 2–3, 7, 79, 113, 265, 311, 442, 501
  everyday reasoning . 11, 37, 47, 49, 50–51, 71–72, 117, 141, 150, 230, 315, 340, 376, 487
  finding distances . 113, 163, 227, 412, 441
  map and scale drawing . 425, 436–437, 441
  mechanical drawing . 137, 156, 164, 197, 207, 441
  miscellaneous situations 273, 306, 441
  nature 165, 481, 482, 484
  navigation 162, 323, 339
  painting 293
  science . . 140, 168, 169, 197, 265, 303, 306, 307, 361, 442, 444
  sculpture 294
  shop work 13
  sports 262, 340
  surveying 170, 264, 265, 495
Approximate numbers, computation with 259
Arc 286
  center of an 286
  major 286

PAGE

Arc (*cont.*)
measurement of an 321
midpoint of an 286
minor 286
Arc degree 321
Archimedes (är′kĭ·mē′dēz) . . 40, 468
Architecture . 133, 203, 293, 312, 437, 453
Archytus (är·kī′tăs) 413
Arcs
comparison of 321
equal 286
Area
of a circle 471
of a circular cylinder 484
of a figure 237
of an irregular polygon 263
of a prism 275
of a rectangle 237
of a right circular cone 485
of a sector of a circle 471
of a segment of a circle 472
of a sphere 486
of a trapezoid 245
of a triangle 243, 256, 259
unit of 237
Aristotle (ăr′ĭs·tŏt′l) 40
Assumptions
in plane geometry . 52, 56, 84, 91, 98, 129, 131, 134, 153, 237, 288, 289, 321, 349, 350, 466, 505–506
in space geometry . 173, 176, 275, 276
Aviation 541–546
Axial symmetry 165
Axioms 52
used in solving equations . . . 63

B

Base
of a parallelogram 202
of a triangle 77
Bases
for proof, *see* Summaries
of a trapezoid 215
Bearing of a point 545
Bisector
of an angle 25
of a line segment 25
Bolyai (bô′lyoi) 40, 135

PAGE

C

Carpentry . 114, 163, 164, 225, 273, 339
Celestial navigation 546
Center
of a circle 26
of a regular polygon 454
Central symmetry 165
Centroid 366
Chord 285
Circle 26, 285
angles related to a 285
arc of a 26, 286
area of a 471
center of a 26
central angle of a 285
chord of a 285
circumference of a 286
circumscribed 290
diameter of a 26, 285
equation of a 537
inscribed 330
inscribed angle of a 285
lines related to a 285
measurement of a 463
radius of a 26, 285
regular polygons and the 450
secant of a 285
sector of a 471
segment of a 367
tangent to a 285
Circles
concentric 354
congruent 285
equal 285
Euler's 116
line of centers of two 285
tangent 308
Circumcenter 362
Circumcircle 366
Circumference 286
Circumradius 366
City planning 19
Coincide 12
Coincidence method 159
Commensurable quantities 386
Complement of an angle 24
Conclusion 45, 48
Concurrent lines 361
Condition, necessary and sufficient 355

PAGE

Cone, right circular 313
Congruent figures 81
  corresponding parts of 98
Constant 464
Constructions in geometry 15
  directions for solving 194
  procedure in making 185
Contrapositive 151
Converse 149
  assuming the 314
Coordinate geometry 518
Coordinates, cartesian 519, 520
Corollary 90
  proving a 90
Cosine 493
Cosines, law of 439, 502
Course 549
Cube 274
Curve 6
  compound 311
Cylinder 312
  area of a 484
  right circular 312
  volume of a 484

D

Dead reckoning 546
Debating 11
Decagon 201
  inscribing a regular 478
Deductive reasoning 44
  in column form 59
Definitions 9
  good 9
  importance of 5
  reversible 9
  used in proofs 65
Degree
  angle 22
  arc 321
Demonstration of a proof 68
Depression, angle of 501
Descartes (dā·kärt′) 40, 519
Designs 338, 483
Diagonal of a polygon 201
Diameter 26, 285
Digits, significant 261
Dihedral angles 228
  edges of 228

PAGE

  faces of 228
Direction
  meaning of 544
  of wind 545
Directions
  for proving theorems 103
  for solving construction problems 194
Distance
  between two points 12
  from a point to a circle 348
  from a point to a line 21
  measured indirectly . 113, 163, 227, 412, 441
Divine proportion 475
Drift angle 549

E

Electrical engineering 7
Elevation, angle of 501
Ellipse 358
Engineering . 2–3, 7, 79, 113, 265, 311, 442, 501
Equal line segments 12
Equation, quadratic 248
Euclid (ū′klĭd) 40, 253
Eudoxus (ū·dŏk′sŭs) 40
Euler's circles (oi′lēr) 116
Extreme and mean ratio 475
Extremes of a proportion 381

F

Fibonacci series 481
Figures
  congruent 81, 98, 285
  determining 278
  equal in area 240
  similar 403
Fix, how to establish a 547
Foot
  of a line 175
  of a perpendicular 20
Formula, Hero's 259
Formulas for reference 574
Functions, trigonometric 493

G

Gauss (gous) 40, 479
Geometry
  analytic 519

PAGE

Geometry (*cont.*)
coordinate 518
space 171, 228, 274, 312, 375, 443, 484
Given, the 68
Golden section 475, 481
Graph of a linear equation 531

H

Harmonic division 395
Hawaii 165, 453
Heptagon 201
Herodotus (hē·rŏd′ō·tŭs) 38
Heron 40
Hero's formula 259
Hexagon 201
Hipparchus (hĭ·pär′kŭs) 40
Hippias (hĭp′pĭ·ăs) 40
Hippochrates (hĭ·pŏk′rȧ·tēz) . . . 40
Historical notes . 38, 135, 142, 155, 196, 253, 259, 325, 395, 413, 417, 468, 479, 519
Hyperbola 358, 359
Hypotenuse 80
Hypothesis 48

I

Identity 53
Incenter 363
Incircle 367
Included sides and angles 80
Incommensurable quantities . . . 386
Indirect method of proof 130
Inductive reasoning 130
Inequalities 504
assumptions on 505
Inradius 367
Intercept form 536
Interception 553
Interpolation 497
Intersect 8
Inverse theorem 150
Inversion 384
Isosceles triangle 77

L

Law
of cosines 502
of sines 503

PAGE

Legs of a right triangle 80
Limits 464
theorems on 465
Line
broken 10
curved 6
inclination of a 520
of centers 285
perpendicular to a plane 175
representing a 6
slope of a 521
straight 6
Line segments 9
addition of 14
bisectors of 25
corresponding 98
harmonic division of 395
internal and external division of 393
subtraction of 14
Lines
associated with triangles 118
auxiliary 105
concurrent 361
equations of 531–536
oblique 21
parallel 129, 524
perpendicular 20, 524
skew 176
Lobachevsky (lō·bȧ·chĕf′skĭ) . 40, 135
Location of points 166
Loci 166, 346
assumptions on 349
compound 356, 538
directions for determining a . . 348
directions for proving a theorem on 351
in space 375
linear 532

M

Major premise 115
Maps and scale drawing . . 425, 436–437, 441
Mass spectrometer 36
Mean proportional 381
Means 381
Measurement
approximate 12
of angles 22, 321

PAGE

of arcs 321
of a circle 463
Mechanical drawing . . 137, 156, 164, 197, 207, 441
Median
of a trapezoid 215
of a triangle 118, 440
Methods of proof 130, 156
Minor premise 115
Minute in angle measurement . . 22

N

n-gon 201
Nature, applications in . . 165, 481, 482, 484
Navigation 162, 323, 339
Nonagon 201
Non-Euclidean geometry 135

O

Octagon 201
Offsets 264
Optical illusions 31
Ordinate 519
Orthocenter 364
Orthographic projection 560
Overdetermine 278

P

Painting 293
Pantograph 441
Parabola 358
Parallel ruler 207
Parallelepiped 274
Parallelogram 202
altitude of a 202
area of a 241
base of a 202
Parts
corresponding 98
of a triangle 77
Pascal 197
Pentadecagon 201
Pentagon 201
Pentagram 18
Perimeter
of a polygon 201
of a triangle 77

PAGE

Periscope 169, 227
Perpendicular 20
foot of 20
to a plane 175
Physics . 168–169, 197, 306–307, 359, 437, 541
Pi (π) 468
computation of 470
Plane 6
fundamental properties of a . . 173
line perpendicular to a 175
Planes
parallel 176
perpendicular 228
Planimeter 263
Planning the proof 104, 210
Plato (plā′tō) 40
Point 6
of tangency 285
Point-slope form 533
Points, location of 166
Polygon 77, 201
adjacent sides of a 77
apothem of a regular 454
area of a 263
center of a regular 454
central angle of a regular . . . 455
circumscribed 330
concave 201
consecutive angles of a 77
convex 201
diagonal of a 201
equiangular 201
equilateral 201
inscribed 290
perimeter of a 201
radius of a regular 454
regular 201, 450
sides of a 201
vertices of a 77, 201
Polygons
and circles 450
areas of 236
classified by angles 201
classified by sides 201
similar 403
Postulate 52
parallel 135
Principle of continuity 332

PAGE

Prisms 274
areas of 275
bases of 274
edges of 274
faces of 274
regular 274
right 274
volumes of 275
Problems 185
famous 196
used in proofs 188
Projection
of a line segment 414
orthographic 560
Projective geometry 129
Proof
analytic method of 156
by coincidence 159
by demonstration 51
by superposition 82
direct method of 130
discovering a 209
formal 68
how to make a formal . 68, 103, 209
in geometry 51
indirect method of 130
one-cycle 61
planning the 104, 210
summaries of principal bases for 74, 123, 179, 199, 233, 280, 316, 342, 378, 399, 446, 489, 563
synthetic 156
Proportion 381
fundamental theorems on . . . 383
line segments in 386
terms of a 381
Proportional
fourth 381
mean 381
third 381
Proportional dividers 441
Proposition 185
Protractor 22
Ptolemy (tŏl′ĕ·mĭ) 40
Ptolemy's theorem 436
Pyramid 276
altitude of a 276
base of a 276
frustum of a 277

PAGE

lateral area of a 276
lateral edges of a 276
regular 276
volume of a 276
Pythagoras (pĭ·thăg′ō·răs) . . 39, 253
Pythagorean theorem 252, 417

Q

Quadratic equations 248
Quadrilateral 201
Quantities
commensurable 386
incommensurable 386
ratio of 386

R

Radar 323
Radicals 247
Radicand 247
Radius
of action 555
of a circle 26, 285
of a regular polygon 454
of a sphere 313
Ramses II 38
Ratio of similitude 404
Ratios 238, 381
extreme and mean 475
of geometric quantities 386
trigonometric 493
Ray, definition of 9
Reasoning
deductive 45
in a circle 487
in everyday life . 11, 37, 47, 49, 50–51, 71, 117, 141, 150, 230, 315, 340, 376, 487
indirect 376
inductive 35
Rectangle 202
area of a 237
Reflection . 168–169, 197, 198, 306–307, 359
Resultant 437, 541
Reviews, chapter . 27, 40, 72, 122, 178, 198, 231, 279, 315, 341, 377, 398, 445, 488
See also Tests *and* Word lists
Rhombus 202

PAGE

Riemann (rē′män) 40, 135
Right triangle 78
  parts of a 80
Roots
  square, how to find the 247
  table of 575

S

Saccheri (săc′kā·rē) 40
Scale drawings 436
Science . 140, 168, 169, 197, 265, 303, 306, 307, 361, 442, 444
Secant 285
  from a point 300
  segment of a 418
Sector 471
Segment
  of a circle 367
  of a line 9
  of a secant 418
  projection of a line 414
Segments
  addition of 14
  bisectors of 25
  equal 12
  harmonic division of 395
  in proportion 386
  internal and external division of 393
  of a circle 367
Semicircle 286
Sides
  adjacent 77
  corresponding 98, 406
  included by angles 80
  of an angle 16
  of a polygon 77
Significant digits 261
Similar polygons 403
  conditions for 403
Sine 493
Sines, law of 503
Size of an angle 17
Skew lines 176
Slope of a line 521
Slope-intercept form 534
Solid 8
  geometric 8
  physical 8

PAGE

Solution
  of a construction problem . 185, 194, 348, 430
  trigonometric 499
Space geometry . 171, 228, 274, 312, 375, 443, 484
Sphere 313
  area and volume of a 486
Sports 262, 340
Square 202
  as a unit of surface 237
Square roots
  how to find 247
  table of 575
Statements
  general 45
  specific 45
Summaries of principal bases for proof . 74, 123, 179, 199, 233, 280, 316, 342, 378, 399, 446, 489, 563
Superposition 82
Supplement of an angle 24
Surface
  measurement of a 237
  plane 172
Surveying 170, 264, 265, 495
Syllogism 115
  three steps of a 115
Symbols viii
Symmetry 165
  axial 165
  central 165
Synthetic method of proof 156

T

Table
  of formulas 574
  of linear measure 574
  of square measure 574
  of square roots 575
  of trigonometric ratios 576
Tangent 285
  common external 308
  common internal 308
  from an external point 300
  of an angle 494
Television, parallel lines in 139
Terms, undefined 5

PAGE

Tests
applications . . 125, 181, 235, 282, 344, 401, 449, 491, 503, 539
completing statements . 317, 448, 517
constructions . 199, 235, 282, 344, 379, 448
drawing the figure 283
matching 43
multiple-choice . . 182, 234, 319, 343
on hypothesis and conclusion 124, 181
on proofs 127
stating the *Given* and *To prove* . 319
supplying reasons . 75, 126, 180, 318
true-false . 29, 124, 281, 379, 400, 447, 490, 503, 516
unclassified 42
Thales (thā'lēz) 38, 325
Theorem(s) 67
assumptions on equivalent . . . 153
contrapositive of a 151
converse of a 149
directions for proving 103
equivalent 152
inverse of a 150
Pythagorean 252, 417
Track 549
Transit 495
Transportation 287
Transversal 129
Trapezoid 202, 215
altitude of a 215
area of a 245
bases of a 215
median of a 215
Trapezoid formula 264
Triangle 77, 201
acute 78
altitude of a 118
area of a 243, 256, 259
base of a 77
centroid of a 366
circumcenter of a 362
circumcircle of a 366
circumradius of a 366
equiangular 78

PAGE

equilateral 77
exterior angle of a 119
incenter of a 363
incircle of a 367
inradius of a 367
isosceles 77
median of a 118
notation in a 369
oblique 78
obtuse 78
orthocenter of a 364
parts of a 77
perimeter of a 77
right 78
scalene 77
vertex of a 77
Triangles
classified 77, 78
overlapping 87
Trigonometry 492
Trisection of an angle 196
Two-point form 534

U

Undefined terms 5
Underdetermine 278
Unit
of angle measurement 22
of arc measurement 321
of area measurement 237

V

Variable 464
Vector 551
Vertices of a polygon 77, 201
Volume
of a circular cone 485
of a circular cylinder 484
of a right prism 275
of a solid 275
of a sphere 486

W

Word lists . 28, 41, 75, 123, 180, 198, 234, 281, 317, 378, 399, 490

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